

ON THE SENSITIVITY OF ANOMALOUS LOCALIZED RESONANCE
PHENOMENA WITH RESPECT TO DISSIPATION

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Abstract. We analyze cloaking due to anomalous localized resonance in the quasi-static regime in the case when a general charge density distribution is brought near a slab superlens. If the charge density distribution is within a critical distance of the slab, then the power dissipation within the slab blows up as certain electrical dissipation parameters go to zero. The potential remains bounded far away from the slab in this limit, which leads to cloaking due to anomalous localized resonance. On the other hand, if the charge density distribution is further than this critical distance from the slab, then the power dissipation within the slab remains bounded and cloaking due to anomalous localized resonance does not occur. The critical distance is shown to strongly depend on the the rate at which the dissipation outside of the slab goes to zero.

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1. Introduction. In this paper, we discuss anomalous localized resonance phenomena observed at the interface between positive index and negative index materials. Such phenomena have been at the center of an interesting cloaking strategy [14, 6, 15, 23, 16, 22, 21, 5, 20, 29, 1, 2, 3, 18, 4, 27].

As illustrated in Figure 1, the (2D) geometry we consider consists of a central layer in $\mathcal{S} \equiv [0, a] \times (-\infty, +\infty)$ bordered by a layer to the left in $\mathcal{C} \equiv (-\infty, 0) \times (-\infty, +\infty)$ and a layer to the right in $\mathcal{M} \equiv (a, +\infty) \times (-\infty, +\infty)$. We work in the nonmagnetic quasistatic regime, i.e., the regime in which the magnetic permeability equals 1 and relevant wavelengths and attenuation lengths are much larger than other dimensions in the problem (such as a , the thickness of the slab \mathcal{S}). In this regime the complex electric potential V satisfies the Laplace equation

$$-\nabla \cdot [\varepsilon(x, y) \nabla V(x, y)] = \rho \quad \text{in } \mathbb{R}^2, \quad (1.1)$$

where ε is the dielectric constant (relative permittivity) and ρ is a given charge density distribution. (The potential V is also subject to certain continuity conditions and conditions at infinity — these are discussed in Section 2.) We assume that the charge density distribution ρ is real valued; we also take $\rho \in \mathcal{P}$, where

$$\mathcal{P} \equiv \{\rho \in L^2(\mathcal{M}) \cap L^\infty(\mathcal{M}) : \rho \text{ has compact support in } \mathcal{M}\}. \quad (1.2)$$

Throughout this paper we also assume

$$0 < |\text{supp } \rho| < \infty, \quad (1.3)$$

where $|U|$ denotes the Lebesgue measure of the set U . Note that this restriction on the support of ρ excludes dipolar sources.

For the purposes of the current paper we assume the layers are occupied by three different materials such that the imaginary parts of their dielectric constants are small (corresponding to small losses) and the real parts of their dielectric constants are equal but with opposite signs. In particular we take the dielectric constant $\varepsilon(x, y)$ to be

$$\varepsilon(x, y) \equiv \begin{cases} \varepsilon_c = 1 + i\mu & \text{if } x < 0, \\ \varepsilon_s = -1 + i\delta & \text{if } 0 \leq x \leq a, \\ \varepsilon_m = 1 & \text{if } x > a, \end{cases} \quad (1.4)$$

where $0 < \delta < 1$ and $\mu = \delta + \lambda\delta^\beta$ for some constants $\lambda \in \mathbb{R}$ and $\beta > 0$. In the limit $\delta \rightarrow 0^+$ the moduli (1.4) are that of a quasistatic two-dimensional superlens (“poor man’s superlens”). The question we address in this paper is to determine those ρ for which the power dissipation in this superlens blows up as $\delta \rightarrow 0^+$. As we shall explain shortly this is closely tied with cloaking due to anomalous resonance. Curiously we will see that the answer depends on the value of β , thus showing the sensitivity of the energy dissipation rate to perturbations. We will say that λ is *feasible* if

$$\lambda > 0 \quad \text{for } 0 < \beta < 1, \quad \lambda \geq -1 \quad \text{for } \beta = 1, \quad \text{or } \lambda \neq 0 \quad \text{for } \beta > 1. \quad (1.5)$$

We define $0 < \delta_\mu(\beta, \lambda) < 1$ such that $\mu \geq 0$ for $0 < \delta \leq \delta_\mu$ (which is required physically — the restrictions we placed on λ ensure that such a δ_μ exists). Note that the materials to the left and right of the slab are both vacuum if $\beta = 1$ and $\lambda = -1$. Given a charge

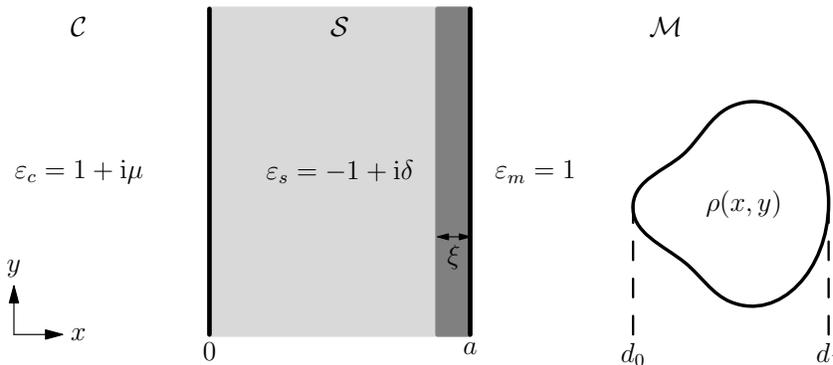


FIG. 1. We consider a slab geometry with a dielectric constant as illustrated in the figure — the slab (shaded light gray) is in the region $\mathcal{S} = [0, a] \times (-\infty, +\infty)$. The charge density ρ has compact support in the region $x > a$. For certain charge densities ρ that are close enough to a , the energy dissipation in the slab (in particular in the darkly shaded region $a - \xi < x < a$) tends to infinity as a sequence δ_j tends to 0.

density $\rho(x, y) \in \mathcal{P}$ with compact support in \mathcal{M} , we define

$$d_0 \equiv \min\{x : (x, y) \in \text{supp}(\rho)\} \quad \text{and} \quad d_1 \equiv \max\{x : (x, y) \in \text{supp}(\rho)\} \quad (1.6)$$

(see Figure 1). Since ρ has compact support in \mathcal{M} , we have

$$\text{supp } \rho \subseteq [d_0, d_1] \times [h_0, h_1] \quad (1.7)$$

for some constants $h_0 < h_1$. In order to enforce charge conservation, we require

$$\int_{d_0}^{d_1} \int_{h_0}^{h_1} \rho(x, y) \, dy \, dx = 0. \quad (1.8)$$

The physical charge density is $\Re(\rho e^{-i\omega t})$ and the physical time-harmonic electric field is given by $\mathbf{E} = \Re(-\nabla V e^{-i\omega t})$.

We say anomalous localized resonance (ALR) occurs if the following two properties hold as $\delta \rightarrow 0^+$ (see [17]):

- (1) $|V| \rightarrow \infty$ in certain localized regions with boundaries that are not defined by discontinuities in the relative permittivity and
- (2) V approaches a smooth limit outside these localized regions.

For example, when ρ is a dipole, $\varepsilon_c = \varepsilon_m = 1$, and when ALR occurs, as the loss in the lens (represented by δ) tends to zero the potential diverges and oscillates wildly in regions that contain the boundaries of the lens. It is important to note that the boundaries of the resonant regions move as the dipole is moved. Outside the resonant regions the potential converges to what we expect from perfect lensing (see [24, 25]). This behavior and its relation to sub-wavelength resolution in imaging (superlensing) were first discovered in [19] and were analyzed in more depth in [17].

In [17], Milton, Nicorovici, McPhedran, and Podolskiy showed that if ρ is a dipole and $\varepsilon_c = \varepsilon_m = 1$, then ALR occurs if $a < d_0 < 2a$, where d_0 is the location of the dipole. In this case there are two locally resonant strips — one centered on each face of

the slab. As mentioned above, outside these regions the potential converges to a smooth function that satisfies mirroring properties of a perfect lens. In particular, to an observer far enough to the right of the lens it will appear only as if there is a dipole at d_0 ; to an observer far enough to the left of the lens it will appear only as if there is a dipole located at $-d_0$ (see [17]). In neither case can the observer determine whether or not a lens is present. (However, if either observer is close to the lens, the presence of the lens will be obvious due to the resonance.) If $d_0 > 2a$, then there is no resonance and again the potential converges to a smooth function that satisfies the mirroring properties expected of a perfect lens. That is, to an observer far enough to the right of the lens (beyond the dipole) it will appear as if there is a dipole at d_0 and no lens, while to an observer to the left of the lens it will appear as if there is a dipole at $d_0 - a$ and no lens — see [24, 17, 31].

Cloaking due to ALR (CALR) can be understood from an energetic perspective. First, consider the quantity

$$E(\delta) \equiv \delta \int_0^a \int_{-\infty}^{\infty} |\nabla V|^2 dy dx; \quad (1.9)$$

$E(\delta)$ is proportional to the time-averaged electrical power dissipated in the slab. Suppose ρ is independent of δ such that, in the limit $\delta \rightarrow 0^+$, we have $E(\delta) \rightarrow \infty$ and $|V|/\sqrt{E(\delta)} \rightarrow 0$ for all $(x, y) \in \mathbb{R}^2$ with $|x| > b$ for some $b > 0$. This blow-up in the power dissipation is not physical, as it implies the fixed source ρ must produce an infinite amount of power in the limit $\delta \rightarrow 0^+$ (see, e.g., [14, 2]). The power dissipation was proved to blow up as $\delta \rightarrow 0^+$ for finite collections of dipolar sources close enough to the slab in [17, 14]; see also Bergman's work in [4].

To make sense out of this we rescale the source ρ by defining $\rho_r \equiv \rho/\sqrt{E(\delta)}$. Since (1.1) is linear, the associated potential will be $V_r \equiv V/\sqrt{E(\delta)}$ and, thanks to (1.9), the rescaled time-averaged electrical power dissipation will be

$$E_r(\delta) \equiv \delta \int_0^a \int_{-\infty}^{\infty} |\nabla V_r|^2 dy dx = \delta \int_0^a \int_{-\infty}^{\infty} \frac{|\nabla V|^2}{E(\delta)} dy dx = 1.$$

Thus the source ρ_r produces constant power independent of δ . Also, the rescaled potential satisfies $|V_r| = |V|/\sqrt{E(\delta)} \rightarrow 0$ as $\delta \rightarrow 0^+$ for $|x| > b$, implying that the source ρ_r becomes invisible in this limit to observers beyond $|x| = b$. This idea was introduced by Milton and Nicorovici [14]; also see [11, 2, 3].

Cloaking due to anomalous localized resonance in the quasistatic regime was first analyzed in [14]. In that paper the authors used separation of variables and rigorous analytic estimates to prove that if $\varepsilon_c = \varepsilon_m = 1$ and a fixed field is applied to the system (e.g., a uniform field at infinity), then a polarizable dipole located in the region $a < d_0 < 3a/2$ causes anomalous localized resonance and is cloaked in the limit $\delta \rightarrow 0^+$; if $\varepsilon_c \neq \varepsilon_m = 1$ (here ε_c has no relation to the value we chose in (1.4)), then the cloaking region becomes $a < d_0 < 2a$.

In [14], Milton and Nicorovici also derived analogous results for circular cylindrical lenses. In that case they assumed the relative permittivity was ε_c for $0 < r < r_c$, $\varepsilon_s = -1 + i\delta$ for $r_c < r < r_s$, and $\varepsilon_m = 1$ for $r_s < r$. With r_0 denoting the distance of the polarizable dipole from the origin, the cloaking region was found to be $r_s < r_0 < r_* = r_s^2/r_c$ if $\varepsilon_c \neq \varepsilon_m$ and $r_s < r_0 < r_{\#} = \sqrt{r_s^3/r_c}$ if $\varepsilon_c = \varepsilon_m$. In particular they proved

that an arbitrary number of polarizable dipoles within the cloaking region will be cloaked — see [23] for numerical verification of this result. They also extended their results to the finite-frequency and three-dimensional cases for the Veselago slab lens [28] (where $\varepsilon_c = \varepsilon_m = 1$).

To summarize, suppose $\varepsilon_c = \varepsilon_m = 1$ and the polarizable dipole is absent and a uniform electric field at infinity is applied to the slab lens configuration. The lens will not perturb this external field in the limit $\delta \rightarrow 0^+$, and, hence, is invisible to external observers [19, 17]. When the polarizable dipole is placed in this uniform field but outside of the cloaking region (so $d_0 > 3a/2$), it will become polarized and create a dipole field of its own which interacts with the lens. If $d_0 > 2a$ as well there will be no resonance in the limit $\delta \rightarrow 0^+$; to an external observer, the lens will be invisible but the dipole will be clearly visible in this limit. If $3a/2 < d_0 < 2a$, resonance will occur as $\delta \rightarrow 0^+$ but it will be localized to strips around the boundaries of the lens — in particular the resonant fields will not interact with the dipole. The dipole will still be visible in this limit but to an observer outside of the resonance region (and outside the lens) the lens will be invisible. Finally, if $d_0 < 3a/2$ (so the polarizable dipole is within the cloaking region), the resonant field will interact with the polarizable dipole and effectively cancel the effect of the external field on it. In other words, the net field at the location of the polarizable dipole will be zero, and, hence, its induced dipole moment will be zero (in the limit as $\delta \rightarrow 0^+$) — both the lens and the dipole will be invisible to external observers. See Figure 3 in [14] and the figures in [23] for dramatic illustrations of this in the circular cylindrical case.

Nicorovici, McPhedran, Enoch, and Tayeb studied CALR for the circular cylindrical superlens in the finite-frequency case in [22]. For physically plausible values of δ they discovered that the cloaking device (the superlens) can effectively cloak a tiny cylindrical inclusion located within the cloaking region but that the superlens does not necessarily cloak itself — they deemed this phenomenon the “ostrich effect.” In the quasistatic (long wavelength) limit, however, the lens can effectively cloak both the inclusion and itself even at rather large values of δ , which was also pointed out in the case of a polarizable dipole in [14].

