ENVELOPING MANIFOLDS

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Abstract. We study the problem of embedding compact subsets of $\mathbb{R}^n$ into C^1 submanifolds of minimal dimension. In [4], we define a generalized tangent space T_xA suitable for a general compact subset A of $\mathbb{R}^n$ and we prove that A may be locally embedded into a C^1 manifold of dimension $\dim(T_xA)$. This result leads naturally to the global conjecture that for a compact subset A of $\mathbb{R}^n$, there exists a C^1 manifold M such that $M \supset A$ and $\dim M = \max_{x\in A} \dim(T_xA)$. We prove that this conjecture is false in general, but true if $\dim(T_xA)$ is constant on A. Applications of these ideas to dimension theory, embedding theory, and dynamical systems are discussed.

1. Introduction

We study the problem of embedding compact subsets of Euclidean space into C^1 submanifolds of the smallest possible dimension. Shchepin and Repovš [6] address the complementary problem of characterizing smooth submanifolds within the class of compact sets. Let $A \subset \mathbb{R}^n$ and fix $x \in A$. What is the smallest integer d for which there exists a C^1 submanifold M such that $M \supset N(x) \cap A$ for some neighborhood $N(x)$ of x? In [4], we define a generalized tangent space T_xA and we prove that the answer to this question is $d = \dim(T_xA)$. A manifold M of dimension $\dim(T_xA)$ such that $M \supset N(x) \cap A$ for some neighborhood $N(x)$ of x is said to be an **enveloping manifold** for A at x. Given the local result, it is natural to conjecture that there exists a C^1 manifold M such that $M \supset A$ and $\dim(M) = \dim_T(A)$, where $\dim_T(A) := \max_{x\in A} \dim(T_xA)$ is the tangent dimension of A. This shall henceforth be referred to as the global enveloping manifold conjecture. We prove that this conjecture holds if $\dim(T_xA)$ is constant on A. However, topological obstructions exist in the heterogeneous case. We describe a low-dimensional counterexample to the global conjecture.

Enveloping manifolds may be profitably used to solve problems in dimension theory, embedding theory, and dynamical systems. We describe two such applications. The Eckmann-Ruelle algorithm (ERA) is used by experimentalists when computing the Lyapunov exponents associated with the invariant measure of a dynamical system. This algorithm produces spurious exponents as well as the correct exponents. One needs an efficient, rigorous method to identify the spurious data. In [4], we use the generalized tangent space and enveloping manifolds to develop such a method.

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Let $f : X \to X$ be a dynamical system and let $\phi_i : X \to \mathbb{R}$, $1 \leqslant i \leqslant m$, be observables. One is interested in the relationship between X and its image under the map $\Phi = (\phi_1, \dots, \phi_m) : X \to \mathbb{R}^m$. In particular, the structure of the image, its dimension, and the possibility of embedding the image into a submanifold of relatively small dimension are of interest.

The Whitney embedding theorem is one of the celebrated results in the theory of singularities. Sauer, Yorke, and Casdagli [5] prove the following powerful generalization.

Theorem 1.1 (Prevalence Whitney Embedding Theorem [5])**.** *Let A be a compact subset of $\mathbb{R}^n$ of box dimension d and let m be an integer greater than $2d$. For almost every smooth map $\phi : \mathbb{R}^n \to \mathbb{R}^m$,*

(1) *ϕ is one to one on A and*
(2) *ϕ is an immersion on each compact subset C of a smooth manifold contained in A.*

Here "almost-every" is interpreted in the sense of prevalence, a generalization of the translation-invariant notion of Lebesgue almost-every to infinite-dimensional spaces. See [1, 2, 3] for details. This theorem is not optimal because its application requires that one have *a priori* knowledge of the box dimension of A. Using enveloping manifolds, one may show that the tangent dimension $\dim_T(A)$ bounds the box dimension of A from above. More generally, the tangent dimension bounds from above any dimension characteristic $D(\cdot)$ with the following properties.

(1) If $A \subset B$, then $D(A) \leqslant D(B)$.
(2) If M is a C^1 submanifold of $\mathbb{R}^n$ of dimension k, then $D(M) = k$.
(3) For each set A and each cover $\{U_i : i = 1, \dots, N\}$ of A, one has $D(A) = \max_i D(A \cap U_i)$.

This crucial observation leads to a Platonic Whitney embedding theorem.

Theorem 1.2 (Platonic Whitney Embedding Theorem [4])**.** *Let $A \subset \mathbb{R}^n$ be compact. For almost every $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, if $\phi(A)$ satisfies $\dim_T \phi(A) < \frac{m}{2}$, then ϕ is a diffeomorphism on A.*

Notice that the dimension hypothesis is observable because it applies to the image of A.

