OBSERVING LYAPUNOV EXPONENTS OF INFINITE-DIMENSIONAL DYNAMICAL SYSTEMS

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Abstract. Can Lyapunov exponents of infinite-dimensional dynamical systems be observed by projecting the dynamics into $\mathbb{R}^N$ using a 'typical' nonlinear projection map? We answer this question affirmatively by developing embedding theorems for compact invariant sets associated with C^1 maps on Hilbert spaces. Examples of such discrete-time dynamical systems include time-T maps and Poincaré return maps generated by the solution semigroups of evolution partial differential equations.

We make every effort to place hypotheses on the projected dynamics rather than on the underlying infinite-dimensional dynamical system. In so doing, we adopt an empirical approach and formulate checkable conditions under which a Lyapunov exponent computed from experimental data will be a Lyapunov exponent of the infinite-dimensional dynamical system under study (provided the nonlinear projection map producing the data is typical in the sense of prevalence).

Contents

1. Introduction

This paper is about observing Lyapunov exponents of infinite-dimensional dynamical systems by projecting the dynamics into $\mathbb{R}^N$. We focus on discrete-time infinite-dimensional dynamics produced by maps on real Hilbert spaces. Important types of such maps include time-T maps and Poincaré return maps generated by the solution semigroups of evolution partial differential equations.

Let H be a real Hilbert space and let $f : H \to H$ be a C^1 (continuously Fréchet-differentiable) map. A ***Lyapunov exponent*** $\omega(x, v)$ is a limit of the form

$$\omega(x,v) = \lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n v\|, \tag{1}$$

where $x \in H$ and $v \in T_x H$ is a tangent vector.

1.1. Lyapunov exponents in finite dimensions.

1.1. Lyapunov exponents in finite dimensions. Lyapunov exponents play a central role in the theory of nonuniformly hyperbolic dynamical systems in finite dimensions (here the domain of f is a compact Riemannian manifold M). They are deeply related to a number of dynamical quantities of interest, including entropy, dimension, and rates of escape in open systems. Although Lyapunov exponents encode information about the infinitesimal behavior of f, a vast array of results demonstrates that local and even global information about the nonlinear dynamics of f can be deduced from them (see *e.g.* [1, 2, 29]).

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The limit in (1) does not necessarily exist for every (x, v) in the tangent bundle TM; nevertheless, Lyapunov exponents exist for almost every $x \in M$ assuming stationarity. Oseledec [18] proves that if μ is an f-invariant Borel probability measure, then for μ almost every $x \in M$, there exist numbers

$$\omega_1(x) > \omega_2(x) > \cdots > \omega_{q(x)}(x)$$

with corresponding multiplicities $m_1(x), \ldots, m_{q(x)}(x)$ such that

(a) for every tangent vector $v \in T_x M$, $\omega(x, v)$ exists and equals $\omega_j(x)$ for some j;
(b) $\sum_{i=1}^{q(x)} m_i(x) = \dim(M)$;
(c) $\sum_{i=1}^{q(x)} \omega_i(x) m_i(x) = \lim_{n\to\infty} \left(\frac{1}{n}\right) \log |\det(Df_x^n)|$.

Further, if f is a diffeomorphism of M, then the tangent space $T_x M$ admits a decomposition

$$T_x M = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_{q(x)}(x)$$

with $\dim(E_i(x)) = m_i(x)$ and $\omega(x, v) = \omega_i(x)$ for every $v \in E_i(x)$. If μ is ergodic, then the $\omega_i(\cdot)$ are constant μ almost everywhere; in this case we call the ω_i the Lyapunov exponents of the system (f, μ).

While the Oseledec multiplicative ergodic theorem makes conclusions about Lyapunov exponents given an invariant measure, the *existence* of important invariant measures for dynamical systems that exhibit some degree of hyperbolicity is another matter entirely. Researchers actively work to identify mechanisms that may produce nonuniform hyperbolicity and then prove that these mechanisms do produce nonuniform hyperbolicity for concrete systems of interest in the physical and biological sciences. This program has been carried out for limit cycles and homoclinic orbits in [28] and [27], respectively.

1.2. **Lyapunov exponents in infinite dimensions.** Here one starts with a dynamical system $\sigma : \Omega \to \Omega$, selects a Banach space B, and then assigns to each $\omega \in \Omega$ a bounded linear operator $\mathcal{L}_\omega$ on B. The assignment $\omega \mapsto \mathcal{L}_\omega$ is known as a cocycle over the dynamical system. Having defined the cocycle, one then hopes to prove a multiplicative ergodic theorem in the spirit of Oseledec that applies to the compositions $\mathcal{L}_\omega^{(n)} = \mathcal{L}_{\sigma^{n-1}(\omega)} \circ \cdots \circ \mathcal{L}_{\sigma(\omega)} \circ \mathcal{L}_\omega$. For a smooth map f on a real Hilbert space H, Ruelle proves a multiplicative ergodic theorem for the derivative cocycle assuming Df_x is compact [23]. Cocycles into operators on Banach spaces (possibly with nontrivial essential spectrum) are treated in [5, 12, 17, 26].

Transfer operator techniques have led to substantial understanding of the statistical properties of deterministic autonomous dynamical systems. With an eye on applications, multiplicative ergodic theorists in recent years have sought to extend transfer operator techniques to nonautonomous and random dynamical systems. This effort has led to beautiful multiplicative ergodic theorems for transfer operator cocycles [5, 6].

The program aimed at deducing global dynamical information about infinite-dimensional systems from Lyapunov exponent data is in its early stages of development. Results in this direction include the existence of Sinai-Ruelle-Bowen (SRB) measures for periodically-kicked supercritical Hopf bifurcations in a concrete PDE context [16] and the existence of horseshoes in a general context [13, 14].