In [5] Bouchitté and Schweizer considered an annular lens with inner and outer radii of 1 and R , respectively, and relative permittivity $\varepsilon_s = -1 + i\delta$ embedded in vacuum. They proved that a small circular inclusion of radius $\gamma(\delta)$ (with $\gamma(\delta) \rightarrow 0$ as $\delta \rightarrow 0^+$) is cloaked in the limit $\delta \rightarrow 0^+$ if it is located within the annulus $R < |x_0| < R_* = R^{3/2}$, where x_0 is the position of the circular source. If $|x_0| > R_*$, then the source is visible but the annular superlens is not. Both of these results are consistent with the results in [14]. Bruno and Lintner considered a similar scenario in [6], where they showed numerically that a small dielectric disk is not perfectly cloaked. They verified (numerically) that an annular superlens embedded in vacuum by itself is invisible to an external applied field in the zero loss limit (assuming the source is at a position further than R_* from the origin) — a fact that was first shown analytically by Nicorovici, McPhedran, and Milton in [19]; however, they also showed that elliptical superlenses can cloak polarizable dipoles that are near enough to the lens but that such lenses are not invisible themselves. That is,

the polarizable dipole is cloaked but it is obvious to external observers that something is being hidden — this is another example of the “ostrich effect” introduced in [22].

Kohn, Lu, Schweizer, and Weinstein used variational principles to derive resonance results in the quasistatic regime in core/shell geometries (where the superlens resides in the shell) that are not necessarily radial in [11]. They assumed the source was supported on the boundary of a disk in \mathbb{R}^2 , and obtained results similar to those described above.

In [2, 3], Ammari, Ciraolo, Kang, Lee, and Milton used properties of certain Neumann-Poincaré operators to prove results analogous to those in [14]. The most general results they derived hold for very general core/shell geometries and charge density distributions ρ with compact support in the quasistatic regime. In the circular cylindrical case their requirements are more explicit and involve gap conditions on the Fourier coefficients of the Newtonian potential of ρ . Although these gap conditions may be difficult to deal with for a given source, they verified that their results are consistent with those in [14] when ρ is a dipole or quadrupole. Their results can be summarized as follows. First, if the support of ρ is completely contained within the cloaking region ($r_s < r_0 < r_*$ if $\varepsilon_c \neq \varepsilon_m = 1$ and $r_s < r_0 < r_{\#}$ if $\varepsilon_c = \varepsilon_m = 1$), and if ρ satisfies the gap property, then CALR occurs. Second, weak CALR (defined by $\limsup_{\delta \rightarrow 0^+} E(\delta) = \infty$ and $|V| < C$ for all δ where $C > 0$ is independent of δ) occurs if the support of ρ is completely inside the cloaking region and the Newtonian potential does not extend harmonically to all of \mathbb{R}^2 . Third, if $\Re(\varepsilon_s) \neq -1$, then CALR does not occur. Fourth, CALR does not occur for any isotropic constant values of ε_c and ε_s when the core and shell are concentric spheres in \mathbb{R}^3 . Using a folded geometry approach (extending that of Leonhardt and Philbin in [12] and Leonhardt and Tyc in [13]), in [1] Ammari, Ciraolo, Kang, Lee, and Milton proved that CALR can occur in 3D when the core and shell are concentric spheres and the shell has a certain anisotropic relative permittivity — see [16] for the analogous problem in 2D.

In [21], Nicorovici, McPhedran, Botten, and Milton asked whether or not one can enlarge the cloaking region by spatially overlapping the cloaking regions of identical circular cylindrical superlenses. Curiously they found that doing so *reduces* the cloaking effect (at least in the quasistatic regime). The cloaking region can be extended by arranging the disks in such a way that their corresponding cloaking regions just touch.

We also point out the works [15, 29]. In [15], Milton and Nicorovici utilized a correspondence (first discovered although not fully exploited by Yaghjian and Hansen in [30]) between the perfect Veselago lens at a fixed frequency in the long time limit and the lossy Veselago lens in the quasistatic limit to show that transverse magnetic dipole sources that generate bounded power eventually become cloaked if they are within the cloaking region ($a < d_0 < 3a/2$). In [29] Xiao, Huang, Dong, and Chan obtained similar results in the case when both the permittivity and permeability of the Veselago lens have a positive imaginary part.

Finally, in [18], Nguyen proved that arbitrary inhomogeneous objects are magnified by properly constructed superlenses in both the quasistatic and finite-frequency regimes in two and three dimensions.

In this paper we consider the scenario sketched in Figure 1 and described by (1.1)–(1.8). We study the behavior of

$$E_\xi(\delta) \equiv \delta \int_{a-\xi}^a \int_{-\infty}^{\infty} |\nabla V|^2 dy dx,$$

where $0 < \xi < a$ is a small parameter. The quantity $E_\xi(\delta)$ is proportional to the time-averaged electrical power dissipated in the strip $R_\xi \equiv \{(x, y) \in \mathbb{R}^2 : a - \xi < x < a\}$, illustrated by the darkened strip in Figure 1; $E_\xi(\delta)$ is also a lower bound on the quantity defined in (1.9). In particular, we derive conditions on ρ that determine whether or not $\limsup_{\delta \rightarrow 0^+} E_\xi(\delta) = \infty$ (*weak* CALR), $\lim_{\delta \rightarrow 0^+} E_\xi(\delta) = \infty$ (*strong* CALR), or $E_\xi(\delta) < C$ for a constant $C > 0$ as $\delta \rightarrow 0^+$ (no CALR).

In order to do this, we begin by taking the Fourier transform of (1.1) in the y -variable and calculating $E_\xi(\delta)$ explicitly in terms of $\widehat{\rho}(x, k)$ (the Fourier transform of ρ in the y -variable). We then derive upper and lower bounds on $E_\xi(\delta)$ to obtain our results. The result for unbounded energy is contained in Corollary 4.4. Essentially, if there is a $d_* \in [d_0, d_1]$ such that

$$\limsup_{k \rightarrow \infty} \left| e^{d_* k} \int_{d_0}^{d_1} \widehat{\rho}(x, k) e^{-kx} dx \right| > 0$$

and $a < d_* < \tau(\beta)a$, where

$$\tau(\beta) = \begin{cases} \frac{\beta + 2}{\beta + 1} & \text{for } 0 < \beta < 1, \\ \frac{3}{2} & \text{for } \beta \geq 1, \end{cases}$$

then $\limsup_{\delta \rightarrow 0^+} E_\xi(\delta) = \infty$. As far as we are aware, there are two novelties to our result. First, the blow-up in energy occurs only if ρ is within a critical distance of the slab that depends non-trivially on β . Second, unlike in [2] (Theorem 5.3) and [3] (Theorem 4.1), we do not assume that the support of ρ is completely contained within the critical distance. In fact, there are examples of charge density distributions ρ that cause a blow-up in energy if only part of the support of ρ is within the critical distance — see Sections 4.1.1 and 4.1.2. (It seems the results in [2] and [3] would hold even if only part of the support of ρ is within the critical distance to the lens — see the Introduction in [3].) In Theorem 5.6 we show that $\lim_{\delta \rightarrow 0^+} E_\xi(\delta) = 0$ if ρ is supported outside the critical distance.

The remainder of this paper is organized as follows. In Section 2 we derive an expression for the potential. In Section 3 we compute the power dissipation $E_\xi(\delta)$. In Section 4 we obtain some lower bounds that are used to prove our result about the blow-up of $E_\xi(\delta)$ as $\delta \rightarrow 0^+$. We then analytically and numerically illustrate our results for two charge density distributions. In Section 5 we prove that $E_\xi(\delta)$ remains bounded (and, in fact, goes to 0) as $\delta \rightarrow 0^+$ if ρ is farther than the critical distance from the slab. Finally, in Section 6 we show that the potential remains bounded far enough away from the slab in the limit as $\delta \rightarrow 0^+$ regardless of the position of the source.

2. Derivation of the Potential. The potential $V \in L^2_{\text{loc}}(\mathbb{R}^2)$ solves the following problem in the quasistatic regime:

$$\left\{ \begin{array}{l} -\nabla \cdot [\varepsilon(x, y) \nabla V(x, y)] = \rho(x, y) \quad \text{in } \mathbb{R}^2, \\ V(x, y), \varepsilon \frac{\partial V}{\partial x}(x, y) \quad \text{continuous across } x = 0, a \text{ for almost every } y \in \mathbb{R}, \\ \frac{\partial V}{\partial x}(x, y) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty \text{ for almost every } y \in \mathbb{R}, \\ V(x, \cdot) \in H^1(\mathbb{R}) \quad \text{for almost every } x \in \mathbb{R}, \\ \frac{\partial V}{\partial x}(x, \cdot) \in L^2(\mathbb{R}) \quad \text{for almost every } x \in \mathbb{R}, \end{array} \right. \quad (2.1)$$

where ε is given in (1.4). In this section, we will take the Fourier transform with respect to the y -variable of the problem (2.1). Since $V \in L^2_{\text{loc}}(\mathbb{R}^2)$, the PDE (2.1) can be understood in a distributional sense (since L^2_{loc} functions are distributions — see [8]). The continuity conditions in (2.1) ensure continuity of the potential and the normal component of the electric displacement field $\mathbf{D} = -\varepsilon \nabla V$ across the left and right edges of the slab. These continuity conditions are typical in quasistatic problems — see, e.g., Section 4.4.2 in [9] and [17]. The condition at infinity in (2.1) ensures that the x -component of the electric field, namely $-\partial V / \partial x$, vanishes as $x \rightarrow -\infty$ and $x \rightarrow \infty$. It turns out that this condition is sufficient for our purposes (for the problem stated in (2.1) one can show that the y -component of the electric field, namely $-\partial V / \partial y$, goes to 0 as $|x| \rightarrow \infty$ as well). We only consider $|x| \rightarrow \infty$, rather than $x^2 + y^2 \rightarrow \infty$, since the slab extends infinitely in the y -direction. The last two requirements are regularity results that we impose to ensure that we can perform the computations in this section.

In the remainder of this section, we sketch a proof of the following theorem; a complete proof can be found in [27].

THEOREM 2.1. There exists a nonempty class of potentials

$$\mathcal{V} \equiv \{V \in L^2_{\text{loc}}(\mathbb{R}^2) : V \text{ satisfies (2.1)}\}. \quad (2.2)$$

We recall the following definitions:

$$\left\{ \begin{array}{l} \mathcal{C} \equiv \{(x, y) \in \mathbb{R}^2 : x < 0\}; \\ \mathcal{S} \equiv \{(x, y) \in \mathbb{R}^2 : 0 < x < a\}; \\ \mathcal{M} \equiv \{(x, y) \in \mathbb{R}^2 : a < x\}. \end{array} \right. \quad (2.3)$$

We then define

$$\left\{ \begin{array}{l} V_c(x, y) \equiv \chi_{\mathcal{C}}(x, y) V(x, y), \\ V_s(x, y) \equiv \chi_{\mathcal{S}}(x, y) V(x, y), \\ V_m(x, y) \equiv \chi_{\mathcal{M}}(x, y) V(x, y), \end{array} \right. \quad (2.4)$$

where

$$\chi_U(x, y) = \begin{cases} 1 & \text{if } (x, y) \in U, \\ 0 & \text{if } (x, y) \notin U, \end{cases} \quad (2.5)$$

is the characteristic function of the set $U \subset \mathbb{R}^2$.

Finally, we will use the convention that the Fourier transform of a function $f(x, y)$ with respect to the variable y is defined by

$$\widehat{f}(x, k) \equiv \int_{-\infty}^{\infty} f(x, y) e^{-iky} dy. \quad (2.6)$$

If f is a distribution, it is well known that

$$\frac{\partial \widehat{f}}{\partial x}(x, k) = \frac{\partial \widehat{f}}{\partial x}(x, k) \quad \text{and} \quad \frac{\partial \widehat{f}}{\partial y}(x, k) = ik \widehat{f}(x, k); \quad (2.7)$$

see, e.g., [8].

REMARK 2.2. As we will see, for each $V \in \mathcal{V}$, $\widehat{V}(x, k)$ is determined for all $k \in \mathbb{R}$ except $k = 0$, where it remains finite. In particular, the quantity $E_\xi(\delta)$ is the same for each potential in \mathcal{V} . The class of potentials \mathcal{V} contains every solution to (2.1) since the Fourier transform is injective on $L^2(\mathbb{R})$. (Because $V(x, \cdot) \in H^1(\mathbb{R})$ and $\frac{\partial V}{\partial x}(x, \cdot) \in L^2(\mathbb{R})$ for $V \in \mathcal{V}$, $\widehat{V}(x, k)$ is well defined and the analysis of the next two sections holds. Solutions to (2.1) are then obtained through the inverse Fourier transform.) Although an infinite slab is unphysical, it is a good approximation to a long, thin slab when edge effects are neglected. Thus one of the potentials in our class of solutions should reasonably approximate the physical potential; hence our expression of power dissipation given in Section 3 should be a good approximation to the power dissipated by a long, thin slab.

We apply the Fourier transform with respect to y in (1.1) and by straightforward calculations (see [27] for the complete derivation) find that the general form of the Fourier transform of V_c is

$$\widehat{V}_c(x, k) = A_k e^{|k|x} \quad (2.8)$$

for arbitrary constants A_k .