The paper is organized as follows. Section 2 contains requisite material from [4] and an outline of the existence proof for enveloping manifolds. In Section 3, we prove the global enveloping manifold conjecture in the homogeneous case and we produce a counterexample to the general conjecture.

2. Local Enveloping Manifolds

Let A be a compact subset of $\mathbb{R}^n$ and fix $x \in A$.

Definition 2.1. We say that a C^1 submanifold M is an **enveloping manifold** for A at x if there exists a neighborhood $N(x)$ of x such that $M \supset N(x) \cap A$ and if M' is another C^1 submanifold such that $M' \supset N'(x) \cap A$ for some neighborhood $N'(x)$, then $\dim(M') \geqslant \dim(M)$.

The existence of a C^1 enveloping manifold M for $x \in A$ follows trivially from the definition because $\mathbb{R}^n \supset A$, but the determination of the dimension of M is a subtle problem. As we explain below, the dimension of this manifold is characterized by the generalized tangent space T_xA, a notion introduced in [4]. One has $\dim(M) = \dim(T_xA)$.

Definition 2.2. Let D_xA be the set of all unit vectors v for which there exist sequences (y_i) and (z_i) in A such that $y_i \to x$, $z_i \to x$, and $(z_i - y_i)/\|z_i - y_i\| \to v$. The **tangent space** at x relative to A, denoted T_xA, is the smallest linear space containing D_xA. The **tangent bundle** TA is the set $\{(x, v) : x \in A, v \in T_xA\}$.

We note that this is one of the two obvious ways to define the tangent space at a point in an arbitrary compact subset of $\mathbb{R}^n$. The other would be to fix $y_i = x$ in the above definition, but the resulting tangent space would be too small for the purpose at hand. There may exist a unit tangent vector $v \in T_xA$ for which there do not exist sequences (y_i) and (z_i) in A such that $y_i \to x$, $z_i \to x$, and $(z_i - y_i)/\|z_i - y_i\| \to v$. Such tangent vectors are not realizable by normalized displacements. In general, neither the tangent space itself nor its dimension varies continuously with $x \in A$. Nevertheless, the tangent space varies upper semicontinuously with $x \in A$.

Lemma 2.3 ([4])**.** *The function* $x \mapsto \dim(T_xA)$ *is upper semicontinuous on* A*. The tangent bundle* TA *is a closed subset of* $\mathbb{R}^n \times \mathbb{R}^n$*. If* T_xA *has constant dimension on a set* $A_0 \subset A$*, then* T_xA *varies continuously on* A_0*.*

Using the generalized tangent space, one may define a new dimension characteristic.

Definition 2.4. The **tangent dimension** of A, denoted $\dim_T(A)$, is given by

$$\dim_T(A) = \max_{x \in A} \dim(T_xA).$$

Example 2.5. In Figure 1 the tangent space T_pA is two-dimensional while T_xA is one-dimensional for all other points $x \in A$. Choosing $(y_i) \subset A$ and $(z_i) \subset A$ such that $y_i \to p$, $z_i \to p$, and y_i and z_i lie on a vertical line for each i, we obtain the tangent vector $v \in T_pA$. Thus $\dim_T(A) = 2$.

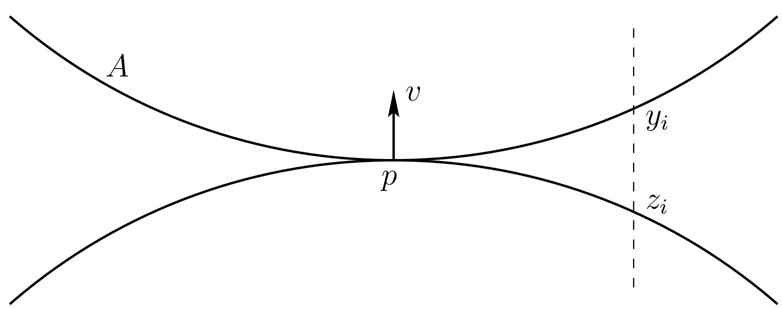

FIGURE 1. A Cusp

The following result characterizes the dimension of local enveloping manifolds.

Theorem 2.6 (Manifold Extension Theorem [4])**.** *For each* $x \in A$ *there exists an enveloping manifold* M *for* A *at* x *with* $T_xM = T_xA$*, and therefore* $\dim(M) = \dim(T_xA)$.