1.3. **Observation of Lyapunov exponents.** Suppose $A \subset H$ satisfies $f(A) = A$ (we call A an invariant set). For example, A may be the global attractor of a dissipative PDE such as the two-dimensional incompressible Navier-Stokes system. We are interested in observing Lyapunov exponents of the restriction $f|A$ by projecting the dynamics into $\mathbb{R}^N$. For a map $\varphi : H \to \mathbb{R}^N$ (we call φ an observable or measurement map), we say that φ induces dynamics on $\varphi(A)$ if there exists a map $f_* : \varphi(A) \to \varphi(A)$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ \varphi(A) & \xrightarrow{f_*} & \varphi(A) \end{array}$$

Question 1.1. For a 'typical' observable φ, if φ induces dynamics on $\varphi(A)$ and if $\omega(z, w)$ is a Lyapunov exponent for f_*, do there exist $x \in A$ and a vector v such that $\omega(x, v)$ is a Lyapunov exponent for $f|A$ and $\omega(x, v) = \omega(z, w)$?

Ott and Yorke [20] develop an affirmative answer to Question 1.1 for the case $H = \mathbb{R}^D$. In this work we treat the infinite-dimensional case by developing an embedding result of the following type: For a 'typical' observable φ, if φ induces dynamics on $\varphi(A)$, then φ embeds A into $\mathbb{R}^N$. We then use this embedding result (Theorem 3.10) to answer Question 1.1 in the affirmative (Corollary 3.11). Keep the following in mind as we develop the theory.

(a) (Placement of hypotheses) Since we develop a theory of observation, we strive to place the hypotheses on the observed set $\varphi(A)$ and the induced dynamics thereon rather than on f and A. Indeed, we view f and A as objects that are not known *a priori*.

(b) (Notion of 'typical') We use the measure-theoretic notion of prevalence [3, 8, 9, 10, 21]. Prevalence is suitable for spaces of observables such as $C^1(H, \mathbb{R}^N)$. See Section 2 for a brief overview.

(c) (Generalized tangent spaces) We expect the set A to have fractal properties. We therefore use a generalized notion of tangent space suitable for such sets (Definition 3.2).

We finish the introduction by briefly examining an alternate approach to the embedding problem: Use assumptions about dimension to embed A into $\mathbb{R}^N$ (in the spirit of the Whitney embedding theorem) rather than formulating results in terms of induced dynamics. As we will see, one encounters an unresolved challenge when using dimension characteristics.

1.4. **Embedding results via dimension characteristics.** Here we discuss a prototype result that makes use of dimension.

Prototype Theorem 1.2. *Let H be a real Hilbert space and let $A \subset H$ be compact. Fix $N \in \mathbb{N}$. For almost every (in the sense of prevalence) $\varphi \in C^1(H, \mathbb{R}^N)$, if $\dim(\varphi(A)) < N/2$, then φ is one-to-one on A.*

Observe that the hypothesis involving dimension is placed on $\varphi(A)$ rather than A. We do not know if there exists a dimension characteristic for which the prototype theorem holds. Natural candidates include box-counting dimension $\dim_{\mathfrak{B}}$ and Hausdorff dimension $\dim_{\mathfrak{H}}$. Sets with finite box-counting dimension project well:

Theorem 1.3 ([7])**.** *Let H be a real Hilbert space and let $A \subset H$ be a compact set with $\dim_{\mathfrak{B}}(A) = d < \infty$ and with thickness exponent $\tau(A)$ (see [7, Definition 3.4] for the definition of thickness exponent). Let $N > 2d$ be an integer and let $\alpha \in \mathbb{R}$ satisfy*

$$0 < \alpha < \frac{N - 2d}{N(1 + \tau(A)/2)}.$$

For almost every (in the sense of prevalence) C^1 map $\varphi : H \to \mathbb{R}^N$, there exists $K > 0$ such that for all $x, y \in A$, we have

$$K \left\| \varphi(x) - \varphi(y) \right\|^{\alpha} \geqslant \left\| x - y \right\|.$$

That is, φ is one-to-one on A with Hölder-continuous inverse.

Remark 1.4. Theorems 1.3 and 1.5 remain true when one replaces the thickness exponent of A with the Lipschitz deviation $\operatorname{dev}(A)$ [22]. Roughly speaking, $\tau(A)$ measures how well A can be approximated by finite-dimensional subspaces of H, while $\operatorname{dev}(A)$ measures how well A can be approximated by the graphs of Lipschitz functions defined on finite-dimensional subspaces of H (with lower values of $\tau(A)$ and $\operatorname{dev}(A)$ indicating better approximability). One always has $\operatorname{dev}(A) \leqslant \tau(A)$.

However, it is difficult to infer the box-counting dimension of a set from that of its images. Sauer and Yorke [25] construct a compact set $Q \subset \mathbb{R}^{10}$ with $\dim_{\mathfrak{B}}(Q) = 5$ such that $\dim_{\mathfrak{B}}(\varphi(Q)) < 4$ for every C^1 map $\varphi : \mathbb{R}^{10} \to \mathbb{R}^6$. See [4, 11] for additional examples in the same spirit. By contrast, Hausdorff dimension is preserved by typical smooth maps (for sets with thickness exponent zero, a condition automatically satisfied when H is finite-dimensional).

