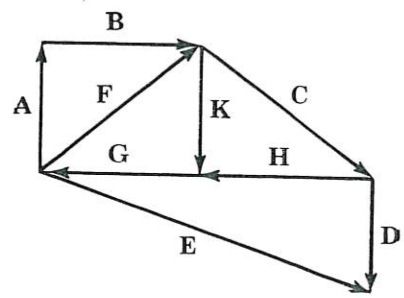

Figure 1.6

If $\mathbf{B}$ is a vector, $-\mathbf{B}$ is defined to be the vector with the same magnitude as $\mathbf{B}$ but opposite direction (fig. 1.5). Subtraction of vectors is defined by adding the negative:

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

The student who ignores this definition and simply memorizes figure 1.5 will inevitably confuse $\mathbf{A} - \mathbf{B}$ with $\mathbf{B} - \mathbf{A}$, which has the opposite direction. A good way of avoiding confusion is to keep in mind that $\mathbf{A} - \mathbf{B}$ is, algebraically, the vector that must be added to $\mathbf{B}$ to produce $\mathbf{A}$; hence it runs from the tip of $\mathbf{B}$ to the tip of $\mathbf{A}$, when $\mathbf{A}$ and $\mathbf{B}$ share a common tail.

The above definitions apply to the vector $\mathbf{0}$ if it is represented by a degenerate line segment. We have $\mathbf{0} = -\mathbf{0}$, $\mathbf{A} - \mathbf{A} = \mathbf{0}$, $\mathbf{A} + \mathbf{0} = \mathbf{A}$, and $\mathbf{0} + \mathbf{A} = \mathbf{A}$ for every vector $\mathbf{A}$. The zero vector (which should be distinguished from the zero scalar) does not have a well-defined direction.

Section 1.2

EXERCISES

The first four problems refer to figure 1.6.

1. Write $\mathbf{C}$ in terms of $\mathbf{E}, \mathbf{D}, \mathbf{F}$.
2. Write $\mathbf{G}$ in terms of $\mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{K}$.
3. Solve for $\mathbf{x} : \mathbf{x} + \mathbf{B} = \mathbf{F}$.
4. Solve for $\mathbf{x} : \mathbf{x} + \mathbf{H} = \mathbf{D} - \mathbf{E}$.
5. If $\mathbf{A}$ and $\mathbf{B}$ are represented by arrows whose initial points coincide, what arrow represents $\mathbf{A} + \mathbf{B}$?
6. By drawing a diagram, show that if $\mathbf{A} + \mathbf{B} = \mathbf{C}$, then $\mathbf{B} = \mathbf{C} - \mathbf{A}$.
7. Is the following statement correct? If $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and $\mathbf{D}$ are distinct nonzero vectors represented by arrows from the origin to the points A, B, C, and D respectively, and if $\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D}$, then $ABCD$ is a parallelogram.

Section 1.2

8. Let the sides of a regular hexagon be drawn as arrows, with the terminal point of each arrow at the initial point of the next.
   (a) If **A** and **B** are vectors represented by consecutive sides, find the other four vectors in terms of **A** and **B.**
   (b) What is the vector sum of all six vectors?

1.3 Multiplication of Vectors by Numbers

The symbol $|\mathbf{A}|$ denotes the *magnitude* of the vector **A.** Although it should not be confused with $|s|$, which denotes (as usual) the absolute value of a number s, it does have many properties that are quite similar. For example, $|\mathbf{A}|$ is never negative, and $|\mathbf{A}| = 0$ if and only if $\mathbf{A} = \mathbf{0}$. Since **A** and $-\mathbf{A}$ have the same magnitude, we can always write $|\mathbf{A}| = |-\mathbf{A}|$ and $|\mathbf{A} - \mathbf{B}| = |\mathbf{B} - \mathbf{A}|$. The "triangle inequality"

$$|\mathbf{A} + \mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}|$$

is the vector expression of the fact that any side of a triangle does not exceed, in length, the sum of the lengths of the other two sides (fig. 1.7).

If s is a number and **A** is a vector, $s\mathbf{A}$ is defined to be the vector having magnitude $|s|$ times that of **A** and pointing in the same direction if s is positive or in the opposite direction if s is negative. Any vector $s\mathbf{A}$ is called a *scalar multiple* of **A** (fig. 1.8).