The continuity conditions at the left boundary of the central slab, i.e., at $x = 0$, together with some algebraic manipulations lead us to the general form of the Fourier transform of V_s , namely

$$\widehat{V}_s(x, k) = \frac{A_k}{2\chi_c} \left[(\chi_c + 1) e^{|k|x} + (\chi_c - 1) e^{-|k|x} \right], \quad (2.9)$$

where

$$\chi_c \equiv \varepsilon_s / \varepsilon_c. \quad (2.10)$$

Next we will show the details of the derivation for the solution in the third layer, \mathcal{M} . From (2.1) we note that in the set \mathcal{M} the potential satisfies

$$\begin{cases} \Delta V_m(x, y) = -\rho(x, y) & \text{for } x > a, \\ \lim_{x \rightarrow a^+} V_m(x, y) = \lim_{x \rightarrow a^-} V_s(x, y) & \text{for almost every } y \in \mathbb{R}, \\ \lim_{x \rightarrow a^+} \varepsilon_m \frac{\partial V_m}{\partial x}(x, y) = \lim_{x \rightarrow a^-} \varepsilon_s \frac{\partial V_s}{\partial x}(x, y) & \text{for almost every } y \in \mathbb{R}, \\ \lim_{x \rightarrow \infty} \frac{\partial V_m}{\partial x}(x, y) = 0 & \text{for almost every } y \in \mathbb{R}. \end{cases}$$

After taking the Fourier transform with respect to y we find that $\widehat{V}_m(x, k)$ satisfies

$$\begin{cases} \frac{\partial^2 \widehat{V}_m}{\partial x^2}(x, k) - k^2 \widehat{V}_m(x, k) = -\widehat{\rho}(x, k) & \text{for } x > a, \\ \lim_{x \rightarrow a^+} \widehat{V}_m(x, k) = \lim_{x \rightarrow a^-} \widehat{V}_s(x, k) & \text{for all } k \in \mathbb{R}, \\ \lim_{x \rightarrow a^+} \varepsilon_m \frac{\partial \widehat{V}_m}{\partial x}(x, k) = \lim_{x \rightarrow a^-} \varepsilon_s \frac{\partial \widehat{V}_s}{\partial x}(x, k) & \text{for all } k \in \mathbb{R}, \\ \lim_{x \rightarrow \infty} \frac{\partial \widehat{V}_m}{\partial x}(x, k) = 0 & \text{for all } k \in \mathbb{R}. \end{cases} \quad (2.11)$$

We make the change of variables $z = x - a$ so that (2.11) becomes

$$\begin{cases} \frac{\partial^2 \widehat{V}_m}{\partial z^2}(z, k) - k^2 \widehat{V}_m(z, k) = -\widehat{\rho}(z, k) & \text{for } z > 0, \\ \lim_{z \rightarrow 0^+} \widehat{V}_m(z, k) = \lim_{z \rightarrow 0^-} \widehat{V}_s(z, k) = A_k \psi_k^+ & \text{for all } k \in \mathbb{R}, \\ \lim_{z \rightarrow 0^+} \frac{\partial \widehat{V}_m}{\partial z}(z, k) = \lim_{z \rightarrow 0^-} \chi_m \frac{\partial \widehat{V}_s}{\partial z}(z, k) = A_k \psi_k^- & \text{for all } k \in \mathbb{R}, \end{cases} \quad (2.12)$$

where $\widehat{\rho}(x, k) = \widehat{\rho}(x - a, k)$; $\widehat{V}_j(x, k) = \widehat{V}_j(x - a, k)$ for $j = m, s$;

$$\psi_k^+ = \frac{1}{2\chi_c} \left[(\chi_c + 1) e^{|k|a} + (\chi_c - 1) e^{-|k|a} \right]; \quad (2.13)$$

$$\psi_k^- = \frac{|k|\chi_m}{2\chi_c} \left[(\chi_c + 1) e^{|k|a} - (\chi_c - 1) e^{-|k|a} \right]; \quad (2.14)$$

$$\chi_m = \varepsilon_s / \varepsilon_m.$$

(We have eliminated the condition at infinity for now — we will return to it later.)

The Laplace transform of $\widehat{V}_m(z, k)$ is defined by

$$u(s, k) \equiv \int_0^\infty \widehat{V}_m(z, k) e^{-sz} dz; \quad (2.15)$$

see, e.g., [26]. We need to solve the ODE in (2.12) for the cases $k = 0$ and $k \neq 0$ separately.

Case 1: $k = 0$

Here the Laplace-transformed version of (2.12) is

$$s^2 u(s, 0) - s A_0 \psi_0^+ - A_0 \psi_0^- = -\mathcal{L} \{ \widehat{\rho}(z, 0) \} (s, 0),$$

where $\mathcal{L}\{g\}$ denotes the Laplace transform of the function g — see (2.15). Thus

$$u(s, 0) = \frac{A_0}{s} - [\mathcal{L} \{ \widehat{\rho}(z, 0) \} (s, 0)] \cdot \frac{1}{s^2}$$

where we have used (2.13) and (2.14) to simplify the expression for $u(s, 0)$. Since $\widehat{V}_m = 0$ for $z < 0$ (see (2.3)–(2.5)), we can use the convolution theorem for Laplace transforms (see [26]) to find

$$\widehat{V}_m(z, 0) = A_0 - \int_0^z (z - z') \widehat{\rho}(z', 0) dz' \Rightarrow \widehat{V}_m(x, 0) = A_0 - \int_0^{x-a} (x - a - z') \widehat{\rho}(z', 0) dz'.$$

Next we make the change of variables $z' = x' - a$ in the above integral to find

$$\widehat{V}_m(x, 0) = A_0 - \int_a^x (x - x') \widehat{\rho}(x' - a, 0) dx' = A_0 + \int_a^x (x' - x) \widehat{\rho}(x', 0) dx'.$$

We now impose the condition as $x \rightarrow \infty$; see (2.11). We need to require

$$\lim_{x \rightarrow \infty} \frac{\partial \widehat{V}_m}{\partial x}(x, 0) = \lim_{x \rightarrow \infty} \left\{ \frac{\partial}{\partial x} \left[A_0 + \int_a^x (s - x) \widehat{\rho}(s, 0) ds \right] \right\} = 0.$$

By the Leibniz Rule (see, e.g., [10, 27]), this is equivalent to the requirement

$$\lim_{x \rightarrow \infty} \left[- \int_a^x \widehat{\rho}(s, 0) ds \right] = 0.$$

For $x > d_1$, by (1.8) we have

$$\int_a^x \widehat{\rho}(s, 0) ds = \int_{d_0}^{d_1} \widehat{\rho}(s, 0) ds = \int_{d_0}^{d_1} \int_{-\infty}^{\infty} \rho(s, y) dy ds = \int_{d_0}^{d_1} \int_{h_0}^{h_1} \rho(s, y) dy ds = 0.$$

Thus the condition at ∞ is automatically satisfied for any choice of A_0 . (Throughout this section, we have assumed that $\widehat{\rho}(x, k)$ is continuous at $k = 0$ — in fact, in Lemma 3.1 we will see that $\widehat{\rho}(x, k)$ is infinitely differentiable on \mathbb{R} as a function of k for almost all $x \in \mathbb{R}$.)

Case 2: $k \neq 0$

Here the Laplace-transformed version of (2.12) is

$$s^2 u(s, k) - s A_k \psi_k^+ - A_k \psi_k^- - k^2 u(s, k) = -\mathcal{L} \{ \widehat{\rho}(z, k) \} (s, k).$$

Therefore

$$u(s, k) = A_k \psi_k^+ \frac{s}{s^2 - k^2} + A_k \psi_k^- \frac{1}{s^2 - k^2} - \frac{\mathcal{L} \{ \widehat{\rho}(z, k) \} (s, k)}{s^2 - k^2}.$$

Recalling that $\widehat{V}_m(z, k) = 0$ for $z < 0$ (see (2.3)–(2.5)), by the convolution theorem for Laplace transforms we have

$$\widehat{V}_m(z, k) = A_k \psi_k^+ \cosh(|k|z) + A_k \psi_k^- \frac{\sinh(|k|z)}{|k|} - \int_0^z \frac{\sinh[|k|(z - z')]}{|k|} \widehat{\rho}(z', k) dz'.$$

This is equivalent to

$$\begin{aligned} \widehat{V}_m(x, k) &= A_k \psi_k^+ \cosh[|k|(x - a)] + A_k \psi_k^- \frac{\sinh[|k|(x - a)]}{|k|} \\ &\quad - \int_0^{x-a} \frac{\sinh[|k|(x - a - z')]}{|k|} \widehat{\rho}(z', k) dz'. \end{aligned}$$

We make the change of variables $z' = x' - a$ in the above integral to find

$$\begin{aligned} \widehat{V}_m(x, k) &= A_k \psi_k^+ \cosh[|k|(x - a)] + \frac{A_k \psi_k^-}{|k|} \sinh[|k|(x - a)] \\ &\quad + \frac{1}{|k|} \int_a^x \sinh[|k|(x' - x)] \widehat{\rho}(x', k) dx', \end{aligned} \tag{2.16}$$

where we have used the fact that $\widehat{\rho}(x - a, k) = \widehat{\rho}(x, k)$.

We now impose the limit condition at infinity — see (2.11). We use the Leibniz Rule to find

$$\begin{aligned}
\lim_{x \rightarrow \infty} \frac{\partial \widehat{V}_m}{\partial x}(x, k) &= \lim_{x \rightarrow \infty} \left(A_k \{ |k| \psi_k^+ \sinh [|k|(x-a)] + \psi_k^- \cosh [|k|(x-a)] \} \right. \\
&\quad \left. - \int_a^x \widehat{\rho}(x', k) \cosh [|k|(x' - x)] dx' \right) \\
&= \lim_{x \rightarrow \infty} \left\{ |k| e^{|k|x} \left[\frac{A_k \psi_k^+ e^{-|k|a}}{2} + \frac{A_k \psi_k^- e^{-|k|a}}{2|k|} - \frac{1}{2|k|} \int_{d_0}^{d_1} \widehat{\rho}(s, k) e^{-|k|s} ds \right] \right. \\
&\quad \left. + |k| e^{-|k|x} \left[-\frac{A_k \psi_k^+ e^{|k|a}}{2} + \frac{A_k \psi_k^- e^{|k|a}}{2|k|} - \frac{1}{2|k|} \int_{d_0}^{d_1} \widehat{\rho}(s, k) e^{|k|s} ds \right] \right\} \\
&\quad \xrightarrow{\text{as } x \rightarrow \infty} 0 \\
&= \lim_{x \rightarrow \infty} \left\{ |k| e^{|k|x} \left[\frac{A_k e^{-|k|a}}{2|k|} (|k| \psi_k^+ + \psi_k^-) - \frac{1}{2|k|} \int_{d_0}^{d_1} \widehat{\rho}(s, k) e^{-|k|s} ds \right] \right\}.
\end{aligned}$$

This limit will be 0 if and only if we choose

$$A_k \equiv \frac{I_k}{e^{-|k|a} (|k| \psi_k^+ + \psi_k^-)}, \quad (2.17)$$

where

$$I_k \equiv \int_{d_0}^{d_1} \widehat{\rho}(s, k) e^{-|k|s} ds. \quad (2.18)$$

By (1.4) and (2.10) we have

$$\frac{\chi_c - 1}{\chi_c + 1} = \frac{2i + \delta - \mu}{\delta + \mu},$$

so by (2.9) the potential in the set \mathring{S} is

$$\widehat{V}_s(x, k) = \begin{cases} A_0 & \text{if } k = 0, \\ \frac{I_k}{|k|g} \left[e^{|k|x} + \left(\frac{2i - \lambda \delta^\beta}{2\delta + \lambda \delta^\beta} \right) e^{-|k|x} \right] & \text{if } k \neq 0, \end{cases} \quad (2.19)$$

where

$$g \equiv \frac{2\chi_c e^{-|k|a} \left(\psi_k^+ + \frac{1}{|k|} \psi_k^- \right)}{\chi_c + 1} = i\delta \left[1 - \frac{(\delta + 2i)(2i - \lambda \delta^\beta)}{\delta(2\delta + \lambda \delta^\beta)} e^{-2|k|a} \right] \quad (2.20)$$

and A_0 is an arbitrary complex constant. In the next section we will see that the power dissipation is independent of A_0 . Finally, it can be shown that

$$|g|^2 \geq 8e^{-4|k|a} \left| \frac{\chi_c}{\chi_c + 1} \right|^2 = 8 \left[\frac{1 + \delta^2}{(\delta + \mu)^2} \right] e^{-4|k|a} > 0 \quad (2.21)$$

for all $k \in \mathbb{R}$ and all $0 < \delta \leq \delta_\mu$ (see [27] for details).

REMARK 2.3. We may add the term $C\delta(k)$, where δ is the Dirac delta distribution and C is a constant, to \widehat{V} . This corresponds to adding a constant to the potential V . However, adding a constant to the potential will not affect any of the results presented in this paper.

3. Derivation of the Power Dissipation. We begin this section by recording some important properties of I_k , defined in (2.18). We omit the proof of the following lemma since it is given in [27].

LEMMA 3.1. Suppose $\rho \in \mathcal{P}$ (where \mathcal{P} is defined in (1.2)) and that I_k is defined as in (2.18). Then

- (1) for almost every $s \in [d_0, d_1]$, $\widehat{\rho}(s, k)$ is infinitely continuously differentiable as a function of k for all $k \in \mathbb{R}$;
- (2) for each $k \in \mathbb{R}$,

$$|I_k|^2 \leq (d_1 - d_0) \|\rho\|_{L^2(\mathcal{M})}^2 e^{-2|k|d_0};$$

- (3) if ρ is real valued, then $I_{-k} = \overline{I_k}$; this implies that $|I_k|^2$ is an even function of k for $k \in \mathbb{R}$;
- (4) the function I_k is continuous at k for each $k \in \mathbb{R}$;
- (5) $\lim_{k \rightarrow 0} I_k = I_0 = 0$;
- (6) $\lim_{k \rightarrow 0} |I_k|/|k| = |C_0| < \infty$, where

$$\begin{aligned} C_0 &= \int_{d_0}^{d_1} -s\widehat{\rho}(s, 0) ds + \int_{d_0}^{d_1} \frac{\partial \widehat{\rho}}{\partial k}(s, 0) ds \\ &= - \int_{d_0}^{d_1} \int_{h_0}^{h_1} s\rho(s, y) dy ds - \int_{d_0}^{d_1} \int_{h_0}^{h_1} iy\rho(s, y) dy ds; \end{aligned}$$

moreover, there is a positive constant C_I such that $|I_k|/|k| \leq C_I$ for all $k \in \mathbb{R}$.

For $0 < \xi < a$, the time-averaged electrical power dissipation in the strip R_ξ is defined as

$$E_\xi(\delta) \equiv \delta \int_{a-\xi}^a \int_{-\infty}^{\infty} |\nabla V(x, y)|^2 dy dx = \delta \int_{a-\xi}^a \int_{-\infty}^{\infty} \left(\left| \frac{\partial V}{\partial x} \right|^2 + \left| \frac{\partial V}{\partial y} \right|^2 \right) dy dx, \quad (3.1)$$

where $V(x, y)$ is the (complex) electric potential in the slab \mathcal{S} due to the charge density ρ and $|z| = \sqrt{(z')^2 + (z'')^2}$ denotes the modulus of the complex number $z = z' + iz''$. Recall that in the quasistatic regime the potential V solves (2.1) with ε given by (1.4). As shown in [27], $V \in H^1(\mathring{\mathcal{S}})$; thus the quantity in (3.1) is well defined and finite.