Theorem 1.5 ([19])**.** *Let H be a real Hilbert space and let $A \subset H$ be a compact set with thickness exponent $\tau(A)$. Let $N \in \mathbb{N}$. For almost every (in the sense of prevalence) C^1 map $\varphi : H \to \mathbb{R}^N$, we have*

$$\dim_{\mathfrak{H}}(\varphi(A)) \geqslant \min\left\{ N, \frac{\dim_{\mathfrak{H}}(A)}{1 + \tau(A)/2} \right\}.$$

In particular, $\dim_{\mathfrak{H}}(\varphi(A)) = \dim_{\mathfrak{H}}(A)$ if $\tau(A) = 0$ and $N \geqslant \dim_{\mathfrak{H}}(A)$.

However, sets with low Hausdorff dimension may be difficult to project in a one-to-one way. Kan [24, Appendix] constructs a set $X \subset \mathbb{R}^D$ with Hausdorff dimension zero such that every linear map $\varphi : \mathbb{R}^D \to \mathbb{R}^N$ fails to be one-to-one on X if $N < D$.

For $H = \mathbb{R}^D$, the difficulties associated with Hausdorff dimension and box-counting dimension can be overcome by using the notion of tangent dimension $\dim_{\mathfrak{T}}(Y)$. Introduced in [20], $\dim_{\mathfrak{T}}(Y)$ is given for $Y \subset \mathbb{R}^D$ by

$$\dim_{\mathfrak{T}}(Y) = \sup_{x \in Y} \dim(T_x Y),$$

where $T_x Y$ denotes the tangent space at x relative to Y (Definition 3.2). Ott and Yorke formulate a 'Platonic' version of the Whitney embedding theorem using tangent dimension.

Theorem 1.6 ([20]). *Let A be a compact subset of $\mathbb{R}^D$ and let $N \in \mathbb{N}$. For almost every (in the sense of prevalence) $\varphi \in C^1(\mathbb{R}^D, \mathbb{R}^N)$, if $\dim_{\mathfrak{T}}(\varphi(A)) < N/2$, then φ is one-to-one on A.*

The proof of Theorem 1.6 uses the fact that if $A \subset \mathbb{R}^D$ is compact, then $\dim_{\mathfrak{B}}(A) \leqslant \dim_{\mathfrak{T}}(A)$. This inequality is a consequence of a manifold extension theorem [20, Theorem 3.5]: For every $x \in A$, there exists a neighborhood $N(x)$ of x in $\mathbb{R}^D$ and a C^1 manifold M such that $M \supset A \cap N(x)$ and $T_x A = T_x M$. The manifold extension theorem, however, does not hold in general for compact subsets of infinite-dimensional real Hilbert spaces.

2. Linear prevalence

Prevalence is a measure-theoretic notion of genericity for infinite-dimensional spaces. We summarize the theory here in the context of complete metric linear spaces. For more information, see [3, 8, 9, 10, 21].

Definition 2.1. Let V be a complete metric linear space. A Borel set $S \subset V$ is said to be ***shy*** if there exists a Borel measure μ on V satisfying

(a) $0 < \mu(K) < \infty$ for some compact set $K \subset V$;
(b) $\mu(S + x) = 0$ for all $x \in V$.

We say that such a measure is transverse to S. More generally, a set S is said to be shy if it is contained in a shy Borel set. The complement of a shy set is said to be a ***prevalent*** set.

Prevalence has the following properties [9].

(LP1) All prevalent sets are dense.
(LP2) Every subset of a shy set is shy.
(LP3) Every translate of a shy set is shy.
(LP4) The union of a countable collection of shy sets is shy.
(LP5) A set $S \subset \mathbb{R}^m$ is shy if and only if it has Lebesgue measure zero.

Property (LP5) shows that prevalence generalizes the translation-invariant notion of *Lebesgue almost every* to infinite-dimensional complete metric linear spaces.

It is useful to view a Borel measure μ on V as an object that defines a family of perturbations. From this point of view, a Borel set $E \subset V$ is prevalent if there exists a Borel measure μ such that for every $x \in V$, $x + y \in E$ for μ almost every y in the support of μ. An often useful choice for μ is Lebesgue measure on a finite-dimensional subspace of V.

Definition 2.2. Let V be a complete metric linear space. A finite-dimensional subspace $P \subset V$ is said to be a ***probe*** for a Borel set $E \subset V$ provided

$$\lambda_P(\{p \in P : x + p \notin E\}) = 0$$

for every $x \in V$, where λ_P denotes Lebesgue measure on P.

Notice that if a Borel set $E \subset V$ has a probe, then E is prevalent.

3. Projection of dynamics: the Hilbert space case

Throughout this section, let H be a real Hilbert space with norm $\|\cdot\|$ induced by the inner product $\langle \cdot, \cdot \rangle$ and let H^* denote the dual of H. Let $f : H \to H$ be a map and let $A \subset H$ satisfy $f(A) = A$ (we call A an invariant set). For a map $\varphi : H \to \mathbb{R}^N$, we say that φ induces dynamics on $\varphi(A)$ if there exists a map $f_* : \varphi(A) \to \varphi(A)$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ \varphi(A) & \xrightarrow{f_*} & \varphi(A) \end{array}$$

We focus on the following question. For a typical observable $\varphi : H \to \mathbb{R}^N$, does the existence of an induced map f_* with specified properties imply that A is 'equivalent' to $\varphi(A)$ and that the dynamical systems (f, A) and $(f_*, \varphi(A))$ are 'equivalent'? This question has been answered affirmatively in the continuous category.