Here are the fundamental properties of the operation of multiplying vectors by numbers:

$$0\mathbf{A} = \mathbf{0} \qquad 1\mathbf{A} = \mathbf{A} \qquad (-1)\mathbf{A} = -\mathbf{A} \tag{1.1}$$

$$(s + t)\mathbf{A} = s\mathbf{A} + t\mathbf{A} \tag{1.2}$$

$$s(\mathbf{A} + \mathbf{B}) = s\mathbf{A} + s\mathbf{B} \tag{1.3}$$

$$s(t\mathbf{A}) = (st)\mathbf{A} \tag{1.4}$$

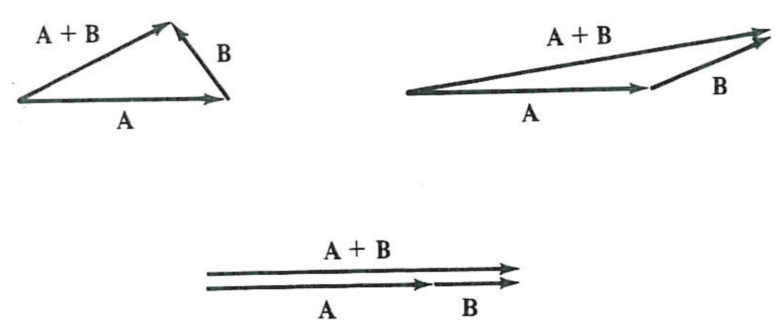

Figure 1.7

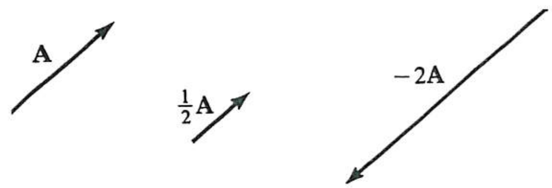

Figure 1.8

A vector whose magnitude is 1 is called a *unit vector*. To get a unit vector in the direction of **A**, divide **A** by $|\mathbf{A}|$ (equivalently, multiply **A** by $|\mathbf{A}|^{-1}$):

$$\left|\frac{\mathbf{A}}{|\mathbf{A}|}\right| = \frac{|\mathbf{A}|}{|\mathbf{A}|} = 1$$

Section 1.3

EXERCISES

1. Is it ever possible to have $|\mathbf{A}| < 0$?
2. If $|\mathbf{A}| = 3$, what is $|4\mathbf{A}|$? What is $|-2\mathbf{A}|$? What can you say about $|s\mathbf{A}|$ if you know that $-2 \leq s \leq 1$?
3. If **A** is a nonzero vector, and if $s = |\mathbf{A}|^{-1}$, what is $|-s\mathbf{A}|$?
4. If **B** is a nonzero vector, and $s = |\mathbf{A}|/|\mathbf{B}|$, what can you say about $|s\mathbf{B}|$?
5. If **A** is a scalar multiple of **B**, is **B** necessarily a scalar multiple of **A**?
6. If $\mathbf{A} - \mathbf{B} = \mathbf{0}$, is it necessarily true that $\mathbf{A} = \mathbf{B}$?
7. If $|\mathbf{A}| = |\mathbf{B}|$, is it necessarily true that $\mathbf{A} = \mathbf{B}$?
8. You are given a plane in space. How many distinct vectors of unit magnitude are perpendicular to this plane?
9. How many distinct vectors exist, all having unit magnitude, perpendicular to a given line in space?
10. If **A** is a nonzero vector, how many distinct scalar multiples of **A** will have unit magnitude?
11. Let **A** and **B** be nonzero vectors represented by arrows with the same initial point to points A and B respectively. Let **C** denote the vector represented by an arrow from this same initial point to the midpoint of the line segment AB. Write **C** in terms of **A** and **B**.
12. Prove that $|\mathbf{A} - \mathbf{B}| \geq |\mathbf{A}| - |\mathbf{B}|$.
13. Find nonzero scalars a, b, and c such that $a\mathbf{A} + b(\mathbf{A} - \mathbf{B}) + c(\mathbf{A} + \mathbf{B}) = \mathbf{0}$ for every pair of vectors **A** and **B**.
14. Derive a formula for a vector that bisects the angle between two vectors **A** and **B**.