Using the definition in (3.1), we compute the power dissipation in the strip R_ξ (see Figure 1) as follows. Note that for any function $f: \mathbb{R}^2 \rightarrow \mathbb{C}$ such that

$$\int_{-\infty}^{\infty} |f(x, y)|^2 dy < \infty,$$

we have the Plancherel Theorem, namely

$$\int_{-\infty}^{\infty} |f(x, y)|^2 dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\widehat{f}(x, k)|^2 dk. \quad (3.2)$$

Using (3.2) together with classical properties of the Fourier transform in (2.7), from (3.1) we obtain

$$\begin{aligned} E_\xi(\delta) &= \delta \int_{a-\xi}^a \left[\int_{-\infty}^{\infty} \left| \frac{\partial V_s}{\partial x}(x, y) \right|^2 dy + \int_{-\infty}^{\infty} \left| \frac{\partial V_s}{\partial y}(x, y) \right|^2 dy \right] dx \\ &= \frac{\delta}{2\pi} \int_{a-\xi}^a \left[\int_{-\infty}^{\infty} \left| \frac{\partial \widehat{V}_s}{\partial x}(x, k) \right|^2 dk + \int_{-\infty}^{\infty} |k|^2 |\widehat{V}_s(x, k)|^2 dk \right] dx. \end{aligned} \quad (3.3)$$

Now, (2.19), (2.21), and Lemma 3.1 imply that \widehat{V}_s and $\partial \widehat{V}_s / \partial x$ are finite at and near $k = 0$; thus we can omit the point $k = 0$ from the integrals in (3.3) without changing the value of $E_\xi(\delta)$. Inserting (2.19) into (3.3) gives (after some straightforward computations)

$$\begin{aligned} E_\xi(\delta) &= \frac{2\delta}{2\pi} \int_{a-\xi}^a \left\{ \int_{k \neq 0} \frac{|I_k|^2}{|g|^2} \left[e^{2|k|x} + \frac{e^{-2|k|x} (\lambda^2 \delta^{2\beta} + 4)}{(2\delta + \lambda \delta^\beta)^2} \right] dk \right\} dx \\ &= \frac{\delta}{\pi} \int_{k \neq 0} \frac{|I_k|^2}{|g|^2} \left\{ \int_{a-\xi}^a \left[e^{2|k|x} + \frac{e^{-2|k|x} (\lambda^2 \delta^{2\beta} + 4)}{(2\delta + \lambda \delta^\beta)^2} \right] dx \right\} dk \\ &= \frac{\delta}{2\pi} \int_{k \neq 0} \frac{|I_k|^2}{|k||g|^2} e^{2|k|a} \left[\left(1 - e^{-2|k|\xi} \right) + \frac{(\lambda^2 \delta^{2\beta} + 4)}{(2\delta + \lambda \delta^\beta)^2} e^{-4|k|a} \left(e^{2|k|\xi} - 1 \right) \right] dk \\ &= \frac{\delta}{\pi} \int_{k > 0} \frac{|I_k|^2}{k|g|^2} e^{2ka} \left[\left(1 - e^{-2k\xi} \right) + \frac{(\lambda^2 \delta^{2\beta} + 4)}{(2\delta + \lambda \delta^\beta)^2} e^{-4ka} \left(e^{2k\xi} - 1 \right) \right] dk \end{aligned} \quad (3.4)$$

$$\geq \widetilde{E}_\xi(\delta) \equiv \int_{k > \widetilde{k}} F dk, \quad (3.5)$$

where $\widetilde{k} > 0$ is arbitrary,

$$F \equiv \left(\frac{\delta |I_k|^2}{\pi k |g|^2} \right) e^{2ka} L, \quad (3.6)$$

and

$$L \equiv \left(1 - e^{-2k\xi} \right) + \frac{(\lambda^2 \delta^{2\beta} + 4)}{(2\delta + \lambda \delta^\beta)^2} e^{-4ka} \left(e^{2k\xi} - 1 \right). \quad (3.7)$$

4. Lower Bound on Power Dissipation. In this section we derive some asymptotic estimates on the function F defined in (3.6). From (2.20) we have

$$|g|^2 = \delta^2 \left\{ \left(1 + \frac{4 + \lambda \delta^{\beta+1}}{2\delta^2 + \lambda \delta^{\beta+1}} e^{-2ka} \right)^2 + \left[\frac{2(\delta - \lambda \delta^\beta)}{2\delta^2 + \lambda \delta^{\beta+1}} e^{-2ka} \right]^2 \right\}. \quad (4.1)$$

Upon inspection of (3.4) and (3.6) we see (heuristically) that if $|g|^2 = O(\delta^2)$ as $\delta \rightarrow 0^+$, we may be able to show that the power dissipation blows up as $\delta \rightarrow 0^+$. To this end we define

$$k_0(\delta) \equiv \frac{1}{2a} \ln \left[\frac{1}{\delta(\delta + \mu)} \right] = \frac{1}{2a} \ln \left(\frac{1}{2\delta^2 + \lambda \delta^{\beta+1}} \right). \quad (4.2)$$

Note that $k_0(\delta) \rightarrow \infty$ as $\delta \rightarrow 0^+$. From (3.5) and recalling (2.20) and (3.6)–(3.7) we see that

$$E_\xi(\delta) \geq \int_{k_0(\delta)}^{\infty} F dk \quad (4.3)$$

for all $0 < \delta \leq \delta_0(\beta, \lambda)$ where $0 < \delta_0 \leq \delta_\mu$ is such that $k_0(\delta) > 0$ for $0 < \delta \leq \delta_0$. (Recall that $\delta_\mu(\beta, \lambda)$ is defined so that $\mu = \delta + \lambda\delta^\beta \geq 0$ for all $\delta \leq \delta_\mu$.)

LEMMA 4.1. Suppose $\beta > 0$, λ is feasible (see (1.5)), and $C_1 > 25$. Then there exists $0 < \delta_g(\beta, \lambda, C_1) \leq \delta_\mu(\beta, \lambda)$ such that if $0 < \delta \leq \delta_g$ and $k \geq k_0(\delta)$ then

$$|g|^2 \leq C_1 \delta^2.$$

Proof. Note that (4.1) is equivalent to

$$|g|^2 = \delta^2 \left[1 + \frac{2(4 + \lambda\delta^{\beta+1})}{2\delta^2 + \lambda\delta^{\beta+1}} e^{-2ka} + \frac{16 + 4\delta^2 + \lambda^2\delta^{2\beta}(4 + \delta^2)}{(2\delta^2 + \lambda\delta^{\beta+1})^2} e^{-4ka} \right].$$

All three terms in the above equation are positive for all $0 < \delta \leq \delta_\mu$. Also, since $k \geq k_0(\delta)$, $e^{-2ka} \leq e^{-2k_0 a} = 2\delta^2 + \lambda\delta^{\beta+1}$. Then for $0 < \delta \leq \delta_\mu$ we have

$$|g|^2 \leq \delta^2 [25 + 2\lambda\delta^{\beta+1} + 4\delta^2 + \lambda^2\delta^{2\beta}(4 + \delta^2)].$$

We then choose $\delta_g(\beta, \lambda, C_1) \leq \delta_\mu(\beta, \lambda)$ small enough to ensure that the term in brackets is less than or equal to C_1 for all $0 < \delta \leq \delta_g$. \square

LEMMA 4.2. Suppose $\beta > 0$, λ is feasible, $0 < \xi < a$, and let $0 < C_L < 1$ be a constant. Then there exists $0 < \delta_L(\beta, \lambda, \frac{\xi}{a}, C_L) \leq \delta_\mu(\beta, \lambda)$ such that if $0 < \delta \leq \delta_L$ and $k \geq k_0(\delta)$, then $L \geq C_L$.

Proof. From (3.7) we have

$$\begin{aligned} L &= (1 - e^{-2k\xi}) + \frac{\lambda^2\delta^{2\beta} + 4}{(2\delta + \lambda\delta^\beta)^2} e^{-4ka} (e^{2k\xi} - 1) \\ &\geq 1 - e^{-2k\xi} \geq 1 - e^{-2k_0\xi} = 1 - (2\delta^2 + \lambda\delta^{\beta+1})^{\frac{\xi}{a}} \geq C_L \end{aligned}$$

for $0 < \delta \leq \delta_L(\beta, \lambda, \frac{\xi}{a}, C_L)$, where $0 < \delta_L \leq \delta_\mu$ is such that $(2\delta^2 + \lambda\delta^{\beta+1})^{\frac{\xi}{a}} \leq 1 - C_L$ for $0 < \delta \leq \delta_L$. \square

For $0 < \delta \leq \min\{\delta_0, \delta_g, \delta_L\}$ we apply the bounds from Lemmas 4.1 and 4.2 to (4.3) and, recalling (3.6)–(3.7) and (4.1), find

$$E_\xi(\delta) \geq \frac{C_L}{\pi C_1 \delta} \int_{k_0(\delta)}^{\infty} \frac{|I_k|^2}{k} e^{2ka} dk. \quad (4.4)$$

Our goal is to show that $E_\xi(\delta)$ tends to infinity as a sequence δ_j tends to 0.

Since $|I_k|^2$ is a continuous function of k (by Lemma 3.1), from (4.4) and the Mean Value Theorem for Integrals we have, for $0 < \delta \leq \min\{\delta_0, \delta_g, \delta_L\}$, that

$$\begin{aligned} E_\xi(\delta) &\geq \frac{C_L}{\pi C_1 \delta} \int_{k_0(\delta)}^{k_0(\delta) + \frac{1}{\ln(\frac{e}{\delta})}} \frac{|I_k|^2}{k} e^{2ka} dk \\ &\geq \left(\frac{C_L}{\pi C_1} \right) \left(\frac{e^{2k_0(\delta)a}}{\delta [k_0(\delta) + 1]} \right) \int_{k_0(\delta)}^{k_0(\delta) + \frac{1}{\ln(\frac{e}{\delta})}} |I_k|^2 dk \\ &= \left(\frac{C_L}{\pi C_1} \right) \left[\frac{e^{2k_0(\delta)a}}{\delta \ln(\frac{e}{\delta}) [k_0(\delta) + 1]} \right] |I_{k_0(\delta) + t(\delta)}|^2 \end{aligned} \quad (4.5)$$

for some $0 \leq t(\delta) \leq 1/\ln(e/\delta) \leq 1$. Note that $t(\delta) \rightarrow 0$ as $\delta \rightarrow 0^+$. So now we must show the lower bound (4.5) tends to infinity as a sequence δ_j tends to 0.

THEOREM 4.3. Let $\rho \in \mathcal{P}$, $\beta > 0$, and λ be feasible. Assume there exist constants $d_* \in [d_0, d_1]$ and $\Lambda \in (0, \infty]$ such that $\limsup_{k \rightarrow \infty} |I_k e^{kd_*}| = \Lambda$. Then there exists a sequence $\{\delta_j\}_{j=1}^\infty$ with $\delta_j \rightarrow 0$ as $j \rightarrow \infty$ and there exist positive constants $C' = \frac{C_L e^{-2d_*}}{2\pi C_1}$, $C_2 = \frac{C' a \Lambda^2 \lambda^{(d_* - a)/a}}{2}$, $C_3 = \ln \lambda$, and $C_4 = \frac{C' a \Lambda^2}{4}$ such that

$$E_\xi(\delta_j) \geq \begin{cases} \frac{C_2 \delta_j^{(\beta+1)(\frac{d_*-a}{a})-1}}{(\ln \delta_j - 1) [C_3 + (\beta+1) \ln \delta_j]} & \text{for } 0 < \beta < 1, \\ \frac{C_4 \delta_j^{2(\frac{d_*-a}{a})-1}}{(\ln \delta_j - 1) \ln \delta_j} & \text{for } \beta \geq 1. \end{cases} \quad (4.6)$$

(The constants C_2 and C_3 are well defined since we require $\lambda > 0$ if $0 < \beta < 1$ — see (1.5).) Moreover, if $\lim_{k \rightarrow \infty} |I_k e^{kd_*}| = \Lambda$, then for δ small enough we have

$$E_\xi(\delta) \geq \begin{cases} \frac{C_2 \delta^{(\beta+1)(\frac{d_*-a}{a})-1}}{(\ln \delta - 1) [C_3 + (\beta+1) \ln \delta]} & \text{for } 0 < \beta < 1, \\ \frac{C_4 \delta^{2(\frac{d_*-a}{a})-1}}{(\ln \delta - 1) \ln \delta} & \text{for } \beta \geq 1. \end{cases} \quad (4.7)$$

Proof. If $0 < \delta \leq \min\{\delta_0, \delta_g, \delta_L\}$, then (4.5) holds. Since $0 \leq t(\delta) \leq 1$ and $k_0(\delta) + 1 \leq 2k_0(\delta)$ for δ small enough (equivalently $k_0(\delta)$ large enough), (4.5) implies

$$\begin{aligned} E_\xi(\delta) &\geq \left(\frac{C_L}{2\pi C_1} \right) \left[\frac{e^{2k_0(\delta)a}}{\delta \ln(\frac{e}{\delta}) k_0(\delta)} \right] \left| [I_{k_0(\delta) + t(\delta)}] e^{[k_0(\delta) + t(\delta)]d_*} \right|^2 e^{-2[k_0(\delta) + t(\delta)]d_*} \\ &\geq \frac{C' e^{-2k_0(\delta)(d_* - a)}}{\delta \ln(\frac{e}{\delta}) k_0(\delta)} \left| I_{k'(\delta)} e^{k'(\delta)d_*} \right|^2, \end{aligned} \quad (4.8)$$

where $k'(\delta) \equiv k_0(\delta) + t(\delta)$.

Since $\limsup_{k \rightarrow \infty} |I_k e^{kd_*}| = \Lambda$ there exists a sequence $\{k_j\}_{j=1}^\infty$ with $k_j \rightarrow \infty$ as $j \rightarrow \infty$ and

$$\lim_{j \rightarrow \infty} |I_{k_j} e^{k_j d_*}| = \Lambda.$$

We choose a sequence $\{\delta_j\}_{j=1}^{\infty}$ such that $\delta_j \rightarrow 0^+$ as $j \rightarrow \infty$ and $k_j = k_0(\delta_j)$ (where $k_0(\delta) = -\frac{1}{2a} \ln(2\delta^2 + \lambda\delta^{\beta+1})$ is defined in (4.2)).