3.1. **Continuous observables.** We establish some notation before stating the result. For a map $g : X \to X$ and $k \in \mathbb{N}$, let $\mathrm{Per}_k(g)$ denote the set of periodic points of g of period k. More precisely,

$$\mathrm{Per}_k(g) = \{x \in X : g^k(x) = x \text{ and } g^i(x) \neq x \text{ for } 1 \leqslant i \leqslant k-1\}.$$

Theorem 3.1 ([15]). *Let H be a separable real Hilbert space and let $f : H \to H$ be a map. Suppose that $A \subset H$ is a compact set satisfying $f(A) = A$. Let $N \in \mathbb{N}$ and let V be any closed subspace of $C^0(H, \mathbb{R}^N)$ that contains the bounded linear functions. For prevalent $\varphi \in V$, if f induces a map f_* on $\varphi(A)$ satisfying $f_* \circ \varphi = \varphi \circ f$ on A and if*

(a) *$f_* : \varphi(A) \to \varphi(A)$ is invertible;*
(b) *$\mathrm{Per}_1(f_*) \cup \mathrm{Per}_2(f_*)$ is countable;*

then $\varphi|A$ is a homeomorphism and the dynamical systems (f, A) and $(f_*, \varphi(A))$ are topologically conjugate.

3.2. **Observing differentiable dynamics.** In order to formulate versions of Theorem 3.1 for differentiable dynamics, we must first define a notion of differentiability suitable for maps defined on arbitrary subsets of real Hilbert spaces. We call this notion quasi-differentiability; it is defined in terms of generalized tangent spaces.

Definition 3.2. Let X be a real Hilbert space and let $E \subset X$. For $x \in E$, let $\Delta_x E$ be the set of all directions $v \in X$ for which there exist sequences $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ in E such that $x_i \to x$, $y_i \to x$, $x_i \neq y_i$ for all i, and

$$\lim_{i\to\infty} \frac{y_i - x_i}{\|y_i - x_i\|} = v.$$

The ***tangent space*** at x relative to E, denoted $T_x E$, is the smallest closed subspace of X that contains $\Delta_x E$. The tangent bundle over E is defined by $TE = \{(x, v) : x \in E,\ v \in T_x E\}$.

Definition 3.3. Let X be a real Hilbert space. A map $f : X \to X$ is said to be ***quasi-differentiable*** on a set $E \subset X$ if for each $x \in E$ there exists a bounded linear operator Df_x on X such that

$$\lim_{i\to\infty} \frac{f(y_i) - f(x_i) - Df_x(y_i - x_i)}{\|y_i - x_i\|} = 0$$

for all sequences $(x_i)_{i=1}^{\infty}$ in E and $(y_i)_{i=1}^{\infty}$ in E satisfying $x_i \to x$, $y_i \to x$, and $x_i \neq y_i$ for all i. We call the operator Df_x a quasi-derivative of f at x.

Now assume for the remainder of Subsection 3.2 that $f : H \to H$ is C^1 and recall that $A \subset H$ satisfies $f(A) = A$. We will address the following question.

(Q1) For a prevalent C^1 observable $\varphi : H \to \mathbb{R}^N$, if f induces a quasi-differentiable map f_* on $\varphi(A)$, does φ embed A into $\mathbb{R}^N$?

As we now explain, care must be taken when choosing a notion of embedding.

3.2.1. ***Notions of embedding.*** Our first notion of embedding is motivated by classical differential topology.

Definition 3.4. A C^1 map $\varphi : H \to \mathbb{R}^N$ is said to be an ***immersion*** on a set $E \subset H$ if $D\varphi_x : T_x E \to T_{\varphi(x)}\varphi(E)$ is injective for every $x \in E$. An injective immersion φ is said to be an ***embedding*** of E if $\varphi|E$ maps E homeomorphically onto $\varphi(E)$.

Suppose that C^1 $\varphi : H \to \mathbb{R}^N$ embeds a set $E \subset H$ into $\mathbb{R}^N$ and let $p \in E$ be an accumulation point of E. If H is finite-dimensional, then the fact that $D\varphi_p : T_p E \to T_{\varphi(p)}\varphi(E)$ is injective implies that it is surjective as well. This follows from the fact that the unit sphere in any finite-dimensional Hilbert space is compact. However, injectivity of $D\varphi_p$ on $T_p E$ does not imply surjectivity of $D\varphi_p$ on $T_p E$ if H is infinite-dimensional because the unit sphere in such an H is no longer compact. The following example illustrates the phenomenon.

Let X be an infinite-dimensional separable real Hilbert space with orthonormal basis $\{\boldsymbol{e}_i : i \in \mathbb{N}\}$ and let $p \in X$. Define $Q = \{p + \boldsymbol{e}_i/i : i \in \mathbb{N}\} \cup \{p\}$. The direction set $\Delta_p Q$ is empty and therefore $T_p Q = \{0\}$ despite the fact that p is an accumulation point of Q. Now suppose that C^1 $\varphi : X \to \mathbb{R}^N$ embeds Q into $\mathbb{R}^N$. Since $\varphi(p)$ is an accumulation point of $\varphi(Q)$, the compactness of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$ implies that $\Delta_{\varphi(p)}\varphi(Q)$ is nonempty and therefore $\dim(T_{\varphi(p)}\varphi(Q)) > 0$.

Motivated by this example, we formulate a second, stronger notion of embedding.

Definition 3.5. A C^1 map $\varphi : H \to \mathbb{R}^N$ is said to be a ***strong embedding*** of a set $E \subset H$ if φ is an embedding of E and if $D\varphi_x : T_x E \to T_{\varphi(x)}\varphi(E)$ is bijective for every $x \in E$.

Note that if H is finite-dimensional, then an embedding of E is a strong embedding of E.

3.2.2. ***Embedding theorems: general case***. We formulate conditions under which (Q1) has an affirmative answer in the sense of Definition 3.4.