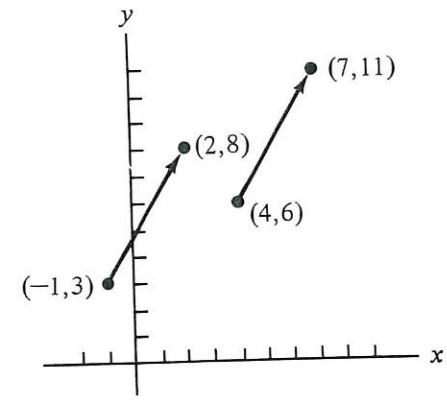

Figure 1.10

Example 1.1 The directed line segment extending from (4,6) to (7,11) is equivalent to the directed line segment extending from $(-1,3)$ to (2,8) because both of these directed line segments represent the vector $3\mathbf{i} + 5\mathbf{j}$ (fig. 1.10).

Section 1.4

EXERCISES

1. What is the x component of $\mathbf{i}$?
2. What is the x component of $\mathbf{j}$?
3. What is the magnitude of $\mathbf{i} + \mathbf{j}$?
4. What is the magnitude of $3\mathbf{i} - 4\mathbf{j}$?
5. With the axes in conventional position (fig. 1.9), directions may be specified in geographical terms. What is the unit vector pointing west? south? northeast?
6. Vector $\mathbf{A}$ is represented by an arrow with initial point (4,2) and terminal point $(5,-1)$. Write $\mathbf{A}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.
7. The direction of a nonzero vector in the plane can be described by giving the angle θ it makes with the positive x direction (see fig. 1.11). This angle is conventionally taken to be positive in the counterclockwise sense. Write A_1 and A_2 in terms of $|\mathbf{A}|$ and this angle θ.
8. In figure 1.11, if $|\mathbf{A}| = 6$ and $\theta = 30°$, determine A_1 and A_2.
9. In terms of $\mathbf{i}$ and $\mathbf{j}$, determine
   (a) the unit vector at positive angle 60° with the x axis.
   (b) the unit vector with $\theta = -30°$ (θ as in exercise 7).
   (c) the unit vector having the same direction as $3\mathbf{i} + 4\mathbf{j}$.
   (d) the unit vectors having x components equal to $\frac{1}{2}$.
   (e) the unit vectors perpendicular to the line $x + y = 0$.
10. Determine $|6\mathbf{i} + 8\mathbf{j}|$, $|-3\mathbf{i}|$, $|\mathbf{i} + s\mathbf{j}|$, $|(\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}|$.
11. In terms of $\mathbf{i}$ and $\mathbf{j}$, determine the vector represented by the arrow extending from the origin to the midpoint of the line segment joining (1,4) with (3,8).

Thus we are not free to assign values to α, β, and γ arbitrarily. (See exercises 20–22.)

There is no way of telling from the direction cosines what the magnitude of the vector may be; the magnitude must be specified separately. For example, *any* vector parallel to the yz plane and making an angle of 45° with the positive y and z directions has direction cosines

$$\cos\alpha = 0 \qquad \cos\beta = \frac{\sqrt{2}}{2} \qquad \cos\gamma = \frac{\sqrt{2}}{2}$$

Section 1.5

EXERCISES

In the first seven problems, let $\mathbf{A} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{B} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, and $\mathbf{C} = 3\mathbf{i} - 4\mathbf{k}$.

