ON A THEOREM OF SHCHEPIN AND REPOVŠ CONCERNING THE SMOOTHNESS OF COMPACTA

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Abstract. We show that a theorem of Shchepin and Repovš concerning the smoothness of compacta follows from the theory of semicontinuous relations.

1. Introduction

Compact subsets of Euclidean space possess a surprising amount of smoothness. Let $A \subset \mathbb{R}^n$ be compact. Using the notion of generalized tangent space, Shchepin and Repovš [7] prove that A contains a residual subset of 'smooth' points. In this note, we show that the result of Shchepin and Repovš follows directly from the theory of semicontinuous relations.

The generalized tangent space illuminates the relationship between compacta and smooth submanifolds. Consider the following two problems.

(1) **The Extension Problem.** Let $A \subset \mathbb{R}^n$ be compact and fix $x \in A$. What is the smallest integer d for which there exists a neighborhood N of x and a C^1 submanifold $M \subset \mathbb{R}^n$ of dimension d such that $M \supset A \cap N$? This local question suggests a global version. Is it possible to embed A into a C^1 submanifold of 'minimal' dimension?

(2) **The Characterization Problem.** Among the compact subsets of $\mathbb{R}^n$, characterize those that have the structure of a smooth submanifold.

We discuss solutions to these problems in the final section. See [5] for applications to dynamical systems and embedding theory.

2. Tangent Regularity

Let S denote the unit sphere in the Euclidean space $\mathbb{R}^n$. For a closed $A \subset \mathbb{R}^n$, we call $s \in S$ a *tangent direction for A at $x \in A$* when there exist sequences $\{y_i\}$ and $\{x_i\}$ in A converging to x, with $y_i \neq x_i$ for all i, such that

$$s = \lim_{i \to \infty} \frac{y_i - x_i}{\|y_i - x_i\|}.$$

If the sequences can be chosen with $x_i = x$ for all i, then s is called a *proper tangent direction for A at x*. We denote by $D_x A$ the set of tangent directions for A at x and by $d_x A$ the subset of proper tangent directions. The *generalized tangent space* $T_x A$ is the smallest linear space containing $D_x A$.

Date: Summer 2004.
2000 *Mathematics Subject Classification.* Primary: 58A05; Secondary: 58A40, 54H99, 57R99.
Key words and phrases. Generalized Tangent Space, Semicontinuous Relation.

We say that $x \in A$ is *tangent regular* if $d_x A = D_x A$. If $d_x A$ is a proper subset of $D_x A$, then we say that x is *singular*. Shchepin and Repovš [7] prove the surprising result that most points of A are tangent regular.

Theorem 2.1. *For every closed subset A of $\mathbb{R}^n$, the set of tangent regular points is a residual subset of A. That is, it contains a G_δ subset which is dense in A.*

Notice that by intersecting A with large closed balls, we may reduce to the case when A is compact. We therefore assume that A is compact from this point forward.

There is a general theory of semicontinuous relations which has proved quite useful in topological dynamics, see e.g. Takens [8] and Akin, Hurley and Kennedy [2]. In this section we will show that Theorem 2.1 follows directly from this general theory. We review the theory following the notation of Akin [1].

For compact metric spaces X and Y, a *relation* $R : X \to Y$ is just a subset of the product $X \times Y$. For $x \in X$, we write $R(x)$ for $\{y : (x, y) \in R\}$. If $A \subset X$, then

$$R(A) = \bigcup_{x \in A} R(x).$$

Given a relation $R : X \to Y$, the *inverse relation* $R^{-1} : Y \to X$ is the set

$$\{(y, x) : (x, y) \in R\}.$$

Thus, for $B \subset Y$ we have $R^{-1}(B) = \{x : R(x) \cap B \neq \emptyset\}$.

A relation $R : X \to Y$ is called a *closed relation* when it is a closed subset of $X \times Y$. It is called a *pointwise closed relation* when $R(x)$ is a closed subset of Y for every $x \in X$. Clearly, a closed relation is pointwise closed.