Lemma 3.6. *Let H be a separable real Hilbert space and let $f : H \to H$ be a C^1 map. Suppose $A \subset H$ is such that $f(A) = A$ and Df_x is injective on T_xA for every $x \in A \setminus \mathrm{Per}_1(f|A)$. Let $N \in \mathbb{N}$. For every ball $\mathcal{B} = B(y, r)$ in H, the set $W_{\mathcal{B}}$ of observables $\varphi \in C^1(H, \mathbb{R}^N)$ satisfying*

(a) *there exists $(x, v) \in TA$ such that $v \neq 0$, $x \notin B(y, 2r)$, $f(x) \in B(y, r)$, and $D\varphi_x v = 0$;*
(b) *for every such element of TA we have $(D\varphi_{f(x)} \circ Df_x)v = 0$;*

is a shy subset of $C^1(H, \mathbb{R}^N)$.

Proof of Lemma 3.6. Assume H is infinite-dimensional. It suffices to consider the case $N = 1$. We will construct a measure μ that is transverse to $W_{\mathcal{B}}$. Choose a C^∞ bump function $\beta : \mathbb{R} \to \mathbb{R}$ such that

$$0 \leqslant \beta \leqslant 1, \qquad \beta \equiv 1 \text{ on } \{|s| < 25/16\}, \qquad \mathrm{supp}(\beta) = \{|s| \leqslant 9/4\}.$$

Define $\beta_{\mathcal{B}} : H \to \mathbb{R}$ by

$$\beta_{\mathcal{B}}(x) = \beta\left(\frac{\|x - y\|^2}{r^2}\right).$$

The function $\beta_{\mathcal{B}}$ has the following properties:

$$0 \leqslant \beta_{\mathcal{B}} \leqslant 1, \qquad \beta_{\mathcal{B}}|B(y, 5r/4) \equiv 1, \qquad \mathrm{supp}(\beta_{\mathcal{B}}) = \overline{B(y, 3r/2)}.$$

Now let $\{\boldsymbol{e}_m^* : m \in \mathbb{N}\}$ be an orthonormal basis for H^*. Define

$$Q = \left\{ \beta_{\mathcal{B}} \sum_{m=1}^{\infty} m^{-1} \gamma_m \boldsymbol{e}_m^* : |\gamma_m| \leqslant 1 \text{ for all } m \right\}.$$

Notice that Q is compact. Let μ be the probability measure on Q that results from choosing the γ_m independently and uniformly on $[-1, 1]$. We claim that μ is transverse to $W_{\mathcal{B}}$.

Let $\psi \in C^1(H, \mathbb{R})$. Suppose that there exists $(x, v) \in TA$ such that $v \neq 0$, $x \notin B(y, 2r)$, $f(x) \in B(y, r)$, and $D\psi_x v = 0$. (If no such element of TA exists, then $\{\eta \in Q : \psi + \eta \in W_{\mathcal{B}}\} = \emptyset$.) Let $z = Df_x v$. We represent z as a sequence $(z_i)_{i=1}^{\infty}$ where $z_i = \langle z, \boldsymbol{e}_i \rangle$. Let $\ell \in \mathbb{N}$ be such that $z_\ell \neq 0$. For $(\gamma_m) \in [-1, 1]^{\mathbb{N}}$, we have

$$\begin{aligned} D\left(\psi + \beta_{\mathcal{B}} \sum_{m=1}^{\infty} m^{-1} \gamma_m \boldsymbol{e}_m^*\right)_{f(x)} z &= D\psi_{f(x)} z + \sum_{m \neq \ell} m^{-1} \gamma_m \langle \boldsymbol{e}_m, z \rangle + \ell^{-1} \gamma_\ell \langle \boldsymbol{e}_\ell, z \rangle \\ &= D\psi_{f(x)} z + \sum_{m \neq \ell} m^{-1} \gamma_m \langle \boldsymbol{e}_m, z \rangle + \ell^{-1} \gamma_\ell z_\ell. \end{aligned} \tag{2}$$

Consequently, if we fix γ_m for all $m \neq \ell$, then the right side of (2) is equal to 0 for at most one value of γ_ℓ. The Fubini/Tonelli theorem therefore implies that $\mu(\{\eta \in Q : \psi + \eta \in W_{\mathcal{B}}\}) = 0$. We conclude that μ is transverse to $W_{\mathcal{B}}$. ∎

Lemma 3.7. *Let H be a separable real Hilbert space and let $f : H \to H$ be a C^1 map. Suppose $A \subset H$ is such that $f(A) = A$ and suppose $x \in \mathrm{Per}_1(f|A)$. Let $N \in \mathbb{N}$. If*

(Op1) *the operator $Df_x|T_xA$ is not a scalar multiple of the identity;*
(Op2) *the real point spectrum σ_p of $(Df_x|T_xA)^*$ is countable;*

then the set Z_x of observables $\varphi \in C^1(H, \mathbb{R}^N)$ satisfying

(Ker1) $\ker(D\varphi_x) \cap T_xA \neq \{0\}$*;*
(Ker2) $Df_x\big(\ker(D\varphi_x) \cap T_xA\big) \subset \ker(D\varphi_x)$*;*

is a shy subset of $C^1(H, \mathbb{R}^N)$.

Proof of Lemma 3.7. Assume H is infinite-dimensional. It suffices to consider the case $N = 1$. If $\dim(T_xA) = 1$, then (Ker1) is satisfied by only a shy subset of $C^1(H, \mathbb{R})$. Condition (Ker1) is always satisfied if $\dim(T_xA) > 1$; in this case we show that (Ker2) is a shy condition.