1. Find $|\mathbf{A}|$, $|\mathbf{B}|$, and $|\mathbf{C}|$.
2. Find $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} - \mathbf{C}$.
3. Determine $|\mathbf{A} - \mathbf{C}|$.
4. For what values of s is $|s\mathbf{B}| = 1$?
5. Find the unit vector having the same direction as $\mathbf{A}$.
6. Let $\mathbf{A}$ and $\mathbf{C}$ be represented by arrows extending from the origin.
   (a) Find the length of the line segment joining their endpoints.
   (b) This line segment is parallel to one of the coordinate planes. Which one?
7. Let α denote the angle between $\mathbf{A}$ and the positive x direction. Determine $\cos\alpha$.
8. Determine all unit vectors perpendicular to the xz plane.
9. Compute $|\mathbf{i} + \mathbf{j} + \mathbf{k}|$.
10. Write the vector represented by P_1P_2 in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, if $P_1 = (3,4,7)$ and $P_2 = (4,-1,6)$.
11. Write down the vector represented by the directed line segment OP, if O is the origin and $P(x,y,z)$ is a general point in space.
12. Let $\mathbf{D} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{E} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, and $\mathbf{F} = \mathbf{i} - \mathbf{j}$. Determine scalars s, t, and r, such that $4\mathbf{i} + 6\mathbf{j} - \mathbf{k} = s\mathbf{D} + t\mathbf{E} + r\mathbf{F}$.
13. What are the direction cosines of the vector $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$?
14. Derive the identity $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.
15. Give a geometrical description of the locus of all points P for which OP represents a vector with direction cosine $\cos\alpha = \frac{1}{2}$ (O is the origin).
16. How many unit vectors are there for which $\cos\alpha = \frac{1}{2}$ and also $\cos\beta = \frac{1}{2}$? Illustrate with a diagram.
17. $\mathbf{A}$ is a vector with direction cosines $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, respectively. What are the direction cosines of the reflected image of $\mathbf{A}$ in the yz plane? (Think of the yz plane as a mirror.)
18. Determine all unit vectors for which $\cos\alpha = \cos\beta = \cos\gamma$.
19. Verify the commutative and associative laws of addition for space vectors by expressing them componentwise.

Section 1.5

20. If one is given the direction angles α and β of a vector, to what extent can γ be determined?

21. Why is it impossible for a vector to have direction angles $\alpha = 30°$ and $\beta = 30°$? Answer this both geometrically and in terms of the constraint eq. (1.5).

22. Generalize exercise 21: show that neither β nor γ can be less than $90° - \alpha$.

1.6 Types of Vectors

A first step in solving some problems in mechanics is to choose a coordinate system. For instance, if the problem involves a particle sliding down an inclined plane, it may be convenient to take one of the axes, say the x axis, parallel to the plane, and another axis, say the z axis, perpendicular to the plane. After we have chosen a particular coordinate system, we can speak of the *position vector* of the particle. This is the vector represented by the directed line segment extending from the origin (0,0,0) to the point (x,y,z) where the particle is located, and (in terms of **i**, **j**, and **k**) it is the vector $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Strictly speaking, we should not say "position vector of a particle" because this might give the false impression that it is an intrinsic property of the particle, whereas it also depends on the location of the origin of the coordinate system.

If a particle moves from an initial position (x_1,y_1,z_1) to another position (x_2,y_2,z_2), the *displacement* of the particle is the vector represented by the directed line segment extending from its initial position to its final position. This vector is $(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$. Notice that if the initial position vector is $\mathbf{R}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and the final position vector is $\mathbf{R}_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$, the displacement is $\mathbf{R}_2 - \mathbf{R}_1$. *The displacement of a particle is the final position vector minus the initial position vector* (fig. 1.15).

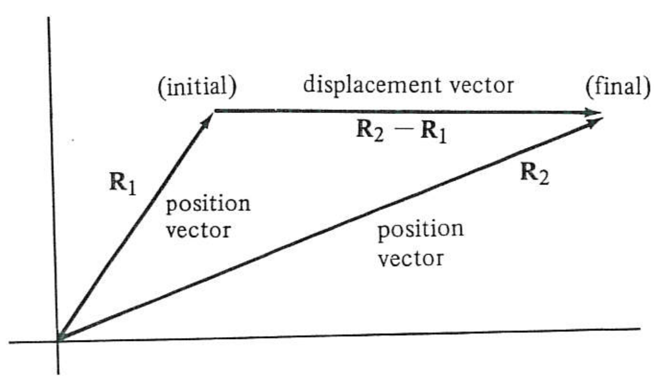

Figure 1.15

performed in the other order (try this with the textbook). In this light, it is even more remarkable that *angular velocity* is, nonetheless, a vector quantity. This matter is discussed in Appendix C.