We outline the proof assuming that $\dim(T_xA) = d$ for all $x \in A$. See [4] for a proof in the general case. Fix $x \in A$ and $U \subset \mathbb{R}^n$. The ambient space $\mathbb{R}^n$ admits the orthogonal decomposition $\mathbb{R}^n = T_xA \oplus E_x$. Let $\pi_x : \mathbb{R}^n \to T_xA$ denote the orthogonal projection onto T_xA and let $\rho_x : \mathbb{R}^n \to E_x$ denote the orthogonal projection onto E_x. Let $U \subset \mathbb{R}^n$. The **tilt** $\tau(U, T_xA)$ of U with respect to T_xA is defined by

$$\tau(U, T_xA) = \sup_{\substack{z,w \in U \\ z \neq w}} \frac{|\rho_x(z) - \rho_x(w)|}{|\pi_x(z) - \pi_x(w)|}.$$

Similarly, we define the **tilt** $\theta(P, T_xA)$ of a subspace P relative to T_xA by

$$\theta(P, T_xA) = \max_{\substack{v \in P \\ |v| \neq 0}} \frac{|\rho_x v|}{|\pi_x v|}.$$

Fix $\eta > 0$ sufficiently small. We choose a ball $B(x)$ centered at x such that $\tau(B(x) \cap A, T_xA) \leqslant \eta$ and $\theta(T_yA, T_xA) \leqslant \eta$ for all $y \in B(x) \cap A$. The set $B(x) \cap A$ may be represented as the graph of a function $\psi : \pi_x(B(x) \cap A) \to E_x$. The map ψ is C^1 on $\pi_x(B(x) \cap A)$ in the sense of Whitney. Applying the Whitney extension theorem, we extend ψ to a C^1 map $\widetilde{\psi} : T_xA \to E_x$. The graph of $\widetilde{\psi}$ constitutes an enveloping manifold for A at x.

The following obvious lemma implies that the dimension of the tangent space is invariant under a diffeomorphism.

Lemma 2.7 (Dimension Invariance)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map and let $A \subset \mathbb{R}^n$ be compact. Fix $x \in A$. If the restriction of $Df(x)$ to T_xA is invertible, then* $\dim(T_{f(x)}f(A)) = \dim(T_xA)$.

If the restriction of $Df(x)$ to T_xA has a nontrivial kernel, then $\dim(T_{f(x)}f(A))$ may be smaller or larger than $\dim(T_xA)$, even if f is one-to-one on A. Therefore, the tangent dimension may increase under a smooth mapping.

Definition 2.8. A **global enveloping manifold** for A is a C^1 submanifold M of dimension $\dim_T A$ which contains A.

Note that M need not be compact or orientable.

3. Global Enveloping Manifolds

We establish the veracity of the conjecture in the homogeneous case.

Theorem 3.1. *Let $A \subset \mathbb{R}^n$ be a compact set such that* $\dim(T_xA) = d$ *for all $x \in A$. Then there exists a global enveloping manifold M for A. In particular,* $\dim(M) = d$.

Proof. For $x \in \mathbb{R}^n$, denote by $B(x, r)$ the open ball of radius r centered at x. Fix $\gamma > 0$ sufficiently small. By Lemma 2.3, there exists $r_0 > 0$ such that $\theta(T_yA, T_xA) < \gamma/2$ if x, $y \in A$ and $|x - y| \leqslant 3r_0$. Using the compactness of A, fix $r < r_0$ and a finite collection $\{\overline{B}(x_i, r) : x_i \in A, i = 1, \ldots, N\}$ such that $\bigcup_{i=1}^N B(x_i, r/2) \supset A$, and

$$\tau(\overline{B}(x_j, r) \cap A, T_{x_i}A) < \frac{\gamma}{2}$$

whenever $|x_j - x_i| \leqslant 2r$. We construct the global enveloping manifold M via an inductive procedure. Arguing as in the proof of the local manifold extension theorem, there exists a C^1 manifold M_1' such that $M_1' \supset \overline{B}(x_1, r) \cap A$. For $\epsilon > 0$, let $F_1(\epsilon)$ be the closure of the ϵ-neighborhood of $A \cap \overline{B}(x_1, r/2)$ in M_1'. Set $A_1(\epsilon) = F_1(\epsilon) \cup A$. We claim that for fixed x_i and x_j, if $|x_j - x_i| \leqslant 2r$, then

$$\tau(\overline{B}(x_j, r) \cap A_1(\epsilon), T_{x_i}A) < \frac{\gamma}{2} + \frac{\gamma}{N}$$

for ϵ sufficiently small. To prove the claim, assume by way of contradiction that there exist sequences $(y_k) \subset A(1/k) \cap \overline{B}(x_j, r)$ and $(z_k) \subset A(1/k) \cap \overline{B}(x_j, r)$ such that

$$\frac{|\rho_{x_i}(z_k) - \rho_{x_i}(y_k)|}{|\pi_{x_i}(z_k) - \pi_{x_i}(y_k)|} \geqslant \frac{\gamma}{2} + \frac{\gamma}{N}.$$