Let $L = Df_x|T_xA$ and assume that L is not a scalar multiple of the identity. Suppose that $0 \neq w^* \in (T_xA)^*$ satisfies $L(\ker(w^*)) \subset \ker(w^*)$. For all $v \in \ker(w^*)$, we have $\langle w, v \rangle = 0$ and $\langle w, Lv \rangle = \langle L^* w, v \rangle = 0$. The vector w is therefore an eigenvector of L^*.

We show that Z_x is shy by using Lebesgue measure on a 1-dimensional subspace of $C^1(H,\mathbb{R})$. For $\gamma \in \sigma_p$, let E_γ be the eigenspace associated with γ. Since L is not a scalar multiple of the identity, neither is L^*. Let

$$y \in T_xA \setminus \bigcup_{\gamma\in\sigma_p} E_\gamma.$$

We view $y^* \in (T_xA)^*$ as an element of H^* by composing y^* with the orthogonal projection π from H onto T_xA: define $y^*(v) = \langle y, \pi(v)\rangle$ for all $v \in H$. Let Y be the 1-dimensional subspace of $C^1(H,\mathbb{R})$ spanned by y^*. Let $\varphi \in C^1(H,\mathbb{R})$. We claim that

$$\lambda_Y(\{c \in \mathbb{R} : \varphi + cy^* \in Z_x\}) = 0. \tag{3}$$

Let $\gamma \in \sigma_p$. Suppose that $c_1, c_2 \in \mathbb{R}$ are such that $\varphi + c_1y^* \in Z_x$ and $\varphi + c_2y^* \in Z_x$. Suppose further that the vectors in T_xA associated with $D\varphi_x \circ \pi + c_1y^*$ and $D\varphi_x \circ \pi + c_2y^*$ via the Riesz representation theorem are both elements of E_γ. This implies that $(c_1 - c_2)y \in E_\gamma$. Since $y \in T_xA \setminus E_\gamma$, we conclude that $c_1 = c_2$. The set σ_p is countable and therefore $\{c \in \mathbb{R} : \varphi + cy^* \in Z_x\}$ is countable. This establishes (3).

■

The following proposition provides a preliminary answer to (Q1).

Proposition 3.8. *Let H be a separable real Hilbert space and let $f : H \to H$ be a C^1 map. Suppose that $A \subset H$ is a compact set such that $f(A) = A$. Assume*

(H1) $\mathrm{Per}_1(f|A) \cup \mathrm{Per}_2(f|A)$ *is countable;*

(H2) $f|A$ *is injective;*

(H3) Df_x *is injective on T_xA for every $x \in A \setminus \mathrm{Per}_1(f|A)$;*

(H4) *for every $x \in \mathrm{Per}_1(f|A)$, the operator $Df_x|T_xA$ is not a scalar multiple of the identity;*

(H5) *for every $x \in \mathrm{Per}_1(f|A)$, the real point spectrum of $(Df_x|T_xA)^*$ is countable.*

Let $N \in \mathbb{N}$. For prevalent $\varphi \in C^1(H,\mathbb{R}^N)$, if f induces a quasi-differentiable map f_* on $\varphi(A)$, then φ embeds A into $\mathbb{R}^N$ in the sense of Definition 3.4.

Proof of Proposition 3.8. Let $N \in \mathbb{N}$. Applying Proposition 3.5 of [15], there exists a prevalent set $\Gamma_1 \subset C^1(H,\mathbb{R}^N)$ such that for $\varphi \in \Gamma_1$, if f induces a map f_* on $\varphi(A)$ satisfying $f_* \circ \varphi = \varphi \circ f$ on A, then φ maps A homeomorphically onto its image.

Let $\{B_i : i \in \mathbb{N}\}$ be a collection of open balls in H that forms a basis for the topology on H. Define the following sets:

$$\Gamma_2 = \bigcap_{i=1}^{\infty} C^1(H,\mathbb{R}^N) \setminus W_{B_i}, \qquad \Gamma_3 = \bigcap_{x\in\mathrm{Per}_1(f|A)} C^1(H,\mathbb{R}^N) \setminus Z_x.$$

The set Γ_2 is prevalent by Lemma 3.6 and property (LP4). The set Γ_3 is prevalent by Lemma 3.7 and (LP4). Property (LP4) applies here because (H1) gives that $\mathrm{Per}_1(f|A)$ is countable.

Let $\Gamma = \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$. For $\varphi \in \Gamma$, if f induces a quasi-differentiable map f_* on $\varphi(A)$ satisfying $f_* \circ \varphi = \varphi \circ f$ on A, then φ embeds A into $\mathbb{R}^N$. ■

We obtain an improved version of Proposition 3.8 by transferring (H1)–(H3) onto the induced dynamics.

Theorem 3.9. *Let H be a separable real Hilbert space and let $f : H \to H$ be a C^1 map. Suppose that $A \subset H$ is a compact set such that $f(A) = A$. Assume (H4) and (H5). Let $N \in \mathbb{N}$. For prevalent $\varphi \in C^1(H,\mathbb{R}^N)$, if f induces a quasi-differentiable map f_* on $\varphi(A)$ satisfying*

(H1)* $\mathrm{Per}_1(f_*) \cup \mathrm{Per}_2(f_*)$ *is countable;*

(H2)* f_* *is injective;*

(H3)* $(Df_*)_z$ *is injective on $T_z\varphi(A)$ for every $z \in \varphi(A) \setminus \mathrm{Per}_1(f_*)$;*

then φ embeds A into $\mathbb{R}^N$ in the sense of Definition 3.4.