Section 1.6

EXERCISES

1. A particle moves from (3,7,8) to (5,2,0). Write its displacement in terms of **i**, **j**, and **k**.
2. Write down the position vector of a particle located at the point (1,2,9).
3. The position vector of a moving particle at time t is $\mathbf{R} = 3\mathbf{i} + 4t^2\mathbf{j} - t^3\mathbf{k}$. Find its displacement during the time interval from $t = 1$ to $t = 3$.
4. What is the *magnitude* of the resultant of the following two displacements: 6 miles east, 8 miles north?
5. Strings are tied to a small metal ring and, by an arrangement of pulleys and weights, four forces are exerted on the ring. One force is directed upward with magnitude 3 lb, another is directed east with magnitude 6 lb, and a third is directed north with magnitude 2 lb. The ring is in equilibrium (i.e., it is not moving). What is the magnitude of the fourth force that is counterbalancing the other three?
6. The *center of mass* of a system of n particles is defined by the position vector

$$\mathbf{R}_{\text{cm}} = \frac{m_1\mathbf{R}_1 + m_2\mathbf{R}_2 + \cdots + m_n\mathbf{R}_n}{m_1 + m_2 + \cdots + m_n}$$

where the ith particle is located at $\mathbf{R}_i$ and has mass m_i. The *mass unbalance* of the system, measured at the position **R**, is defined to be

$$m_1(\mathbf{R}_1 - \mathbf{R}) + m_2(\mathbf{R}_2 - \mathbf{R}) + \cdots + m_n(\mathbf{R}_n - \mathbf{R})$$

Show that the mass unbalance, measured at the center of mass, is zero. (*Hint:* Try it first for $n = 1$ and $n = 2$.)

7. Suppose a particle of electrical charge q_1 is located at $\mathbf{R}_1$, and q_2 is located at $\mathbf{R}_2$. The *Coulomb force* on particle 1 due to particle 2 is proportional to q_1 and q_2, and inversely proportional to the square of the distance between them; it is directed along the line from q_2 to q_1. Write down a vector formula for this force.

1.7 Some Problems in Geometry

To avoid circumlocution, practically everybody who works with vectors makes no distinction between vectors and directed line segments. It is easier to say "the vector **A**" than to say "the vector represented by the directed line segment **A**." When we do this, it is still important to recognize that the concept of a vector is an *abstraction* from the concept of a directed line segment, in which we ignore the actual location of the directed line segment: we say "**A** equals **B**" when we really mean "the directed

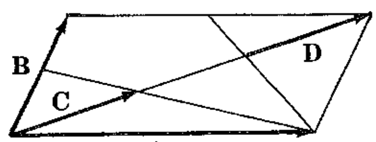

Figure 1.17

To conclude that $TUVW$ is a parallelogram, we need to show that $TU = -VW$. From the figure, TU may be expressible in terms of **A** and **B**; in fact, TU equals the "tip half" of **A** plus the "tail half" of **B**. Thus

$$TU = \tfrac{1}{2}\mathbf{A} + \tfrac{1}{2}\mathbf{B} = \tfrac{1}{2}(\mathbf{A} + \mathbf{B})$$

Similarly,

$$VW = \tfrac{1}{2}(\mathbf{C} + \mathbf{D})$$

But our basic relationship shows that $\mathbf{A} + \mathbf{B} = -(\mathbf{C} + \mathbf{D})$. Thus $TU = -VW$.

Example 1.3 Line segments are drawn from a vertex of a parallelogram to the midpoints of the opposite sides. Show that they trisect a diagonal.

Solution We have diagrammed the situation in figure 1.17, labeling certain vectors for convenience. Since the diagonal is $\mathbf{A} + \mathbf{B}$, the problem reduces to showing $\mathbf{C} = \mathbf{D} = \frac{1}{3}(\mathbf{A} + \mathbf{B})$. Let us try to express **C** in terms of **A** and **B**. First of all, certainly $\mathbf{C} = s(\mathbf{A} + \mathbf{B})$ for some scalar s. Also, since the tip of **C** lies on the line connecting the tip of **A** to the tip of $\frac{1}{2}\mathbf{B}$, we have $\mathbf{C} - \mathbf{A} = t(\frac{1}{2}\mathbf{B} - \mathbf{A})$ for some scalar t. If we equate the two expressions for **C**,

$$s(\mathbf{A} + \mathbf{B}) = \mathbf{A} + t(\tfrac{1}{2}\mathbf{B} - \mathbf{A})$$

we derive

$$(s - \tfrac{1}{2}t)\mathbf{B} = (1 - s - t)\mathbf{A}$$

Since **A** and **B** are not parallel, this equation can be true only if the scalars are zero:

$$s - \tfrac{1}{2}t = 0$$
$$1 - s - t = 0$$

Solving, we obtain $s = \frac{1}{3}$, so $\mathbf{C} = \frac{1}{3}(\mathbf{A} + \mathbf{B})$.

