Matrices, Norms and Quadratic forms.

$$\|A\|_p = \sup_{\|x\|_p = 1} \|Ax\|_p .$$

$$\|x\|_p = \left(\sum |x_i|^p\right)^{1/p}.$$

- A symetric if $A = A^T$
- A skew-symetric $A = -A^T$.
- Quadratic form associated with matrix A. is $q : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$q(x) = \langle Ax, x \rangle = \sum_{j,k=1}^{n} a_{jk} x_j x_k$$

$$(1) \begin{cases} \nabla q(x) = (A + A^T)\, x \\ \Delta^2 q(x) = A + A^T \end{cases}$$

If $A = -A^T$ then $\nabla q = 0$ on $\mathbb{R}^n$

$\Downarrow$

$q = 0$ on $\mathbb{R}^n$.

- Symetric part of A is the matrix

$$A_s = \frac{(A + A^T)}{2}$$

Thus (1) becomes $\begin{cases} \nabla q(x) = 2 A_s x \\ \Delta^2 q(x) = 2 A_s \end{cases}$

Without loss of generality ~~A~~ may be assumed symmetric since.

$\langle Ax, x\rangle = \langle A_s x, x\rangle$.

- A is positive semidefinite, $A \in M_{n\times n}$ if.

$q(x) = \langle Ax, x\rangle \geq 0 \quad \forall x \in \mathbb{R}^n$.

- A is positive definite if $q(x) > 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}$

- $A = A^T$, $A \in M_{n\times n}$. The function $q : \mathbb{R}^n \to \mathbb{R}$

$q(x) = \langle Ax, x\rangle$ is convex iff.

A is positive semi-definite.

q is strictly convex iff A is positive definite.

q convex iff

$$q(y) \geq q(x) + \langle \nabla q(x), y - x\rangle \quad \forall x, y \in \mathbb{R}$$

$$\Updownarrow$$

$$q(y) \geq q(x) + 2\langle Ax, y - x\rangle \quad \forall x, y \in \mathbb{R}^n.$$

$$\Updownarrow$$

$$\langle AY, Y\rangle \geq \langle AX, X\rangle + 2\langle AX, Y-X\rangle.$$
$$\Updownarrow$$
$$\langle A(Y-X), Y\rangle \geq \langle AX, Y\rangle - \langle AX, X\rangle.$$
$$\Updownarrow$$
$$\langle A(Y-X), Y\rangle \geq \langle X, AY\rangle - \langle X, AX\rangle$$
$$\Updownarrow$$
$$\langle A(Y-X), Y-X\rangle \geq 0 \qquad \forall X, Y \in \mathbb{R}^n$$
$$\Downarrow$$
$$\langle AZ, Z\rangle \geq 0 \qquad \forall Z \in \mathbb{R}^n$$

If $\langle AZ, Z\rangle > 0$ then q is strictly convex since $\nabla q(\cancel{z}) = AZ$

- Bilinear form $B: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$.

  $B(X, Y) = \langle AX, Y\rangle = [X, Y]_A$.

Assume A is a positive definite symmetric $n\times n$ matrix. Then $B(X,Y) = \langle AX, Y\rangle$ defines an inner product on $\mathbb{R}^n$ and

$\exists\ 0 < c_1 \leq c_2$ s.t.
$$c_1 \|X\|_2^2 \leq B(X, X) \leq c_2 \|X\|_2^2 \qquad \forall X \in \mathbb{R}^n$$

For inner product verification it is enough to check $B(x,x) \geq 0$. with equality only if $x=0$. This is guaranteed by positive-definiteness character of A.

Consider $C_1 = \min_{x \in S_1} q(x)$, $C_2 = \max_{x \in S_1} q(x)$.

Obviously $C_1 > 0$ due to A being positive definite.

$$C_1 \|x\|_2^2 \leq q(x) \leq C_2 \|x\|_2^2$$

$f: C \to \mathbb{R}$ (C^2) on C and define for some $x, y \in \mathbb{R}^n$.

$$\varphi(t) = f((1-t)x + ty)$$

$$\varphi'(t) = \langle \nabla f(x(t)), y-x \rangle.$$

$$\varphi''(t) = \langle D^2 f(x(t))(y-x), y-x \rangle.$$

$$f(y) = f(x) + \langle \nabla f(x), y-x \rangle + \frac{1}{2} \langle D^2 f(x(\tau))(y-x), y-x \rangle$$

for some $\tau \in (0,1)$.

Second order derivative criterion for convexity.

- Let $f: C \to \mathbb{R}$, $f \in C^2$, and C convex. Then f is convex on C iff $D^2 f(x)$ is positive semi-definite on C.

  If $D^2 f(x)$ is positive definite on C, then f is strictly convex.

<u>Energy principles for linear equations</u>

$$Ax = b, \qquad A \in M_{n \times n}, \qquad A = A^T.$$

$$\mathcal{E} : \mathbb{R}^n \to \mathbb{R},$$

$$\mathcal{E}(x) = \langle Ax, x \rangle - 2\langle b, x \rangle =$$

$$= \sum_{j,k=1}^{n} a_{jk} x_j x_k - 2 \sum_{j=1}^{n} b_j x_j$$

<u>Energy principle</u>

Find minimizers of $\mathcal{E}$ on $\mathbb{R}^n$ and

$$\alpha(\mathcal{E}) = \inf_{x \in \mathbb{R}^n} \mathcal{E}(x)$$

$\nabla \mathcal{E}(x) = 2(Ax-b)$. So.