Proof of Theorem 3.9. If (H1)–(H3) hold, then Theorem 3.9 follows from Proposition 3.8. If (H1) does not hold, then $\mathrm{Per}_1(f|A) \cup \mathrm{Per}_2(f|A)$ is uncountable. For prevalent $\varphi \in C^1(H,\mathbb{R}^N)$, $\varphi(\mathrm{Per}_1(f|A) \cup \mathrm{Per}_2(f|A))$ is uncountable (see Proposition 2.6 of [15]); for any such φ, f cannot induce a map on $\varphi(A)$ satisfying (H1)*. If (H2) (respectively (H3)) fails to hold, then a quasi-differentiable induced map satisfying (H2)* (respectively (H3)*) cannot exist for prevalent $\varphi \in C^1(H,\mathbb{R}^N)$. ■

3.2.3. ***Embedding theorems: strong case.*** We now formulate conditions under which (Q1) has an affirmative answer in the sense of Definition 3.5. The key idea here is to place a mild hypothesis on the tangent dimension of the image $\varphi(A)$.

Theorem 3.10. *Let H be a separable real Hilbert space and let $f : H \to H$ be a C^1 map. Suppose that $A \subset H$ is a compact set such that $f(A) = A$. Assume (H5). Let $N \in \mathbb{N}$. For prevalent $\varphi \in C^1(H, \mathbb{R}^N)$, if*

$$\text{(DimT)} \qquad \dim_{\mathfrak{T}}(\varphi(A)) < N,$$

and if f induces a quasi-differentiable map f_* on $\varphi(A)$ satisfying (H1)–(H3)* as well as*

(H4)* *for every $z \in \mathrm{Per}_1(f_*)$, the operator $Df_*|T_z\varphi(A)$ is not a scalar multiple of the identity,*

then φ strongly embeds A into $\mathbb{R}^N$.

Proof of Theorem 3.10. First assume that for every $q \in A$ and for every pair of sequences $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ in A with $x_i \to q$, $y_i \to q$, and $x_i \neq y_i$ for all i, the sequence of normalized differences $((y_i - x_i)/\|y_i - x_i\|)$ in the unit sphere S of H has a converging subsequence. Under this assumption, if $\varphi \in C^1(H, \mathbb{R}^N)$ is an embedding of A, then φ is a strong embedding of A.

If (H4) holds as well, then the proof of Theorem 3.9 works for Theorem 3.10 as well. If (H4) does not hold, there exists $p \in \mathrm{Per}_1(f|A)$ such that $Df_p|T_pA$ is a scalar multiple of the identity. We consider two cases. First, if $\dim(T_pA) \geqslant N$, then for prevalent $\varphi \in C^1(H, \mathbb{R}^N)$, we have that $\dim(T_{\varphi(p)}\varphi(A)) = N$ and $D\varphi_p$ maps T_pA surjectively onto $T_{\varphi(p)}\varphi(A)$. Any such φ cannot induce a quasi-differentiable map on $\varphi(A)$ satisfying (H4)*. Second, if $\dim(T_pA) < N$, then for prevalent $\varphi \in C^1(H, \mathbb{R}^N)$, $D\varphi_p$ maps T_pA injectively (and therefore bijectively by our sequential precompactness assumption) onto $T_{\varphi(p)}\varphi(A)$. If any such φ induces a quasi-differentiable map f_* on $\varphi(A)$, we would have $(Df_*)_{\varphi(p)} = (D\varphi \circ Df \circ D\varphi^{-1})_{\varphi(p)}$ on $T_{\varphi(p)}\varphi(A)$. This precludes the possibility of (H4)*.

For the second part of the proof, assume that there exist $\hat{q} \in A$ and sequences $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ in A such that $x_i \to \hat{q}$, $y_i \to \hat{q}$, $x_i \neq y_i$ for all i, and the sequence of normalized differences $(v_i = (y_i - x_i)/\|y_i - x_i\|)$ has no converging subsequences. This implies that

$$\rho := \lim_{M \to \infty} \inf_{\substack{i,j \geqslant M \\ i \neq j}} \angle(v_i, v_j) > 0.$$

By passing to a subsequence, we may assume that $\angle(v_i, v_{i'}) \geqslant \rho/2$ for all $i \neq i'$.

We use the sequence (v_i) to construct a probe. Let $V = \overline{\mathrm{span}\{v_i : i \in \mathbb{N}\}}$ and let $\pi_V : H \to V$ denote orthogonal projection onto V. Let $L_0 : V \to V$ be a bounded linear map such that $\langle L_0 v_i, L_0 v_{i'} \rangle = 0$ for all $i \neq i'$. Define $L = L_0 \circ \pi_V$. Let $\{\boldsymbol{e}_n : 1 \leqslant n \leqslant N\}$ be an orthonormal basis for $\mathbb{R}^N$. Define the bounded linear map $\psi : H \to \mathbb{R}^N$ by

$$\psi = \sum_{n=1}^{N} \left(\sum_{m=0}^{\infty} (L(y_{mN+n} - x_{mN+n}))^* \circ L \right) \boldsymbol{e}_n, \tag{4}$$

where we may assume (by passing to a subsequence if necessary) that $\|L(y_i - x_i)\|$ decreases monotonically to zero as $i \to \infty$ and that this happens quickly enough to guarantee that the sums in (4) converge.