The reader should try to complete the solution as an exercise, manipulating **D** in an analogous manner.

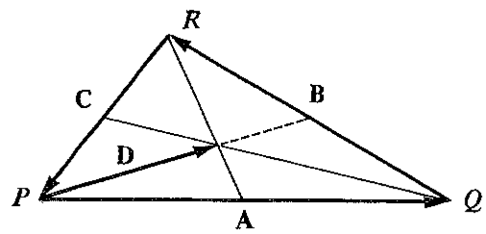

Figure 1.18

Example 1.4 Prove that the medians of a triangle intersect at a single point.

Solution In figure 1.18, **D** is the vector from the corner P to the point of intersection of the medians from Q and R. We must show that **D** lies along the median from P (i.e., that it is a multiple of $\mathbf{A} + \frac{1}{2}\mathbf{B}$). The condition that **D** lies along the median from R is expressed as

$$\mathbf{C} + \mathbf{D} = s(\mathbf{C} + \tfrac{1}{2}\mathbf{A})$$

for some number s, while the fact that **D** lies on the median from Q implies that, for some number t,

$$\mathbf{A} - \mathbf{D} = t(\tfrac{1}{2}\mathbf{C} + \mathbf{A})$$

Solving for **D** and equating the expressions, we derive

$$(s + \tfrac{1}{2}t - 1)\mathbf{C} = (1 - t - \tfrac{1}{2}s)\mathbf{A}$$

As in example 1.3, we conclude that both coefficients must vanish; thus

$$s = t = \tfrac{2}{3}$$

Using this in either equation for **D** and writing **C** in terms of **A** and **B**, we find

$$\mathbf{D} = \tfrac{2}{3}(\mathbf{A} + \tfrac{1}{2}\mathbf{B})$$

which is the form that we sought. Exercise 11 provides a delightful generalization of this example.

Example 1.5 Let θ denote the angle between two nonzero vectors **A** and **B**. Show that

$$\cos\theta = \frac{A_1B_1 + A_2B_2 + A_3B_3}{|\mathbf{A}||\mathbf{B}|} \tag{1.6}$$

(*Note: This is one of the most important identities in vector algebra.* Its significance is revealed in section 1.9.)

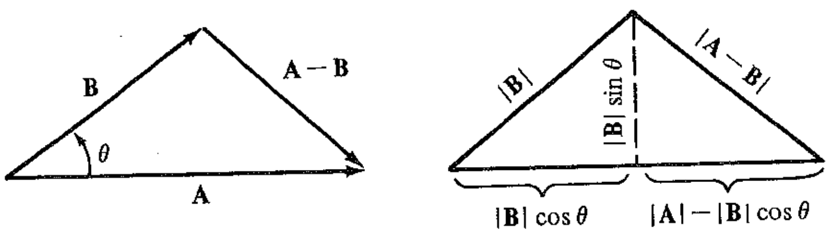

Figure 1.19

Solution This formula will "pop out" if we compare two expressions for $|\mathbf{A} - \mathbf{B}|^2$, one derived componentwise and one derived geometrically. Using components, we know that

$$|\mathbf{A} - \mathbf{B}|^2 = (A_1 - B_1)^2 + (A_2 - B_2)^2 + (A_3 - B_3)^2$$

Expanding powers and regrouping terms, we can write this as

$$|\mathbf{A} - \mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2(A_1B_1 + A_2B_2 + A_3B_3)$$

Now for the geometric formula. **A**, **B**, and θ are depicted in figure 1.19; also, the perpendicular from the tip of **B** to **A** is drawn, with the lengths of the appropriate segments indicated. We can visualize $\mathbf{A} - \mathbf{B}$ as the hypotenuse of a right triangle and, according to Pythagoras,

$$\begin{aligned}|\mathbf{A} - \mathbf{B}|^2 &= (|\mathbf{B}| \sin \theta)^2 + (|\mathbf{A}| - |\mathbf{B}| \cos \theta)^2 \\ &= |\mathbf{B}|^2 (\sin^2 \theta + \cos^2 \theta) + |\mathbf{A}|^2 - 2|\mathbf{A}||\mathbf{B}| \cos \theta \\ |\mathbf{A} - \mathbf{B}|^2 &= |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}| \cos \theta \qquad (1.7)\end{aligned}$$

Comparing this with the componentwise expression, we conclude

$$|\mathbf{A}||\mathbf{B}| \cos \theta = A_1B_1 + A_2B_2 + A_3B_3 \qquad (1.8)$$

which is equivalent to the desired identity.