Since $|I_k e^{kd_*}|$ is a continuous function of k and $t(\delta_j) \rightarrow 0$ as $j \rightarrow \infty$ (i.e., as $\delta_j \rightarrow 0^+$), we have

$$\lim_{j \rightarrow \infty} |I_{k'_j} e^{k'_j d_*}| = \Lambda,$$

where $k'_j = k_0(\delta_j) + t(\delta_j) = k_j + t(\delta_j) \rightarrow \infty$ as $j \rightarrow \infty$. Thus, for j large enough (i.e., δ_j small enough), $|I_{k'_j} e^{k'_j d_*}| \geq \frac{\Lambda}{2}$. Hence for large enough j we have

$$E_{\xi}(\delta_j) \geq \left(\frac{C' \Lambda^2}{4} \right) \frac{e^{-2k_0(\delta_j)(d_*-a)}}{\delta_j \ln\left(\frac{e}{\delta_j}\right) k_0(\delta_j)} = \left(\frac{C' \Lambda^2}{4} \right) \frac{(2\delta_j^2 + \lambda\delta_j^{\beta+1})^{(d_*-a)/a}}{\delta_j \ln\left(\frac{e}{\delta_j}\right) k_0(\delta_j)}. \quad (4.9)$$

Now (4.6) is obtained by applying the inequality

$$2\delta_j^2 + \lambda\delta_j^{\beta+1} \geq \begin{cases} \lambda\delta_j^{\beta+1} & \text{for } 0 < \beta < 1, \\ \delta_j^2 & \text{for } \beta \geq 1, \end{cases}$$

which holds for j large enough, to (4.9).

Similarly, if the stronger condition $\lim_{k \rightarrow \infty} |I_k e^{kd_*}| = \Lambda$ holds, since $k'(\delta) \rightarrow \infty$ as $\delta \rightarrow 0^+$ we have $|I_{k'(\delta)} e^{k'(\delta) d_*}| \geq \frac{\Lambda}{2}$ and

$$E_{\xi}(\delta) \geq \left(\frac{C' \Lambda^2}{4} \right) \frac{(2\delta^2 + \lambda\delta^{\beta+1})^{(d_*-a)/a}}{\delta \ln\left(\frac{e}{\delta}\right) k_0(\delta)} \quad (4.10)$$

for δ small enough; this is the continuous analog of (4.9) and is a direct consequence of (4.8). Finally, (4.7) is obtained by inserting the inequality

$$2\delta^2 + \lambda\delta^{\beta+1} \geq \begin{cases} \lambda\delta^{\beta+1} & \text{for } 0 < \beta < 1, \\ \delta^2 & \text{for } \beta \geq 1, \end{cases}$$

which holds for δ small enough, into (4.10). \square

The next corollary follows immediately.

COROLLARY 4.4. Let $\rho \in \mathcal{P}$, $\beta > 0$, and λ be feasible. Assume there exist constants $d_* \in [d_0, d_1]$ and $\Lambda \in (0, \infty]$ such that

(a) $\limsup_{k \rightarrow \infty} |I_k e^{kd_*}| = \Lambda$; or

(b) $\lim_{k \rightarrow \infty} |I_k e^{kd_*}| = \Lambda$.

If $d_* < \tau(\beta)a$, where τ is the continuous function

$$\tau(\beta) \equiv \begin{cases} \frac{\beta+2}{\beta+1} & \text{if } 0 < \beta < 1, \\ \frac{3}{2} & \text{if } \beta \geq 1, \end{cases} \quad (4.11)$$

then $\limsup_{\delta \rightarrow 0^+} E_{\xi}(\delta) = \infty$ if (a) holds (weak CALR) and $\lim_{\delta \rightarrow 0^+} E_{\xi}(\delta) = \infty$ if (b) holds (strong CALR).

REMARK 4.5. According to the previous corollary, the region of influence, i.e., the region in which the charge density ρ should be placed to cause the anomalous localized resonance near the inner right edge of the slab, is the interval $(a, \tau(\beta)a)$. In particular we can take $d_1 < \tau(\beta)a$ to guarantee that ρ is completely inside this region (assuming the support of ρ is small enough so that $d_0 > a$ as well). This region of influence is the same as that found in the cloaking paper by Milton and Nicorovici [14] and also in the superlensing paper by Milton, Nicorovici, McPhedran, and Podolskiy in [17] in the particular case when ρ is a dipole source. Also see Bergman's work [4].

4.1. *Numerical Discussion.* In this section, we study the behavior of two charge density distributions ρ . In particular, we show they satisfy the conditions of Theorem 4.3 that lead to weak CALR, i.e., they satisfy $\limsup_{k \rightarrow \infty} |I_k e^{kd_*}| = \Lambda$. We also provide plots illustrating the blow-up of the dissipated electrical power as $\delta \rightarrow 0^+$ for these charge density distributions.

4.1.1. *Rectangle.* The first charge density distribution we consider has support in a rectangle centered at (x_0, y_0) . The left and right edges of the rectangle are at $d_0 = x_0 - d$ and $d_1 = x_0 + d$, respectively, where $d > 0$. The bottom and top edges are at $h_0 = y_0 - h$ and $h_1 = y_0 + h$, respectively, where $h > 0$. These parameters are chosen so $d_0 > a$. We define the charge density distribution as

$$\rho(x, y) = \begin{cases} Q & \text{for } (x, y) \in [d_0, d_1] \times (y_0, h_1], \\ -Q & \text{for } (x, y) \in [d_0, d_1] \times [h_0, y_0), \\ 0 & \text{otherwise,} \end{cases}$$

where $Q \neq 0$. Since $\rho \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$, we can use calculus and (2.6) and (2.18) to find

$$\tilde{\rho}(x, k) = -\frac{4Q}{k} [\sin(y_0 k) + i \cos(y_0 k)] \sin^2\left(\frac{hk}{2}\right)$$

and

$$|I_k| = \frac{4|Q|}{k^2} \sin^2\left(\frac{hk}{2}\right) e^{-d_0 k} (1 - e^{-2dk}).$$

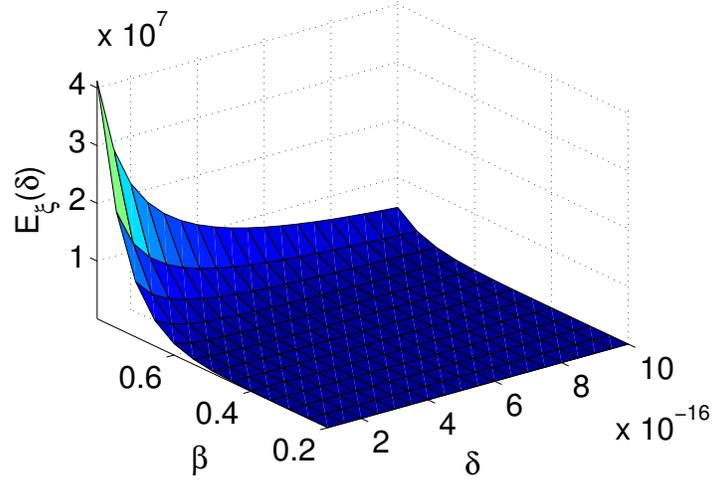
If we take $k_j = \frac{(2j-1)\pi}{h}$ for $j = 1, 2, \dots$ and $d_* = d_0 + \alpha$ for $\alpha > 0$ we have

$$|I_{k_j} e^{d_* k_j}| = \frac{4|Q|}{k_j^2} e^{\alpha k_j} (1 - e^{-2d k_j}) \rightarrow \infty \quad \text{as } j \rightarrow \infty.$$

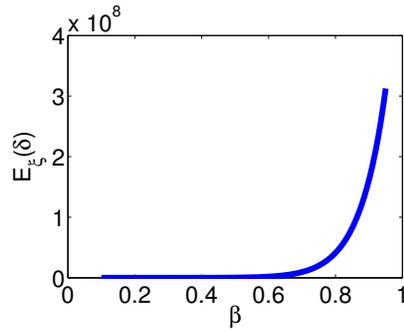
This implies $\limsup_{k \rightarrow \infty} |I_k e^{d_* k}| = \infty$, so ρ satisfies the conditions of Theorem 4.3. Thus there is a sequence $\delta_j \rightarrow 0$ as $j \rightarrow \infty$ such that $E_\xi(\delta_j) \rightarrow \infty$ as $j \rightarrow \infty$ if $d_0 + \alpha < \tau(\beta)a$; according to Theorem 5.6 in the next section, if $d_0 > \tau(\beta)a$, then $E_\xi(\delta) \rightarrow 0$ as $\delta \rightarrow 0^+$.

Since $\alpha > 0$ is arbitrary, the limit superior of the power dissipation blows up as the dissipation in the lens tends to 0 as long as any part of the charge density distribution ρ is within the region of influence $(a, \tau(\beta)a)$.

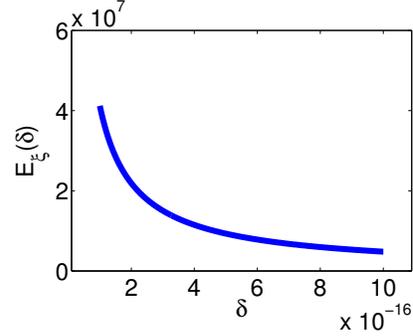
In Figure 2 we plot $E_\xi(\delta)$ for the rectangular charge density ρ studied above for various values of β and δ . The support of ρ is centered at $(6, 6)$, and has width and height 2; thus $d_0 = h_0 = 5$ and $d_1 = h_1 = 7$. We take $0 < \beta < 1$ and $a = d_1/\tau(\beta) = d_1[(\beta + 1)/(\beta + 2)]$,



(a)



(b)



(c)

FIG. 2. (Rectangular ρ) In all of these subfigures we take $a = d_1/\tau(\beta)$ so ρ is completely within the region of influence. (a) A plot of $E_\xi(\delta)$ versus β and δ — the z -axis scale is 10^7 ; (b) a plot of $E_\xi(\delta)$ for $\delta = 10^{-16}$ as a function of β — the y -axis scale is 10^8 ; (c) a plot of $E_\xi(\delta)$ for $\beta = 0.8$ as a function of δ — the y -axis scale is 10^7 .

so the support of ρ is completely inside the region of influence (see (4.11) and the remark following it). Figure 2(a) is a plot of the power dissipation $E_\xi(\delta)$ as a function of β and δ . We observe the divergence of $E_\xi(\delta)$ as $\delta \rightarrow 0^+$ for $0 < \beta < 1$; in particular the divergence appears to be more severe for larger values of β . In Figure 2(b) we fix $\delta = 10^{-16}$ and plot $E_\xi(\delta)$ as a function of β . Note the strong dependence of the divergence of $E_\xi(\delta)$ on the relative dissipation parameter β . Finally, in Figure 2(c) we plot $E_\xi(\delta)$ as a function of δ for $\beta = 0.8$.

4.1.2. *Circle.* We now consider a charge density distribution with support in a circle of radius R centered at (x_0, y_0) . In this case we have $d_0 = x_0 - R$, $d_1 = x_0 + R$, $h_0(x) = y_0 - \sqrt{R^2 - (x - x_0)^2}$, and $h_1(x) = y_0 + \sqrt{R^2 - (x - x_0)^2}$. Again we choose the parameters so that $d_0 > a$. We define the charge density distribution as

$$\rho(x, y) = \begin{cases} Q & \text{for } d_0 \leq x \leq d_1, y_0 < y \leq h_1(x), \\ -Q & \text{for } d_0 \leq x \leq d_1, h_0(x) \leq y < y_0, \\ 0 & \text{otherwise,} \end{cases}$$

where $Q \neq 0$. Again, $\rho \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$, so (2.6) and (2.18) imply

$$\hat{\rho}(x, k) = -\frac{4Q}{k} [\sin(y_0 k) + i \cos(y_0 k)] \sin^2 \left[\frac{k}{2} \sqrt{R^2 - (x - x_0)^2} \right]$$

and

$$|I_k| = \frac{4|Q|}{k} \int_{d_0}^{d_1} \sin^2 \left[\frac{k}{2} \sqrt{R^2 - (x - x_0)^2} \right] e^{-kx} dx.$$

Claim: If $d_* = x_0 + \alpha$ for $\alpha > 0$, then $\limsup_{k \rightarrow \infty} |I_k e^{d_* k}| = \infty$.

Proof of Claim. Let $\{k_j\}_{j=1}^\infty$ be the sequence whose j^{th} term is given by

$$k_j = \frac{2}{R} \left(\frac{\pi}{2} + 2\pi j \right).$$

Then

$$|I_{k_j}| \geq \frac{4|Q|}{k_j} \int_{x_0}^{x_0 + \gamma_j} \sin^2 \left[\frac{k_j}{2} \sqrt{R^2 - (x - x_0)^2} \right] e^{-k_j x} dx, \quad (4.12)$$

where $\gamma_j = \frac{R}{j}$ for $j = 1, 2, \dots$

For $x \in [x_0, x_0 + \gamma_j]$ we have

$$\frac{k_j}{2} \sqrt{R^2 - \gamma_j^2} \leq \frac{k_j}{2} \sqrt{R^2 - (x - x_0)^2} \leq \frac{k_j R}{2}. \quad (4.13)$$

We also have

$$\frac{k_j}{2} \sqrt{R^2 - \gamma_j^2} = \left(\frac{\pi}{2} + 2\pi j \right) \sqrt{1 - \frac{1}{j^2}} = \frac{\pi}{2} - \zeta_j + 2\pi j,$$

where

$$\zeta_j \equiv \frac{\frac{\pi}{2} + 2\pi j}{j^2 \left(1 + \sqrt{1 - \frac{1}{j^2}} \right)} = \left(\frac{\pi}{2} + 2\pi j \right) \left(1 - \sqrt{1 - \frac{1}{j^2}} \right).$$

Note $\zeta_j \rightarrow 0^+$ as $j \rightarrow \infty$ so that $0 < \zeta_j < \pi/2$ for j large enough. In combination with (4.13) this implies

$$2\pi j < \frac{\pi}{2} - \zeta_j + 2\pi j \leq \frac{k_j}{2} \sqrt{R^2 - (x - x_0)^2} \leq \frac{k_j R}{2} = \frac{\pi}{2} + 2\pi j \quad (4.14)$$

for j large enough. Since $\sin \theta$ is monotone increasing for $\theta \in (0, \pi/2)$, (4.12) and (4.14) imply

$$\begin{aligned} |I_{k_j}| &\geq \frac{4|Q|}{k_j} \sin^2 \left(\frac{\pi}{2} - \zeta_j + 2\pi j \right) \int_{x_0}^{x_0 + \gamma_j} e^{-k_j x} dx \\ &= \frac{4|Q|}{k_j^2} \sin^2 \left(\frac{\pi}{2} - \zeta_j \right) e^{-x_0 k_j} (1 - e^{-\gamma_j k_j}). \end{aligned}$$

Hence for j large enough we have

$$|I_{k_j} e^{d_* k_j}| \geq \frac{4|Q|}{k_j^2} \sin^2 \left(\frac{\pi}{2} - \zeta_j \right) e^{\alpha k_j} (1 - e^{-\gamma_j k_j}) \geq |Q| (1 - e^{-4\pi}) \frac{e^{\alpha k_j}}{k_j^2};$$

this expression goes to ∞ as $j \rightarrow \infty$. Thus $\limsup_{k \rightarrow \infty} |I_k e^{d_* k}| = \infty$. \square

Again we note that ρ need not be completely within the region of influence for the limit superior of the power dissipation to blow-up as the dissipation in the lens goes to 0. In particular, according to the above analysis, ρ only needs to be slightly more than halfway inside the region of influence for the blow-up to occur. However, numerical results seem to indicate that the power dissipation due to this charge density distribution blows up even if ρ is just inside the region of influence (as is the case for the rectangular charge density distribution analyzed in Section 4.1.1).