Let $\varphi \in C^1(H, \mathbb{R}^N)$. We claim that the set

$$Z_\varphi = \left\{ c \in \mathbb{R} : \dim(T_{(\varphi + c\psi)(\hat{q})}(\varphi + c\psi)(A)) < N \right\}$$

is countable. To see this, let $c_0 \in \mathbb{R}$ be such that there exist N distinct vectors $w_1, \ldots, w_N$ in the direction set $\Delta_{(\varphi + c_0\psi)(\hat{q})}(\varphi + c_0\psi)(A) \subset \mathbb{S}^{N-1}$ and N strictly increasing sequences $(m_j^{(n)})_{j=1}^{\infty}$ in $\mathbb{Z}^+$ satisfying

$$\lim_{j \to \infty} \frac{(\varphi + c_0\psi)(y_{m_j^{(n)}N+n}) - (\varphi + c_0\psi)(x_{m_j^{(n)}N+n})}{\left\| (\varphi + c_0\psi)(y_{m_j^{(n)}N+n}) - (\varphi + c_0\psi)(x_{m_j^{(n)}N+n}) \right\|} = w_n$$

for every $1 \leqslant n \leqslant N$. Note that any sufficiently large c_0 will have this property. Using c_0 as a starting point, define maps $s \mapsto w_n(s)$ by

$$w_n(s) = \lim_{j \to \infty} \frac{(\varphi + s\psi)(y_{m_j^{(n)}N+n}) - (\varphi + s\psi)(x_{m_j^{(n)}N+n})}{\left\| (\varphi + s\psi)(y_{m_j^{(n)}N+n}) - (\varphi + s\psi)(x_{m_j^{(n)}N+n}) \right\|}.$$

Each map $s \mapsto w_n(s)$ is defined on all of $\mathbb{R}$ except for perhaps one exceptional value of s.

Let $1 \leqslant n_1 < n_2 \leqslant N$. Our choice of ψ implies that $s \mapsto \angle(\boldsymbol{e}_{n_1}, w_{n_1}(s))$ is decreasing (and strictly decreasing on the preimage of $(0,\pi)$), while $s \mapsto \angle(\boldsymbol{e}_{n_1}, w_{n_2}(s))$ is increasing. Similarly, $s \mapsto \angle(\boldsymbol{e}_{n_2}, w_{n_2}(s))$ is decreasing (and strictly decreasing on the preimage of $(0,\pi)$), while $s \mapsto \angle(\boldsymbol{e}_{n_2}, w_{n_1}(s))$ is increasing. It follows that $w_{n_1}(s) = w_{n_2}(s)$ for at most one value of s. The vectors $w_1(s), \ldots, w_N(s)$ are therefore all distinct except for at most $N(N-1)/2$ values of s. We have shown that Z_φ is finite.

The set $\{\varphi \in C^1(H, \mathbb{R}^N) : \dim_{\mathfrak{T}}(\varphi(A)) = N\}$ is prevalent. Every map in this set fails to satisfy (DimT). ■

3.2.4. ***Implications for Lyapunov exponents***. We answer the question that motivates this work - Question 1.1 - using Theorem 3.10.

Corollary 3.11. *Let H be a separable real Hilbert space and let $f : H \to H$ be a C^1 map. Suppose that $A \subset H$ is a compact set such that $f(A) = A$. Assume (H5). Let $N \in \mathbb{N}$. For prevalent $\varphi \in C^1(H, \mathbb{R}^N)$, if $\dim_{\mathfrak{T}}(\varphi(A)) < N$ and if f induces a quasi-differentiable map f_* on $\varphi(A)$ satisfying (H1)*–(H4)*, then Lyapunov exponents of f_* correspond to Lyapunov exponents of f as follows. If $z \in \varphi(A)$ and $\omega(z,w)$ is a Lyapunov exponent of f_* with $w \in T_z\varphi(A)$, then $\omega(\varphi^{-1}z, (D\varphi^{-1})_z w)$ is a Lyapunov exponent of $f|A$ and $\omega(\varphi^{-1}z, (D\varphi^{-1})_z w) = \omega(z,w)$.*

Remark 3.12. A Lyapunov exponent $\omega(z,w)$ of f_* with $w \in T_z\mathbb{R}^N \setminus T_z\varphi(A)$ may be spurious - it may be an artifact of φ that does not correspond to a Lyapunov exponent of f.

4. Discussion

Invariant sets associated with evolution PDEs often live in finite-dimensional submanifolds of the ambient function space, such as inertial manifolds or center manifolds. For example, the genuinely nonuniformly hyperbolic attracting sets produced when certain parabolic PDEs are forced periodically live in two-dimensional center manifolds [16]. It is interesting to consider if observational data can be used to determine whether or not a given invariant set of interest is contained in a finite-dimensional submanifold of the ambient Hilbert space. More precisely:

Definition 4.1. Let H be a real Hilbert space. A subset $E \subset H$ is said to be ***locally embeddable*** if for every $x \in E$, there exists a neighborhood U of x in H and a finite-dimensional C^1 submanifold M of H (without boundary) such that $U \cap E \subset M$. If a finite-dimensional C^1 submanifold M contains every element of E that lies within some neighborhood of x and if the dimension of M is minimal with respect to this property, then we call M a ***local enveloping manifold*** for E at x (see [20, Section 3] for more about local enveloping manifolds when $H = \mathbb{R}^D$).

Question 4.2. Let H be a real Hilbert space. Let $f : H \to H$ be a C^1 map and suppose that $A \subset H$ is a compact set satisfying $f(A) = A$. Let $N \in \mathbb{N}$. For prevalent $\varphi \in C^1(H, \mathbb{R}^N)$, if f induces a quasi-differentiable map f_* on $\varphi(A)$, does it follow that A is locally embeddable?

This question may well have an affirmative answer given the nature of existing theorems on the regularity of embeddings of subsets of infinite-dimensional spaces into Euclidean spaces. Theorem 1.3, for example, guarantees only Hölder continuity of φ^{-1} on $\varphi(A)$, and therefore guarantees only Hölder continuity for an induced map $f_* = \varphi \circ f \circ \varphi^{-1}$ induced by a C^1 map $f : H \to H$.

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