If $|y_k - z_k| \nrightarrow 0$, then by passing to subsequences we may assume that $y_k \to y$ and $z_k \to z$, where y, $z \in A \cap \overline{B}(x_j, r)$ and $y \neq z$. We have

$$\frac{|\rho_{x_i}(z) - \rho_{x_i}(y)|}{|\pi_{x_i}(z) - \pi_{x_i}(y)|} \geqslant \frac{\gamma}{2} + \frac{\gamma}{N},$$

contradicting the choice of r. If $|y_k - z_k| \to 0$, then by passing to subsequences we may assume that for some $w \in A \cap \overline{B}(x_j, r)$ we have $y_k \to w$, $z_k \to w$, and $(y_k - z_k)/|y_k - z_k| \to v$ where $v \in T_w A_1(\epsilon) = T_w A$. We have $|\rho_{x_i} v|/|\pi_{x_i} v| \geqslant \gamma/2 + \gamma/N$, contradicting our choice of $r < r_0$.

Using the claim, there exists $\epsilon_0 > 0$ such that

$$\tau(\overline{B}(x_j, r) \cap A_1(\epsilon_0), T_{x_i}A) < \frac{\gamma}{2} + \frac{\gamma}{N},$$

whenever $|x_j - x_i| \leqslant 2r$, and

$$\theta(T_y A_1(\epsilon_0), T_x A_1(\epsilon_0)) < \frac{\gamma}{2} + \frac{\gamma}{N}.$$

for all x, $y \in A_1(\epsilon_0)$ such that $|x - y| \leqslant 3r$. Set $A_1 = A_1(\epsilon_0)$. At step k of the construction, we obtain a C^1 submanifold $M_k' \supset \overline{B}(x_k, r) \cap A_{k-1}$, $F_k(\epsilon)$ and A_k so that

$$\tau(\overline{B}(x_j, r) \cap A_k, T_{x_i}A) < \frac{\gamma}{2} + \frac{k\gamma}{N},$$

whenever $|x_j - x_i| \leqslant 2r$, and

$$\theta(T_y A_k, T_x A_k) < \frac{\gamma}{2} + \frac{k\gamma}{N}.$$

for all x, $y \in A_k$ such that $|x - y| \leqslant 3r$. The set A_N is a d-dimensional C^1 submanifold containing A. $\square$

In the general case, $\dim(T_x A)$ may vary from point to point. A global enveloping manifold may fail to exist due to this heterogeneity. We construct an example of a compact subset of $\mathbb{R}^4 \subset \mathbb{R}^n$ for which a global enveloping manifold in $\mathbb{R}^n$ cannot be constructed. Let $A = D^2 \cup \Sigma$, where D^2 denotes the closed unit disk

$$D^2 = \{(x_1, x_2, 0, 0) : x_1^2 + x_2^2 \leqslant 1\}$$

and Σ denotes the Möbius strip

$$\Sigma = \left\{ \left(\cos(\theta), \sin(\theta), t\cos\left(\frac{\theta}{2}\right), t\sin\left(\frac{\theta}{2}\right) \right) : \theta \in [0, 2\pi],\ t \in [-1, 1] \right\}.$$

Observe that $D^2 \cap \Sigma = S^1 = \{(x_1, x_2, 0, 0) : x_1^2 + x_2^2 = 1\}$.

Proposition 3.2. *The set A has the following properties.*

(1) $\dim_T A = 3$.

(2) *There exists no global enveloping* 3*-manifold for* A *in* $\mathbb{R}^n$*.*

Proof. To prove (1), observe that $\dim(T_x A) = 2$ for $x \in A \setminus S^1$ since $A \setminus S^1$ is a 2-manifold. For $x \in S^1$ we apply Lemma 2.7. There exists a neighborhood $N(x)$ of x and a diffeomorphism $f : N(x) \to f(N(x))$ such that $f(N(x) \cap A)$ is contained in the union of two orthogonal 2-planes H_1 and H_2 intersecting in a line. Since $\dim(T_{f(x)}(H_1 \cup H_2) = 3$, we have that $\dim(T_x A) = 3$ by Lemma 2.7. We conclude that $\dim_T(A) = 3$.

To prove (2), suppose by way of contradiction that such a manifold $M^3 \supset A$ exists. Let $T^1 M^3$ denote the unit tangent bundle of M^3. Since $D^2 \subset A \subset M^3$, for each $x \in D^2$ the unit sphere $S^2 = T^1_x M^3$ has a canonically defined equator $E(x) = T^1_x D^2$. Observe that $v(\theta) = (0, 0, \cos\theta/2, \sin\theta/2)$, $0 \leqslant \theta \leqslant 2\pi$, is a continuous curve in $T^1 M^3$ such that $v(2\pi) = -v(0)$ and $v(\theta) \notin E((\cos\theta, \sin\theta, 0, 0))$ for each θ. This contradiction establishes the proposition. $\square$

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