In Figure 3 we plot $E_\xi(\delta)$ as a function of β and δ for the circular charge distribution discussed above. We assume ρ is centered at $(6, 6)$ so $d_0 = 5$ and $d_1 = 7$ as in the rectangular case. The only other difference between Figures 3 and 2 are the values of δ we used to construct the plots.

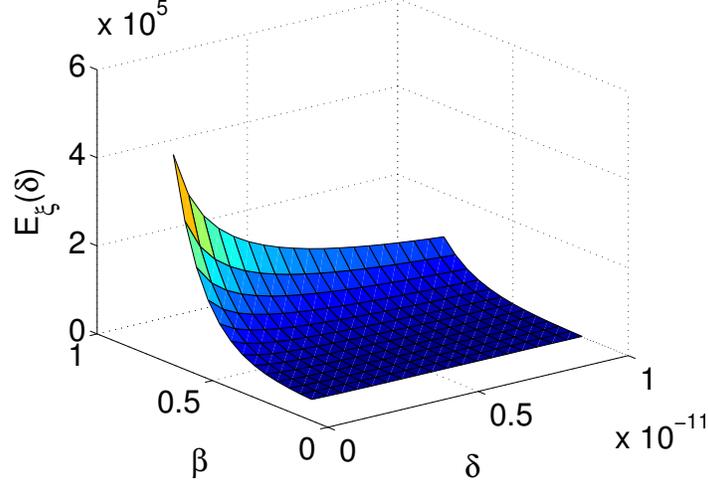
5. Upper Bound Power Dissipation. In this section, we discuss what happens when $d_0 > \tau(\beta)a \geq (3/2)a$. Recall that ρ has compact support, so $\text{supp}(\rho) \subseteq [d_0, d_1] \times [h_0, h_1]$ for some constants $h_0 < h_1$. The power dissipation is given *exactly* by

$$E_\xi(\delta) = \int_{k>0} F dk;$$

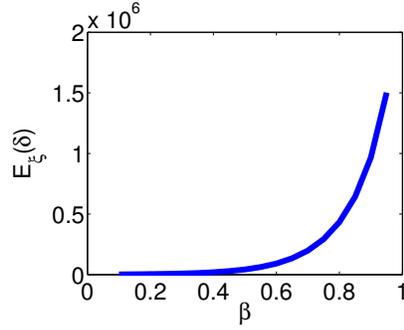
see (3.4) and (3.6)–(3.7). We will now prove a series of lemmas that will lead to an upper bound on $E_\xi(\delta)$.

LEMMA 5.1. Suppose $\beta > 0$ and λ is feasible, and let $k_0(\delta)$ be defined as in (4.2). Then for every $0 < \delta \leq \delta_0$

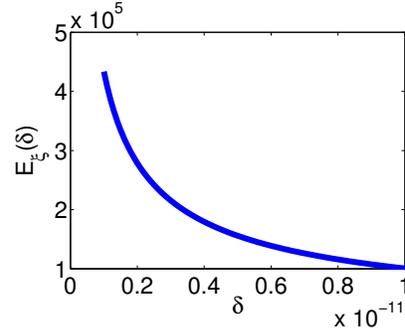
$$|g|^2 \geq \begin{cases} 9e^{-4ka} \frac{\delta^2}{(2\delta^2 + \lambda\delta^{\beta+1})^2} & \text{for } 0 \leq k \leq k_0(\delta), \\ e^{-ka} \frac{\delta^2}{(2\delta^2 + \lambda\delta^{\beta+1})^{\frac{1}{2}}} & \text{for } k \geq k_0(\delta). \end{cases}$$



(a)



(b)



(c)

FIG. 3. (Circular ρ) In all of these subfigures we take $a = d_1/\tau(\beta)$ so ρ is completely within the region of influence. (a) A plot of $E_\xi(\delta)$ versus β and δ — the z -axis scale is 10^5 ; (b) a plot of $E_\xi(\delta)$ for $\delta = 10^{-12}$ as a function of β — the y -axis scale is 10^5 ; (c) a plot of $E_\xi(\delta)$ for $\beta = 0.8$ as a function of δ — the y -axis scale is 10^5 .

Proof. From (4.1) we have

$$\begin{aligned}
 |g|^2 &= \delta^2 \left\{ \left(1 + \frac{4 + \lambda\delta^{\beta+1}}{2\delta^2 + \lambda\delta^{\beta+1}} e^{-2ka} \right)^2 + \left[\frac{2(\delta - \lambda\delta^\beta)}{2\delta^2 + \lambda\delta^{\beta+1}} e^{-2ka} \right]^2 \right\} \\
 &\geq \delta^2 \left(1 + \frac{4 + \lambda\delta^{\beta+1}}{2\delta^2 + \lambda\delta^{\beta+1}} e^{-2ka} \right)^2.
 \end{aligned} \tag{5.1}$$

For $0 < \delta \leq \delta_0 \leq \delta_\mu < 1$ (which implies $\mu = \delta + \lambda\delta^\beta \geq 0$) we have $4 + \lambda\delta^{\beta+1} \geq 4 - \delta^2 \geq 4 - \delta_\mu^2 \geq 3$. Then, from (5.1), for fixed $\delta \leq \delta_0$, and for all $k \in \mathbb{R}$ we have

$$|g|^2 \geq \delta^2 \left(\frac{3}{2\delta^2 + \lambda\delta^{\beta+1}} e^{-2ka} \right)^2 = 9e^{-4ka} \frac{\delta^2}{(2\delta^2 + \lambda\delta^{\beta+1})^2}.$$

In particular this bound holds for $0 \leq k \leq k_0(\delta)$.

To prove the second part of the lemma we note (5.1) implies $|g|^2 \geq \delta^2$ when $0 < \delta \leq \delta_\mu$. If $k \geq k_0(\delta)$ holds as well we have

$$e^{-ka} \frac{\delta^2}{(2\delta^2 + \lambda\delta^{\beta+1})^{\frac{1}{2}}} \leq e^{-k_0(\delta)a} \frac{\delta^2}{(2\delta^2 + \lambda\delta^{\beta+1})^{\frac{1}{2}}} = \delta^2 \leq |g|^2.$$

□

Combining the computations from Lemmas 3.1 and 5.1 we find, for $0 < \delta \leq \delta_0$, that (3.4) implies

$$\begin{aligned} E_\xi(\delta) &\leq \frac{\delta}{\pi} \int_0^{k_0(\delta)} \frac{(d_1 - d_0) \|\rho\|_{L^2(\mathcal{M})}^2 e^{-2kd_0} e^{4ka} (2\delta^2 + \lambda\delta^{\beta+1})^2}{9k\delta^2} e^{2ka} L dk \\ &\quad + \frac{\delta}{\pi} \int_{k_0(\delta)}^\infty \frac{(d_1 - d_0) \|\rho\|_{L^2(\mathcal{M})}^2 e^{-2kd_0} e^{ka} (2\delta^2 + \lambda\delta^{\beta+1})^{\frac{1}{2}}}{k\delta^2} e^{2ka} L dk \\ &= C_5 \delta \int_0^{k_0(\delta)} \frac{e^{-2k(d_0-3a)}}{k} (2\delta + \lambda\delta^\beta)^2 L dk \\ &\quad + 9C_5 \delta^{-\frac{1}{2}} \int_{k_0(\delta)}^\infty \frac{e^{-2k(d_0-\frac{3}{2}a)}}{k} (2\delta + \lambda\delta^\beta)^{\frac{1}{2}} L dk, \end{aligned}$$

where

$$C_5 \equiv \frac{(d_1 - d_0) \|\rho\|_{L^2(\mathcal{M})}^2}{9\pi}.$$

Using (3.7) we can rewrite the above upper bound as

$$E_\xi(\delta) \leq T_1 + T_2 + T_3 + T_4, \quad (5.2)$$

where

$$T_1 \equiv C_5 \delta (2\delta + \lambda\delta^\beta)^2 \int_0^{k_0(\delta)} e^{-2k(d_0-3a)} \left(\frac{1 - e^{-2k\xi}}{k} \right) dk; \quad (5.3a)$$

$$T_2 \equiv C_5 \delta (\lambda^2 \delta^{2\beta} + 4) \int_0^{k_0(\delta)} e^{-2k(d_0-3a)} e^{-4ka} \left(\frac{e^{2k\xi} - 1}{k} \right) dk; \quad (5.3b)$$

$$T_3 \equiv 9C_5 \delta^{-\frac{1}{2}} (2\delta + \lambda\delta^\beta)^{\frac{1}{2}} \int_{k_0(\delta)}^\infty e^{-2k(d_0-\frac{3}{2}a)} \left(\frac{1 - e^{-2k\xi}}{k} \right) dk; \quad (5.3c)$$

$$T_4 \equiv 9C_5 \delta^{-\frac{1}{2}} (2\delta + \lambda\delta^\beta)^{-\frac{3}{2}} (\lambda^2 \delta^{2\beta} + 4) \int_{k_0(\delta)}^\infty e^{-2k(d_0-\frac{3}{2}a)} e^{-4ka} \left(\frac{e^{2k\xi} - 1}{k} \right) dk. \quad (5.3d)$$

We derive estimates of these integrals in the next four lemmas. Recall that $0 < \delta_0 \leq \delta_\mu$ is such that $k_0(\delta) > 0$ for $0 < \delta \leq \delta_0$; we will assume $0 < \delta \leq \delta_0$ for the remainder of this section.

LEMMA 5.2. Suppose $\beta > 0$, λ is feasible, $0 < \xi < a$, and $d_0 \geq \tau(\beta)a$. Then

$$\lim_{\delta \rightarrow 0^+} T_1 = \begin{cases} C_6 \lambda^{[2+(d_0-3a)/a]} & \text{if } 0 < \beta < 1 \text{ and } d_0 = \tau(\beta)a, \\ C_6 (2 + \lambda)^{[2+(d_0-3a)/a]} & \text{if } \beta = 1 \text{ and } d_0 = \tau(\beta)a, \\ C_6 2^{[2+(d_0-3a)/a]} & \text{if } \beta > 1 \text{ and } d_0 = \tau(\beta)a, \\ 0 & \text{if } d_0 > \tau(\beta)a, \end{cases}$$

where

$$C_6 = \xi C_5 (d_0 - 3a)^{-1}.$$

Proof. We begin by noting that (4.11) implies that $(3/2)a \leq \tau(\beta)a < 2a$ for all $\beta > 0$. Next, the function $k^{-1}(1 - e^{-2k\xi})$ tends to 0 as k goes to infinity and is continuous and decreasing for $k \in [0, \infty)$ as long as we define it to be equal to 2ξ at $k = 0$. Thus $k^{-1}(1 - e^{-2k\xi}) \leq 2\xi$ for all $k \geq 0$. If $d_0 \neq 3a$, then this implies

$$T_1 \leq 2\xi C_5 \delta (2\delta + \lambda \delta^\beta)^2 \int_0^{k_0(\delta)} e^{-2k(d_0-3a)} dk \quad (5.4)$$

$$\begin{aligned} &= \frac{2\xi C_5}{2(d_0 - 3a)} \delta (2\delta + \lambda \delta^\beta)^2 \left[1 - e^{-2k_0(\delta)(d_0-3a)} \right] \\ &= C_6 \delta (2\delta + \lambda \delta^\beta)^2 - C_6 \delta (2\delta + \lambda \delta^\beta)^2 e^{-2k_0(\delta)(d_0-3a)}. \end{aligned} \quad (5.5)$$

The first term in (5.5) goes to 0 as $\delta \rightarrow 0^+$. The second term is equal to

$$C_6 \delta (2\delta + \lambda \delta^\beta)^2 (2\delta^2 + \lambda \delta^{\beta+1})^{(d_0-3a)/a}. \quad (5.6)$$

If $0 < \beta < 1$ we rewrite this as

$$C_6 (2\delta^{1-\beta} + \lambda)^2 (2\delta^{1-\beta} + \lambda)^{(d_0-3a)/a} \delta^{[1+2\beta+(\beta+1)(d_0-3a)/a]}.$$

This expression goes to 0 as $\delta \rightarrow 0^+$ if and only if

$$1 + 2\beta + (\beta + 1) \left(\frac{d_0 - 3a}{a} \right) > 0 \Leftrightarrow d_0 > \left(\frac{\beta + 2}{\beta + 1} \right) a = \tau(\beta)a,$$

and it goes to $C_6 \lambda^{[2+(d_0-3a)/a]}$ as $\delta \rightarrow 0^+$ if and only if $d_0 = \tau(\beta)a$.

If $\beta \geq 1$ we rewrite (5.6) as

$$C_6 (2 + \lambda \delta^{\beta-1})^2 (2 + \lambda \delta^{\beta-1})^{(d_0-3a)/a} \delta^{[3+2(d_0-3a)/a]}.$$

This term goes to 0 as $\delta \rightarrow 0^+$ if and only if

$$3 + 2(d_0 - 3a)/a > 0 \Leftrightarrow d_0 > \frac{3}{2}a = \tau(\beta)a,$$

and if $d_0 = \tau(\beta)a$ it goes to $C_6 2^{[2+(d_0-3a)/a]}$ if $\beta > 1$ and $C_6 (2 + \lambda)^{[2+(d_0-3a)/a]}$ if $\beta = 1$.