Incidentally, by referring to figure 1.19, the alert reader will recognize eq. (1.7) as the *law of cosines* from trigonometry.

As applications of this formula, consider examples 1.6 and 1.7.

Example 1.6 Show that the vectors $\mathbf{A} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and $\mathbf{B} = \mathbf{i} + 7\mathbf{j} + \mathbf{k}$ are perpendicular.

Solution

$$\cos \theta = \frac{2 - 7 + 5}{\sqrt{30}\,\sqrt{51}} = 0$$

Hence $\theta = 90°$.

Example 1.7 Find the angle between
(a) $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} - \mathbf{j}$
(b) $\mathbf{i}$ and $\mathbf{i} + \mathbf{j} + \mathbf{k}$

Solutions

(a) $\cos\theta = \dfrac{1-2}{\sqrt{9}\sqrt{2}} = -0.2357; \theta = \cos^{-1}(-0.2357) = 103.63°$

(b) $\cos\theta = \dfrac{1}{\sqrt{1}\sqrt{3}} = 0.57735; \theta = \cos^{-1}(0.57735) = 54.74°$

Notice that the natural interpretation of the angle between two vectors always lies between 0° and 180°. Thus the principal value of the arccosine is perfectly appropriate for these calculations.

Summary: Geometrical and analytical descriptions

Now is a good time to catch our breath and get an overview of what we have learned. There are two ways of looking at vectors—geometrically and analytically. Geometric descriptions are more physical; a vector has magnitude and direction, and relationships are described in terms of lengths and angles. But it is often difficult to compute with these quantities, especially if the problem is three-dimensional and hard to sketch. Thus, to solve such problems as finding the resultant of several forces, we introduce a cartesian coordinate system and represent all vectors by their components. Then a vector becomes an ordered triple of numbers. (Another reason for using this rather unphysical component description is communication. How, for instance, does an astronaut on the moon convey information to his earthbound colleagues about a quantity with magnitude and direction? The astronaut must describe its components in some coordinate system common to both, as determined by, for instance, the fixed stars.)

Let us summarize the equations we have derived; they tell us how to relate one description to the other. The geometrical concept of length of a vector is computed in terms of components by

$$|\mathbf{A}| = (A_1^2 + A_2^2 + A_3^2)^{1/2}$$

The angle θ between two vectors $\mathbf{A}$ and $\mathbf{B}$ is computed from components using eq. (1.6):

$$\cos\theta = \frac{A_1B_1 + A_2B_2 + A_3B_3}{|\mathbf{A}||\mathbf{B}|}$$

In particular, the direction cosines of **A**, which are the cosines of the angles between **A** and the positive coordinate axes, can be computed by substituting **i**, **j**, or **k** for **B** in the above; thus

$$\cos\alpha = \frac{A_1}{|\mathbf{A}|} \qquad \cos\beta = \frac{A_2}{|\mathbf{A}|} \qquad \cos\gamma = \frac{A_3}{|\mathbf{A}|}$$

Viewed another way, these equations can be used to compute the component description of a vector from its geometric characteristics; we have

$$A_1 = |\mathbf{A}|\cos\alpha \qquad A_2 = |\mathbf{A}|\cos\beta \qquad A_3 = |\mathbf{A}|\cos\gamma$$

Table 1.1 Geometrical and analytical descriptions of vectors

Geometrical Quantities: Lengths, Angles, Cosines	Analytical Quantities: Cartesian Coordinates
Length of **A** = $\|\mathbf{A}\|$	$\sqrt{(A_1^2 + A_2^2 + A_3^2)}$
Angle between **A** and **B**	$\cos^{-1}[(A_1B_1 + A_2B_2 + A_3B_3)/\|\mathbf{A}\|\|\mathbf{B}\|]$
Direction cosines	$A_1/\|\mathbf{A}\|, A_2/\|\mathbf{A}\|, A_3/\|\mathbf{A}\|$

Hence the cycle is complete, and we are free to exploit whichever description, geometrical or analytical, is more convenient. Exercises 1 through 5 illustrate these ideas; see table 1.1.