$\hat{x}$ is a critical point of $\mathcal{E}$ iff it is a solution of $Ax=b$.

- Assume A is positive semi-definite symmetric matrix and $\mathcal{E}$ is defined by $\mathcal{E}(x) = \langle Ax, x\rangle - 2\langle b, x\rangle$. $\hat{x}$ minimizes $\mathcal{E}$ on $\mathbb{R}^n$ iff $\hat{x}$ is a solution of $Ax=b$.

- A is positive semi-definite then $\mathcal{E}$ is convex and then any critical point is a minimizer

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What if A is not positive semi-definite?

$\mathcal{E}$ is not convex, $\mathcal{E}(td) \longrightarrow -\infty$, $t \to \infty$

for d s.t $\langle Ad, d\rangle < 0$.

- let A, positive definite symmetric matrix. Then there is a unique minimizer of $\mathcal{E}$ on $\mathbb{R}^n$. which is the unique solution of $Ax=b$ and A has an inverse A^{-1} that is positive definite and symmetric.

$$\|A^{-1}b\| \leq c_1^{-1}\|b\|_2 \quad \text{and}$$

$$\min_{x\in\mathbb{R}^n} \mathcal{E} = -\langle A^{-1}b, b\rangle.$$

<u>Proof</u>. A positive definite then

$\exists\, c_1 > 0$ s.t $\langle Ax, x\rangle \geq c_1\|x\|_2^2 \quad \forall x\in\mathbb{R}^n$.

$\mathcal{E}(x) \geq c_1\|x\|_2^2 - 2\|b\|\,\|x\|_2 \quad \forall x\in\mathbb{R}^n$

$\hookrightarrow$ Coercive

Then $\exists\, \hat{x}$ minimizer of $\mathcal{E}$ on $\mathbb{R}^n$.

If A is positive definite then $\mathcal{E}$ has a unique minimizer. which is the unique sol of $Ax=b$.

$b = e_j$ - canonical j-th element in $\mathbb{R}^n$.

Let ξ_j be the unique solution of

$$Ax = b \quad \text{in this case}$$

$$A\xi_j = e_j$$

A^{-1} is the matrix whose column vectors are these ξ_j

Then observe that solution of $Ax = b$ will be $\underline{\underline{A^{-1}b}}$

$\cancel{\langle A^{-1}b}$ Let $A\tilde{x} = d$.

$$\langle A^{-1}b, d\rangle = \langle \hat{x}, d\rangle = \langle \hat{x}, A\tilde{x}\rangle = \langle A\hat{x}, \tilde{x}\rangle$$
$$= \langle b, A^{-1}d\rangle$$

$$\Downarrow$$

A^{-1} symmetric

$$\langle A\hat{x}, \hat{x}\rangle = \langle b, \hat{x}\rangle. \Rightarrow c_1 \|\hat{x}\|_2^2 \le \|b\|_2 \|\hat{x}\|_2.$$

$$\mathcal{E}(\hat{x}) = -\langle b, \hat{x}\rangle. = -\langle A^{-1}b, b\rangle.$$

If $b \neq 0$, $-\langle b, \hat{x}\rangle < 0$ otherwise if it is zero. $\hat{x} = 0$ is a minimizer.

- If A is positive semi-definite then a vector $\hat{x}$ minimises q on $\mathbb{R}^n$ iff it obeys $Ax = 0$. The set of minimisers is a subspace of $\mathbb{R}^n$.

<u>Proof</u>. A - positive semi-definite then

$$\min_{x \in \mathbb{R}^n} q(x) = 0. \quad \text{and it is attained at } x = 0.$$

Also q is convex. $\Rightarrow$. minimisers are solutions of

$$\nabla q(x) = 0$$
$$\Updownarrow$$
$$Ax = 0$$

The set of minimisers $N(A)$ - kernel of A

- A is positive definite iff it is positive semi-definite and 0 is not an eigenvalue of A.

Let $\dim N(A) = m \geq 1$. , $W = (N(A))^{\perp}$. in $\mathbb{R}^n$.

$\forall x \in \mathbb{R}^n$, $x = y + z$, $y \in N(A)$, $z \in W$.

$$\mathcal{E}(x) = \mathcal{E}(z) - 2 \langle b, y \rangle.$$

Proof. $\min_{x \in \mathbb{R}^n} \mathcal{E}(x) = \min_{x \in W} \mathcal{E}(x)$.

$z \neq 0$, $z \in W$, $q(z) \neq 0$. so A is positive definite on W.

Thus q is coercive on W. and thus $\mathcal{E}$ is coercive on W. Thus $\mathcal{E}$ attains on min on W.

- Suppose $A \in M_{n \times n}$, symmetric and $\mathcal{E}$ defined as above, $\mathcal{E}(x) = \langle Ax, x \rangle - 2\langle b, x \rangle$. If $\mathcal{E}$ is convex and bounded below on $\mathbb{R}^n$. then there exists at least one minimizer. of $\mathcal{E}$ on $\mathbb{R}^n$. If $\mathcal{E}$ is strictly convex. this min is unique

Proof $\mathcal{E}(x)$ convex $\Rightarrow q$ convex. $\Rightarrow A$ is positive semi-definite

$\mathcal{E}$ bounded below $\Rightarrow \langle b, y \rangle = 0 \ \forall y \in N(A)$

Then the previous result implies that there are minimisers of $\mathcal{E}$ on $\mathbb{R}^n$.