If $d_0 = 3a$, then from (5.4) we have

$$T_1 \leq 2\xi C_5 \delta (2\delta + \lambda \delta^\beta)^2 k_0(\delta) = a^{-1} \xi C_5 \delta (2\delta + \lambda \delta^\beta)^2 \ln \left(\frac{1}{2\delta^2 + \lambda \delta^{\beta+1}} \right);$$

this expression goes to 0 as $\delta \rightarrow 0^+$ for all $\beta > 0$. □

LEMMA 5.3. Suppose $\beta > 0$, λ is feasible, $0 < \xi < a/2$, and $d_0 \geq \tau(\beta)a$. Then

$$\lim_{\delta \rightarrow 0^+} T_2 = 0.$$

Proof. We begin by noting that the function $k^{-1}(e^{2k\xi} - 1)$ is continuous for $k \in [0, \infty)$ if we define it to be equal to 2ξ at $k = 0$. Also, since $d_0 \geq \tau(\beta)a \geq (3/2)a$, we have $e^{-2k(d_0-3a)}e^{-4ka} \leq e^{-ka}$ for all $k \geq 0$. This implies the integral

$$\int_0^\infty e^{-2k(d_0-3a)}e^{-4ka} \left(\frac{e^{2k\xi} - 1}{k} \right) dk$$

converges to a positive constant C as long as $0 < \xi < a/2$. Then (5.3b) implies that

$$T_2 \leq CC_5\delta(\lambda^2\delta^{2\beta} + 4) \rightarrow 0 \quad \text{as } \delta \rightarrow 0^+.$$

□

LEMMA 5.4. Suppose $\beta > 0$, λ is feasible, $0 < \xi < a$, and $d_0 > (3/2)a$. Then

$$\lim_{\delta \rightarrow 0^+} T_3 = \begin{cases} C_7\lambda^{[\frac{1}{2}+(d_0-\frac{3}{2}a)/a]} & \text{if } 0 < \beta < 1 \text{ and } d_0 = \tau(\beta)a, \\ 0 & \text{if } d_0 > \tau(\beta)a, \end{cases}$$

where

$$C_7 = \frac{9C_5\xi}{d_0 - \frac{3}{2}a} > 0.$$

Proof. As in the proof of Lemma 5.2 we have $k^{-1}(1 - e^{-2k\xi}) \leq 2\xi$ for all $k \geq 0$. Thus (5.3c) implies

$$\begin{aligned} T_3 &\leq 18C_5\xi\delta^{-\frac{1}{2}}(2\delta + \lambda\delta^\beta)^{\frac{1}{2}} \int_{k_0(\delta)}^\infty e^{-2k(d_0-\frac{3}{2}a)} dk \\ &= \frac{18C_5\xi}{2(d_0 - \frac{3}{2}a)} \delta^{-\frac{1}{2}}(2\delta + \lambda\delta^\beta)^{\frac{1}{2}} \left[-e^{-2k(d_0-\frac{3}{2}a)} \right]_{k_0(\delta)}^\infty \\ &= C_7\delta^{-\frac{1}{2}}(2\delta + \lambda\delta^\beta)^{\frac{1}{2}} e^{-2k_0(\delta)(d_0-\frac{3}{2}a)} \\ &= C_7\delta^{-\frac{1}{2}}(2\delta + \lambda\delta^\beta)^{\frac{1}{2}} (2\delta^2 + \lambda\delta^{\beta+1})^{(d_0-\frac{3}{2}a)/a}. \end{aligned} \quad (5.7)$$

If $0 < \beta < 1$, note that $\tau(\beta)a > (3/2)a$ — this implies that the above analysis holds as long as $d_0 \geq \tau(\beta)a$. We rewrite (5.7) as

$$C_7(2\delta^{1-\beta} + \lambda)^{\frac{1}{2}} (2\delta^{1-\beta} + \lambda)^{(d_0-\frac{3}{2}a)/a} \delta^{[-\frac{1}{2} + \frac{\beta}{2} + (\beta+1)(d_0-\frac{3}{2}a)/a]}.$$

This expression will go to 0 as $\delta \rightarrow 0^+$ if and only if

$$-\frac{1}{2} + \frac{\beta}{2} + (\beta+1) \left(\frac{d_0 - \frac{3}{2}a}{a} \right) > 0 \Leftrightarrow d_0 > \tau(\beta)a,$$

and if $d_0 = \tau(\beta)a$ it goes to $C_7\lambda^{\{1/2+[d_0-(3/2)a]/a\}}$ as $\delta \rightarrow 0^+$.

If $\beta \geq 1$ we note that the analysis leading to (5.7) can only be applied if $d_0 > \tau(\beta)a = (3/2)a$. In this case we rewrite (5.7) as

$$C_7(2 + \lambda\delta^{\beta-1})^{\frac{1}{2}} (2 + \lambda\delta^{\beta-1})^{(d_0-\frac{3}{2}a)/a} \delta^{2(d_0-\frac{3}{2}a)/a},$$

which goes to 0 as $\delta \rightarrow 0^+$ if and only if $2[d_0 - (3/2)a]/a > 0 \Leftrightarrow d_0 > \tau(\beta)a = (3/2)a$. □

LEMMA 5.5. Suppose $\beta > 0$, λ is feasible, $0 < \xi < a$, and $d_0 \geq \tau(\beta)a$. Then

$$\lim_{\delta \rightarrow 0^+} T_4 = 0.$$

Proof. We have, from (5.3d), that

$$\begin{aligned}
T_4 &= 9C_5 \delta^{-\frac{1}{2}} (2\delta + \lambda \delta^\beta)^{-\frac{3}{2}} (\lambda^2 \delta^{2\beta} + 4) \int_{k_0(\delta)}^{\infty} e^{-2k(d_0 - \frac{3}{2}a)} e^{-4ka} \left(\frac{e^{2k\xi} - 1}{k} \right) dk \\
&= 9C_5 \delta^{-\frac{1}{2}} (2\delta + \lambda \delta^\beta)^{-\frac{3}{2}} (\lambda^2 \delta^{2\beta} + 4) \int_{k_0(\delta)}^{\infty} e^{-k(2d_0 + a)} \left(\frac{e^{2k\xi} - 1}{k} \right) dk \\
&\leq 9C_5 \frac{\delta^{-\frac{1}{2}} (2\delta + \lambda \delta^\beta)^{-\frac{3}{2}} (\lambda^2 \delta^{2\beta} + 4)}{k_0(\delta)} \int_{k_0(\delta)}^{\infty} e^{-k(2d_0 + a - 2\xi)} dk \\
&= \left[\frac{9C_5 (\lambda^2 \delta^{2\beta} + 4)}{2d_0 + a - 2\xi} \right] \left[\frac{\delta^{-\frac{1}{2}} (2\delta + \lambda \delta^\beta)^{-\frac{3}{2}}}{k_0(\delta)} \right] e^{-k_0(\delta)(2d_0 + a - 2\xi)} \\
&= C_8 (\lambda^2 \delta^{2\beta} + 4) \left[\frac{\delta^{-\frac{1}{2}} (2\delta + \lambda \delta^\beta)^{-\frac{3}{2}}}{k_0(\delta)} \right] (2\delta^2 + \lambda \delta^{\beta+1})^{(2d_0 + a - 2\xi)/(2a)}, \tag{5.8}
\end{aligned}$$

where

$$C_8 \equiv \frac{9C_5}{2d_0 + a - 2\xi} > 0.$$

If $0 < \beta < 1$ we rewrite (5.8) as

$$\left[\frac{C_8 (\lambda^2 \delta^{2\beta} + 4)}{k_0(\delta)} \right] (2\delta^{1-\beta} + \lambda)^{-\frac{3}{2}} (2\delta^{1-\beta} + \lambda)^{(2d_0 + a - 2\xi)/(2a)} \delta^{[-\frac{1}{2} - \frac{3}{2}\beta + (\beta+1)(2d_0 + a - 2\xi)/(2a)]}.$$

This expression will go to 0 as $\delta \rightarrow 0^+$ if and only if

$$-\frac{1}{2} - \frac{3}{2}\beta + \frac{(\beta+1)(2d_0 + a - 2\xi)}{2a} \geq 0 \Leftrightarrow d_0 \geq \left(\frac{\beta}{\beta+1} \right) a + \xi.$$

We note that $[\beta/(\beta+1)]a + \xi < \tau(\beta)a$ since $0 < \beta < 1$ and $\xi < a$. Thus if $0 < \beta < 1$ and $d_0 \geq \tau(\beta)a$ we have $T_4 \rightarrow 0$ as $\delta \rightarrow 0^+$.

If $\beta \geq 1$ we rewrite (5.8) as

$$\left[\frac{C_8 (\lambda^2 \delta^{2\beta} + 4)}{k_0(\delta)} \right] (2 + \lambda \delta^{\beta-1})^{-\frac{3}{2}} (2 + \lambda \delta^{\beta-1})^{(2d_0 + a - 2\xi)/(2a)} \delta^{[-2 + (2d_0 + a - 2\xi)/a]}.$$

This expression goes to 0 as $\delta \rightarrow 0^+$ if and only if

$$-2 + (2d_0 + a - 2\xi)/a \geq 0 \Leftrightarrow d_0 \geq \frac{a}{2} + \xi.$$

Since $\beta \geq 1$ and $0 < \xi < a$ we have $a/2 + \xi < (3/2)a = \tau(\beta)a$; thus if $\beta \geq 1$ and $d_0 \geq \tau(\beta)a$ we have $T_4 \rightarrow 0$ as $\delta \rightarrow 0^+$. \square

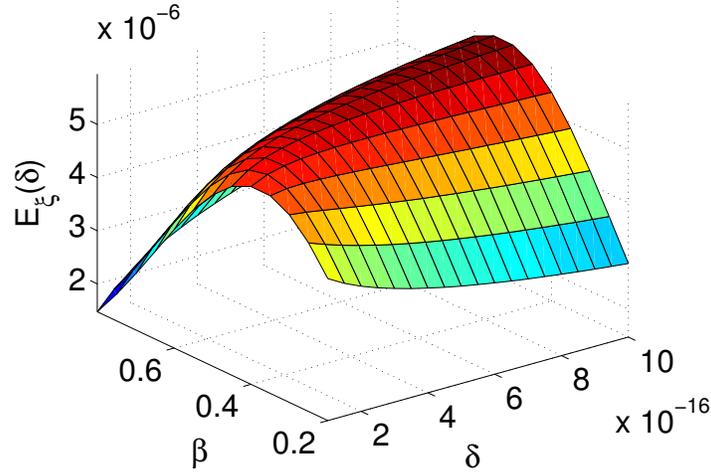
We summarize our result from this section in the following theorem.

THEOREM 5.6. Let $\beta > 0$ and λ feasible be fixed. Suppose also that $0 < \xi < a/2$ and $\rho \in \mathcal{P}$. If $d_0 > \tau(\beta)a$, then $\lim_{\delta \rightarrow 0^+} E_\xi(\delta) = 0$.

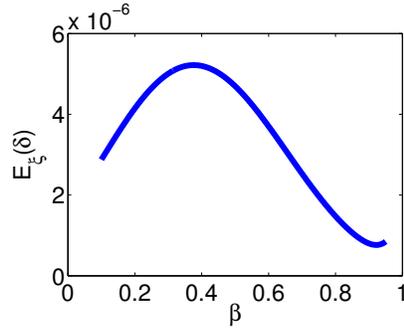
Proof. If the hypotheses of the theorem hold and if $\delta \leq \delta_0$, then (5.2) and Lemmas 5.1–5.5 imply

$$0 \leq E_\xi(\delta) \leq T_1 + T_2 + T_3 + T_4 \rightarrow 0 \quad \text{as } \delta \rightarrow 0^+.$$

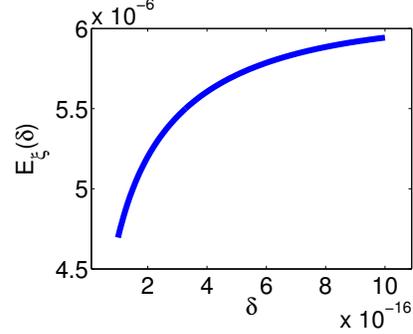
\square



(a)



(b)

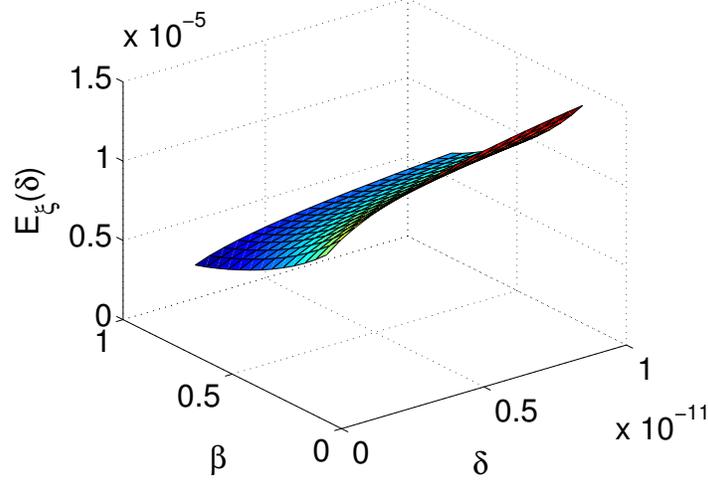


(c)

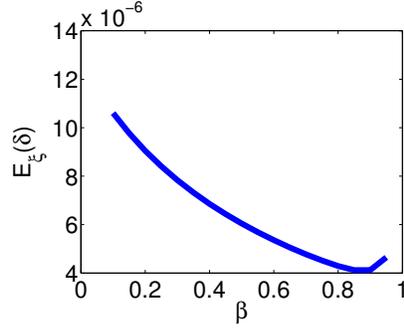
FIG. 4. (Rectangular ρ) In all of these subfigures we take $a = d_0/\tau(\beta)$ so ρ is completely outside the region of influence. (a) A plot of $E_\xi(\delta)$ versus β and δ — the z -axis scale is 10^{-6} ; (b) a plot of $E_\xi(\delta)$ for $\delta = 10^{-16}$ as a function of β — the y -axis scale is 10^{-6} ; (c) a plot of $E_\xi(\delta)$ for $\beta = 0.5$ as a function of δ — the y -axis scale is 10^{-6} .

Figures 4 and 5 are supporting numerical plots; they are the same as Figures 2 and 3, respectively, except in this case we have taken $a = d_0/\tau(\beta)$ so ρ just touches the region of influence (in order to accomplish this we have taken $\beta = 0.5$ in Figures 4(c) and 5(c) rather than $\beta = 0.8$ as in Figures 2(c) and 3(c)).

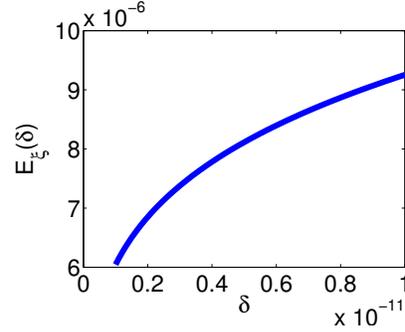
6. Boundedness of the Potential. In this section, we derive bounds on the potential in regions far away from the slab. In particular, we prove that the potentials V_c and



(a)



(b)



(c)

FIG. 5. (Circular ρ) In all of these subfigures we take $a = d_0/\tau(\beta)$ so ρ is completely outside the region of influence. (a) A plot of $E_\xi(\delta)$ versus β and δ — the z -axis scale is 10^{-5} ; (b) a plot of $E_\xi(\delta)$ for $\delta = 10^{-12}$ as a function of β — the y -axis scale is 10^{-6} ; (c) a plot of $E_\xi(\delta)$ for $\beta = 0.5$ as a function of δ — the y -axis scale is 10^{-6} .