EXERCISES

1. Find the angle between $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $3\mathbf{i} - 4\mathbf{k}$.
2. Find the angle between the x axis and $\mathbf{i} + \mathbf{j} + \mathbf{k}$.
3. Find the three angles of the triangle with vertices $(2,-1,1)$, $(1,-3,-5)$, $(3,-4,-4)$.
4. Find the angle between the xy plane and $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. (Note that $\mathbf{k}$ is perpendicular to the xy plane. You will have to decide what is meant by the angle between a vector and a plane.)
5. Show that $\mathbf{i} + \mathbf{j} + \mathbf{k}$ is perpendicular to the plane $x + y + z = 0$. (*Hint:* This plane passes through the origin. Show that $\mathbf{i} + \mathbf{j} + \mathbf{k}$ is perpendicular to every vector extending from the origin to a point in the plane.)
6. Imitate the solution of example 1.2, but instead of proving that $TU = -VW$, prove that $UV = -WT$.
7. Using vector methods, prove directly that if two sides of a quadrilateral are parallel and equal in magnitude, the other two sides are also.
8. By vector methods, show that the line segment joining the midpoints of two sides of a triangle is parallel to the third side, and has length equal to one half the length of the third side.

9. Show that the diagonals of a parallelogram bisect each other.
10. Construct another proof of the fact that the medians of a triangle intersect at a point, based on the following observation: if **D**, **E**, and **F** are vectors drawn from some fixed point to the corners of the triangle, then

$$\mathbf{D} + \tfrac{3}{2}[\tfrac{1}{3}(\mathbf{D} + \mathbf{E} + \mathbf{F}) - \mathbf{D}] = \tfrac{1}{2}(\mathbf{E} + \mathbf{F})$$

Verify this algebraically and then interpret it geometrically. [*Hint:* The tip of the vector $\frac{1}{3}(\mathbf{D} + \mathbf{E} + \mathbf{F})$ *is* this point of intersection.]

The following simple exercises are inserted here to help you recall some of the basic ideas of analytic geometry.

11. A treasure map has n villages marked on it, and it contains the following instructions. Start at village A, go $\frac{1}{2}$ of the way to village B, $\frac{1}{3}$ of the way to village C, $\frac{1}{4}$ of the way to village D, and so forth. The treasure is buried at the last stop. *Problem:* You lose the instructions, and don't know in what order to select the villages. *Show that it doesn't matter!* Then relate this to example 1.4 for $n = 3$.
12. Let PQR be a triangle. By vector methods, show there exists a triangle whose sides are parallel and equal in length to the medians of PQR.
13. True or false: $3x - 4y + 5z = 0$ represents a plane passing through the origin.
14. True or false: The yz plane is represented by the equation $x = 0$.
15. True or false: The locus of points for which $x = 3$ and $y = 4$ is a line parallel to the z axis whose distance from the z axis is 5.
16. True or false: $x^2 + y^2 + z^2 = 9$ is the equation of a sphere centered at the origin having radius 9.
17. Write down the equation of a sphere centered at the point (2,3,4) having radius 3.
18. Write down an equation for the cylinder concentric with the z axis having radius 2.
19. Do the equations $x = y = z$ represent a *line* or a *plane?*
20. What is the locus of points for which $x^2 + z^2 = 0$?
21. What is the locus of points for which $(x - 2)^2 + (y + 3)^2 + (z - 4)^2 = 0$?
22. What geometrical figure is represented by the equation $xyz = 0$? (Keep in mind that a product of numbers is zero if and only if at least one of the numbers is zero.)
23. What is the distance between the points (2,3,4) and (5,3,8)?
24. What is the distance between the point (3,8,9) and the xz plane? (Distance in such cases always means *shortest distance* or *perpendicular distance.*)
25. What is the distance between the point (0,3,0) and the cylinder $x^2 + y^2 = 4$? (You probably won't find a formula for this in any of your books. Just use some common sense.)
26. The expression $x^2 + y^2$ gives the square of the distance between (x,y,z) and the z axis. In view of this, what figure is represented by $x^2 + y^2 = z^2$?
27. Do you know what figure is represented by the equation $(x/2)^2 + (y/3)^2 + (z/4)^2 = 1$? (If so, you know more analytic geometry than is required to read this book.)