A pointwise closed relation $R : X \to Y$ can be regarded as a function from X to the space $C(Y)$ of closed subsets of Y. That is, we identify R with the map given by $x \mapsto R(x)$. Equipped with the Hausdorff metric, $C(Y)$ is itself a compact metric space (see e.g. Akin [1, Chapter 7]).

Definition 2.2. Let $R : X \to Y$ be a pointwise closed relation with X and Y compact metric spaces. The relation R is called *upper semicontinuous* at $x \in X$ if for every open subset $O \subset Y$, $R(x) \subset O$ implies that $\{x_1 \in X : R(x_1) \subset O\}$ is a neighborhood of x. We say that R is an *upper semicontinuous relation* when it is upper semicontinuous at every point of X.

Definition 2.3. The relation R is called *lower semicontinuous* at $x \in X$ if for every open subset $O \subset Y$, $R(x) \cap O \neq \emptyset$ implies that $\{x_1 \in X : R(x_1) \cap O \neq \emptyset\}$ is a neighborhood of x. We say that R is a *lower semicontinuous relation* when it is lower semicontinuous at every point of x.

Definition 2.4. The relation R is called *continuous* at $x \in X$ if R is both upper and lower semicontinuous at x. We say that R is a *continuous relation* when it is continuous at every point of x or, equivalently, when it is both an upper and a lower semicontinuous relation.

We will use the abbreviations *usc* and *lsc* for 'upper semicontinuous' and 'lower semicontinuous.' With these definitions in place we collect the results we will need.

Theorem 2.5. *Let $R : X \to Y$ be a pointwise closed relation with X and Y compact metric spaces.*

(1) *The relation R is continuous at $x \in X$ if and only if the associated mapping from X to $C(Y)$ is continuous at x.*

(2) *The relation R is an usc relation if and only if it is a closed relation, i.e. a closed subset of $X \times Y$.*
(3) *Let $\overline{R}$ denote the closure of $R \subset X \times Y$ so that $\overline{R} : X \to Y$ is an usc relation. Assume that R is a lsc relation. The relation R is continuous at $x \in X$ if and only if $R(x) = \overline{R}(x)$.*
(4) *If R is either a lsc or an usc relation, then the set of continuity points of R is a dense G_δ subset of X.*

Parts (1), (2), and (3) are elementary consequences of the definitions of semicontinuous relations. In the exposition in Chapter 7 of Akin [1], parts (1) and (2) occur in Proposition 7.11 and part (3) is Corollary 7.13. Part (4), due to Kuratowski [4], is the principal tool in this subject. It is proved in Takens [8] where its usefulness for dynamical systems is demonstrated. Part (4) is also proved as Theorem 7.19 in Akin [1].

Proof of Theorem 2.1. For the compact set $A \subset \mathbb{R}^n$, let

$$1_A = \{(x, x) : x \in A\}$$

denote the diagonal in $A \times A$. We define the *direction maps* $\sigma : A \times A \setminus 1_A \to S$ and $\Sigma : A \times A \setminus 1_A \to A \times S$ by

$$\sigma(x, y) = \frac{y - x}{\|y - x\|},$$
$$\Sigma(x, y) = (x, \sigma(x, y)).$$

For $\epsilon > 0$, let V_ϵ denote the ϵ-neighborhood of the diagonal. That is,

$$V_\epsilon = \{(x, y) \in A \times A : \|y - x\| < \epsilon\}.$$

Define the relation $Q^\epsilon : A \to S$ by

$$Q^\epsilon = \Sigma(V_\epsilon \setminus 1_A).$$

Now let $m^\epsilon A$ be the smallest pointwise closed relation which contains Q^ϵ. For each $x \in A$,

$$m^\epsilon A(x) = \overline{Q^\epsilon(x)} = \overline{\sigma(V_\epsilon(x) \setminus x)}.$$

Let $M^\epsilon A$ be the smallest closed relation which contains Q^ϵ. That is,

$$M^\epsilon A = \overline{Q^\epsilon} = \overline{\Sigma(V_\epsilon \setminus 1_A)}.$$