V_m to the left and right of the slab, respectively, are bounded by constants that are independent of δ (for $|x|$ large enough). As discussed in the Introduction, this is the second requirement for cloaking by anomalous localized resonance to occur. At this point we do not address questions regarding which portions of the (rescaled) charge distribution $\rho/\sqrt{E_\xi(\delta)}$ will be cloaked. For example, if the (rescaled) rectangular charge distribution from Section 4.1.1 is halfway inside the cloaking region (so $x_0 = \tau(\beta)a$), we have not

yet determined whether it will be completely cloaked or if only the leading half will be cloaked.

We begin with some technical results. The proofs of the next two lemmas are straightforward and can be found in [27].

LEMMA 6.1. Let ψ_k^+ and ψ_k^- be defined as in (2.13) and (2.14), respectively. Then for each $k \in \mathbb{R}$ and all $0 < \delta \leq \delta_\mu$,

$$\|k|\psi_k^+ + \psi_k^-\|^2 \geq 2|k|^2 e^{-2|k|a}.$$

LEMMA 6.2. Let ψ_k^+ and ψ_k^- be defined as in (2.13) and (2.14), respectively. Then there exists $0 < \delta_{\psi^-}(\beta, \lambda) \leq \delta_\mu$ such that

$$\left| \psi_k^+ - \frac{1}{|k|} \psi_k^- \right|^2 \leq \frac{5}{2} (\delta + \mu)^2 e^{2|k|a}$$

for all $k \in \mathbb{R}$ and all $0 < \delta \leq \delta_{\psi^-}$.

6.1. *The Potential V_c .* Note that V_c is harmonic for $x < 0$ by (2.1) and (1.4). In addition, since $V \in L^2_{\text{loc}}(\mathbb{R}^2)$, $V \in L^1_{\text{loc}}(\mathbb{R}^2)$ as well. Hence the Weyl Theorem (see, e.g., Theorem 18.G in [32]) implies that V is infinitely differentiable for $x < 0$ (after modification on a set of measure 0), so we can examine pointwise values of V_c . The next lemma states that, far enough away from the slab, the potential V_c is bounded for all $\delta \leq \delta_\mu$.

LEMMA 6.3. Suppose $\rho \in \mathcal{P}$. Then there is a positive constant C_9 such that $V_c(x, y) \leq C_9$ for all $x < -3a$ and for all $0 < \delta \leq \delta_\mu$.

Proof. From (2.8) and (2.17) we have

$$|\widehat{V}_c(x, k)|^2 = |A_k|^2 e^{2|k|x} = \frac{|I_k|^2 e^{2|k|x}}{e^{-2|k|a} \|k|\psi_k^+ + \psi_k^-\|^2}. \quad (6.1)$$

In combination with Lemma 6.1, this implies that

$$|\widehat{V}_c(x, k)|^2 \leq \frac{|I_k|^2}{2|k|^2} e^{2|k|(x+2a)} \quad (6.2)$$

for $x < 0$, for all $k \in \mathbb{R}$, and for all $0 < \delta \leq \delta_\mu$. In particular, note that the expression in (6.2) is an even function of k if ρ is real-valued due to Lemma 3.1. Then for $x < 0$ (6.2) implies that

$$\begin{aligned} \int_{-\infty}^{\infty} |\widehat{V}_c(x, k)|^2 dk &\leq \frac{1}{2} \int_{-\infty}^{\infty} \frac{|I_k|^2}{|k|^2} e^{2|k|(x+2a)} dk \\ &= \int_0^{\infty} \frac{|I_k|^2}{|k|^2} e^{2|k|(x+2a)} dk \\ &= \int_0^1 \frac{|I_k|^2}{k^2} e^{2k(x+2a)} dk + \int_1^{\infty} \frac{|I_k|^2}{k^2} e^{2k(x+2a)} dk \\ &= \int_0^1 \frac{|I_k|^2}{k^2} e^{2k(x+2a)} dk + (d_1 - d_0) \|\rho\|_{L^2(\mathcal{M})}^2 \int_1^{\infty} \frac{e^{2k(x+2a-d_0)}}{k^2} dk, \end{aligned} \quad (6.3)$$

thanks to Lemma 3.1. Since

$$\frac{|I_k|^2}{k^2} \leq C_I^2$$

for $k \geq 0$ by Lemma 3.1, the first integral in (6.3) converges for any $x \in \mathbb{R}$. The second integral in (6.3) converges if and only if $x \leq d_0 - 2a$ (note that $d_0 - 2a > -a$ since $d_0 > a$). Then if $x < -2a$ we have, from (6.3), that

$$\int_{-\infty}^{\infty} |\widehat{V}_c(x, k)|^2 dk \leq \int_0^1 C_I^2 dk + (d_1 - d_0) \|\rho\|_{L^2(\mathcal{M})}^2 \int_1^{\infty} \frac{1}{k^2} dk = C_I^2 + (d_1 - d_0) \|\rho\|_{L^2(\mathcal{M})}^2.$$

Then the Plancherel Theorem (3.2) implies that for each $x < -2a$ we have

$$\int_{-\infty}^{\infty} |V_c(x, y)|^2 dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\widehat{V}_c(x, k)|^2 dk \leq \frac{1}{2\pi} \left[C_I^2 + (d_1 - d_0) \|\rho\|_{L^2(\mathcal{M})}^2 \right]. \quad (6.4)$$

Since $V_c(x, y)$ is harmonic for $x < -2a$, it satisfies the mean value property (see, e.g., Chapter 2 in [7]): for any point (x, y) with $x < -3a$ we have

$$V(x, y) = \frac{1}{|B_a((x, y))|} \int_{B_a((x, y))} V(x', y') dy' dx',$$

where $B_a((x, y))$ is the ball of radius a centered at the point (x, y) ; note that all points $(x', y') \in B_a((x, y))$ satisfy $x' < -2a$ since $x < -3a$. Finally by the Cauchy–Schwarz inequality and (6.4) we have

$$\begin{aligned} |V_c(x, y)| &= \frac{1}{|B_a((x, y))|} \left| \int_{B_a((x, y))} V(x', y') dy' dx' \right| \\ &\leq \frac{1}{|B_a((x, y))|} \int_{B_a((x, y))} |V(x', y')| dy' dx' \\ &\leq \frac{1}{|B_a((x, y))|} \left[\int_{B_a((x, y))} |V(x', y')|^2 dy' dx' \right]^{\frac{1}{2}} \left[\int_{B_a((x, y))} dy' dx' \right]^{\frac{1}{2}} \\ &\leq \frac{1}{|B_a((x, y))|^{\frac{1}{2}}} \left[\int_{x-a}^{x+a} \int_{-\infty}^{\infty} |V(x', y')|^2 dy' dx' \right]^{\frac{1}{2}} \\ &\leq \int_{x-a}^{x+a} \frac{1}{2\pi^{3/2}a} \left[C_I^2 + (d_1 - d_0) \|\rho\|_{L^2(\mathcal{M})}^2 \right] dx' \\ &= C_9, \end{aligned}$$

where $C_9 = \pi^{-3/2} \left[C_I^2 + (d_1 - d_0) \|\rho\|_{L^2(\mathcal{M})}^2 \right]$. \square

6.2. The Potential V_m . We will now show that $|V_m(x, y)|$ is bounded for x large enough. In particular, we at least assume that $x > d_1$. We begin with a lemma that is very similar to Lemma 3.1. For $x > d_1$ we define

$$J_k(x) \equiv \int_{d_0}^{d_1} \widehat{\rho}(s, k) e^{-|k|(x-s)} ds. \quad (6.5)$$

The proof of the following lemma can be found in [27].

LEMMA 6.4. Suppose $\rho \in \mathcal{P}$ (where \mathcal{P} is defined in (1.2)) and that, for $x > d_1$, $J_k(x)$ is defined as in (6.5). Then, for every $x > d_1$, $J_k(x)$ satisfies the following properties:

- (1) for all $k \in \mathbb{R}$, $|J_k(x)|^2 \leq (d_1 - d_0) \|\rho\|_{L^2(\mathcal{M})}^2 e^{-2k(x-d_1)}$;
- (2) if ρ is real-valued, then $|J_k(x)|^2$ is an even function of k for $k \in \mathbb{R}$;
- (3) $J_k(x)$ is continuous at k for each $k \in \mathbb{R}$;
- (4) $\lim_{k \rightarrow 0} J_k(x) = J_0(x) = 0$;
- (5) for each $x > d_1$,

$$\lim_{k \rightarrow 0} \frac{|J_k(x)|}{|k|} = |C_0| < \infty,$$

where C_0 is defined in Lemma 3.1; moreover, there is a positive constant C_J , independent of x , such that $|J_k(x)|/|k| \leq C_J$ for all $x > d_1$ and all $k \in [0, 1]$.

LEMMA 6.5. Suppose $\rho \in \mathcal{P}$. Then there is a positive constant C_{10} such that $|V_m(x, y)| \leq C_{10}$ for all $x > a + \max\{d_1, 4a\}$ and for all $\delta \leq \delta_{\psi^-}$ (where δ_{ψ^-} is defined in Lemma 6.2).

Proof. Based on our choice of A_k and I_k in (2.17) and (2.18), respectively, for $x > d_1$ we have

$$\widehat{V}_m(x, k) = e^{-|k|x} \left(\frac{A_k \psi_k^+ e^{|k|a}}{2} - \frac{A_k \psi_k^- e^{|k|a}}{2|k|} \right) + \frac{J_k(x)}{2|k|}; \quad (6.6)$$

see (2.16). Then (2.17), the triangle inequality, and the fact that $(p+q)^2 \leq 2p^2 + 2q^2$ for real numbers p and q imply, for $x > d_1$, that

$$\begin{aligned} |\widehat{V}_m(x, k)|^2 &= \left| e^{-|k|x} \left(\frac{A_k \psi_k^+ e^{|k|a}}{2} - \frac{A_k \psi_k^- e^{|k|a}}{2|k|} \right) + \frac{J_k(x)}{2|k|} \right|^2 \\ &\leq \frac{e^{-2|k|(x-a)}}{2} |A_k|^2 \left| \psi_k^+ - \frac{1}{|k|} \psi_k^- \right|^2 + \frac{|J_k(x)|^2}{2|k|^2}. \end{aligned}$$

Then (6.1), Lemma 6.1, and Lemma 6.2 imply, for $0 < \delta \leq \delta_{\psi^-}$, that

$$\begin{aligned} |\widehat{V}_m(x, k)|^2 &\leq \frac{e^{-2|k|(x-a)}}{2} \frac{|I_k|^2 e^{4|k|a}}{2|k|^2} \left| \psi_k^+ - \frac{1}{|k|} \psi_k^- \right|^2 + \frac{|J_k(x)|^2}{2|k|^2} \\ &\leq \frac{5e^{-2|k|(x-3a)} |I_k|^2}{8|k|^2} (\delta + \mu)^2 e^{2|k|a} + \frac{|J_k(x)|^2}{2|k|^2} \\ &\leq \frac{5}{8} (\delta + \mu)^2 \frac{|I_k|^2}{|k|^2} e^{-2|k|(x-4a)} + \frac{|J_k(x)|^2}{2|k|^2}. \end{aligned} \quad (6.7)$$

Note that the expression in (6.7) is even as a function of k by Lemmas 3.1 and 6.4. Then we have

$$\begin{aligned} \int_{-\infty}^{\infty} |\widehat{V}_m(x, k)|^2 dk &\leq \frac{5}{8} (\delta + \mu)^2 \int_{-\infty}^{\infty} \frac{|I_k|^2}{|k|^2} e^{-2|k|(x-4a)} dk + \int_{-\infty}^{\infty} \frac{|J_k(x)|^2}{2|k|^2} dk \\ &= \frac{5}{4} (\delta + \mu)^2 \left[\int_0^1 \frac{|I_k|^2}{k^2} e^{-2k(x-4a)} dk + \int_1^{\infty} \frac{|I_k|^2}{k^2} e^{-2k(x-4a)} dk \right] \\ &\quad + \int_0^1 \frac{|J_k(x)|^2}{k^2} dk + \int_1^{\infty} \frac{|J_k(x)|^2}{k^2} dk. \end{aligned}$$

Then Lemmas 3.1 and 6.4 imply

$$\begin{aligned} \int_{-\infty}^{\infty} |\widehat{V}_m(x, k)|^2 dk &\leq \frac{5}{4}(\delta + \mu)^2 C_0^2 \int_0^1 e^{-2k(x-4a)} dk + C_J^2 \\ &+ (d_1 - d_0) \|\rho\|_{L^2(\mathcal{M})}^2 \left[\frac{5}{4}(\delta + \mu)^2 \int_1^{\infty} \frac{e^{-2k(x-4a+d_0)}}{k^2} dk + \int_1^{\infty} \frac{e^{-2k(x-d_1)}}{k^2} dk \right]. \end{aligned} \quad (6.8)$$

If $x > \max\{d_1, 4a\}$, then all of the integrals in (6.8) converge. In particular, the integral from 0 to 1 and both of the integrals from 1 to ∞ converge to numbers less than or equal to 1 in that case. Therefore (6.8) becomes

$$\int_{-\infty}^{\infty} |\widehat{V}_m(x, k)|^2 dk \leq \frac{5}{4}(\delta + \mu)^2 C_0^2 + C_J^2 + (d_1 - d_0) \|\rho\|_{L^2(\mathcal{M})}^2 \left[\frac{5}{4}(\delta + \mu)^2 + 1 \right] \equiv \widetilde{C}_{10}.$$

If we define $b \equiv a + \max\{d_1, 4a\}$, for example, then for $x > b$ each point $(x', y') \in B_a((x, y))$ satisfies $x' > \max\{d_1, 4a\}$. Since V_m is harmonic in the region where $x' > d_1$, it satisfies the mean value property there. Using this in combination with the Plancherel Theorem (just as in the proof of Lemma 6.3) gives

$$|V_m(x, y)| \leq \int_{x-a}^{x+a} \frac{\widetilde{C}_{10}}{2\pi^{3/2}a} dx' \equiv C_{10},$$

where $C_{10} = \pi^{-3/2}\widetilde{C}_{10}$. □

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