Since $m^\epsilon A$ lies between Q^ϵ and its closure $M^\epsilon A$, we have

$$M^\epsilon A = \overline{m^\epsilon A}. \tag{2.1}$$

Finally, let

$$mA = \bigcap_{\epsilon > 0} m^\epsilon A,$$
$$MA = \bigcap_{\epsilon > 0} M^\epsilon A.$$

These are both pointwise closed relations and MA is closed. Because the dependence on ϵ is monotone, it suffices to restrict the intersection to rational ϵ. For each $x \in A$, $MA(x)$ is the set $D_x A$ of tangent directions for A at x and $mA(x)$ is the set $d_x A$ of proper tangent directions for A at x as defined above. We leave the simple verification as an exercise.

Now let $O \subset S$ be open and let ϵ be a fixed positive number. If $\sigma(x,y) \in O$ with $0 < d(x,y) < \epsilon$, then for all $x_1 \in A$ sufficiently close to x, $0 < d(x_1,y) < \epsilon$ and $\sigma(x_1,y) \in O$. This implies that the relation $m^\epsilon A$ is lsc for every positive ϵ.

Define

$$A_0 = \{x \in A : m^\epsilon A \text{ is continuous at } x \text{ for every positive rational } \epsilon\}.$$

By the Kuratowski Theorem (2.5.4) and the Baire Category Theorem, A_0 is a dense G_δ subset of A. By (2.1) and Theorem 2.5.3, we have $m^\epsilon A(x) = M^\epsilon A(x)$ for every $x \in A_0$. Intersecting over the positive rationals, we have

$$d_x A = mA(x) = MA(x) = D_x A$$

for all $x \in A_0$, as required. □

3. Compacta and Submanifolds

We now discuss the relationship between compacta and smooth submanifolds. The local extension problem admits a solution in terms of the generalized tangent space.

Theorem 3.1 (Manifold Extension Theorem [5, 3])**.** *Let $A \subset \mathbb{R}^n$ be compact and let $x \in A$. There exists a neighborhood N of x and a C^1 submanifold M such that $M \supset A \cap N$ and $T_x A = T_x M$. In particular,* $\dim(M) = \dim(T_x A)$.

This local result suggests a natural global question.

Definition 3.2. The *tangent dimension* of $A \subset \mathbb{R}^n$, denoted $\dim_T A$, is given by

$$\dim_T A = \max_{x \in A} \dim(T_x A).$$

Question 3.3. Let $A \subset \mathbb{R}^n$ be compact. Does there exist a C^1 submanifold M of dimension $\dim_T A$ such that $M \supset A$?

If $\dim T_x A$ is the same for all $x \in A$, then such a manifold exists [3]. However, topological obstructions may exist in the heterogeneous case. See [3] for an example.

We present two solutions to the characterization problem. The first result characterizes C^1 submanifolds in terms of tangent regularity.

Theorem 3.4 ([7])**.** *Let $A \subset \mathbb{R}^n$ be compact. For every $d \in \{0, 1, \dots, n\}$, the following statements are equivalent.*

(1) *The set A is a smooth submanifold of $\mathbb{R}^n$ of class C^1 and dimension d.*
(2) *Every point in A is tangent regular and* $\dim(T_x A) = d$ *for all $x \in A$.*

This characterization result and Theorem 2.1 imply a second characterization in terms of ambient homogeneity.

Definition 3.5. The set $A \subset \mathbb{R}^n$ is said to be *ambiently C^1-homogeneous* if for every pair of points x, $y \in A$, there exist neighborhoods O_x and O_y in $\mathbb{R}^n$ and a C^1 diffeomorphism

$$h : (O_x, O_x \cap A, x) \to (O_y, O_y \cap A, y).$$

Theorem 3.6 ([6])**.** *Let $A \subset \mathbb{R}^n$ be compact. Then A is ambiently C^1-homogeneous if and only if A is a C^1 submanifold of $\mathbb{R}^n$.*

The original proof of this result [6] requires the Rademacher theorem concerning the differentiability of Lipschitz functions. Shchepin and Repovš [7] simplify the proof by eliminating the need to invoke Rademacher.

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