HOMOCLINIC LOOPS, HETEROCLINIC CYCLES, AND RANK ONE DYNAMICS

ANUSHAYA MOHAPATRA AND WILLIAM OTT

Abstract. We prove that genuine nonuniformly hyperbolic dynamics emerge when flows in $\mathbb{R}^N$ with homoclinic loops or heteroclinic cycles are subjected to certain time-periodic forcing. In particular, we establish the emergence of strange attractors and SRB measures with strong statistical properties (central limit theorem, exponential decay of correlations, *et cetera*). We identify and study the mechanism responsible for the nonuniform hyperbolicity: saddle point shear. Our results apply to concrete systems of interest in the biological and physical sciences, such as May-Leonard models of Lotka-Volterra dynamics.

Contents

1. Introduction

This paper is about saddle point shear, a mechanism that can produce sustained, observable chaos in concrete models of physical phenomena. Shear-induced chaos has received substantial recent attention in the context of periodically-kicked limit cycles [22, 24, 32, 40, 41]. Here we study the effects of periodic forcing on certain flows that admit homoclinic orbits or heteroclinic cycles. We formulate hypotheses that imply the existence of sustained, observable chaos for a set of forcing amplitudes of positive Lebesgue measure. By *sustained, observable chaos* we refer to an array of precisely defined dynamical, geometric, and statistical properties that are made precise in Section 4.

Date: December 30, 2013.
2010 *Mathematics Subject Classification.* 37C29, 37C40, 37D25, 37D45, 37G20, 37G35.
Key words and phrases. heteroclinic bifurcation, heteroclinic cycle, homoclinic loop, nonuniformly hyperbolic dynamics, rank one dynamics, saddle point shear, SRB measure, strange attractor.

1.1. **Background: uniform hyperbolicity and beyond.** When a dynamical system possesses some degree of hyperbolicity, individual orbits are typically unstable. Utilizing a probabilistic point of view often yields insight. The following questions are fundamental.

(Q1) Does the dynamical system admit an invariant measure that describes the asymptotic distribution of a large set (positive Riemannian volume) of orbits? If so, is this measure unique?

(Q2) What are the geometric and ergodic properties of the invariant measure(s)? For example, is a central limit theorem satisfied? At what rate do correlations decay? Large deviation principle? Weak or almost-sure invariance principle (approximation by Brownian motion)?

The Birkhoff ergodic theorem applies directly to a conservative system; that is, a system preserving a measure μ that is equivalent to Riemannian volume. If μ is ergodic, then almost every orbit with respect to μ and therefore with respect to Riemannian volume is asymptotically distributed according to μ. By contrast, invariant measures associated with dissipative (volume-contracting) systems are necessarily singular with respect to Riemannian volume. Direct application of the Birkhoff ergodic theorem yields no information about (Q1) in the dissipative context. Question (Q1) remains a major challenge.

It is natural in the dissipative context to focus on special invariant sets on which the core dynamics evolve: attractors. Let M be a compact Riemannian manifold and let $F : M \to M$ be a C^2 embedding. A compact set Ω satisfying $F(\Omega) = \Omega$ is called an ***attractor*** if there exists an open set U called its ***basin*** such that $F^n(x) \to \Omega$ as $n \to \infty$ for every $x \in U$. The attractor Ω is said to be

(a) ***irreducible*** if it cannot be written as a union of two disjoint attractors;

(b) ***uniformly hyperbolic*** if the tangent bundle over Ω splits into two DF-invariant subbundles E^s and E^u such that $DF|E^s$ is uniformly contracting, E^u is nontrivial, and $DF|E^u$ is uniformly expanding.

The geometry and ergodic theory of uniformly hyperbolic discrete-time systems is well-understood. In particular, an irreducible, uniformly hyperbolic attractor Ω supports a unique F-invariant Borel probability measure ν with the following property: there exists a set $S \subset U$ with full Riemannian volume in U such that for every continuous observable $\varphi : U \to \mathbb{R}$ and for every $x \in S$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(F^i(x)) = \int_M \varphi \, \mathrm{d}\nu. \tag{1}$$

The measure ν is known as a Sinai/Ruelle/Bowen measure (SRB measure). It is natural to link sets of positive Riemannian volume with observable events. If we do so, then the SRB measure ν is observable because temporal and spatial averages coincide for a set of initial data of full Riemannian volume in the basin.

Few physical processes have a uniformly hyperbolic character. There are many reasons for this, among them discontinuities and singularities (*e.g.* the Lorentz gas), transient effects, neutral directions, and nonuniform effects. We make no attempt here to survey the vast literature that pushes beyond uniform hyperbolicity; we simply direct the reader to a few points of entry [4, 9]. The concept of SRB measure has evolved as the theory of nonuniform hyperbolicity has developed. The following definition is state of the art.

Definition 1.1. Let M be a compact Riemannian manifold and let $F : M \to M$ be a C^2 embedding. An F-invariant Borel probability measure ν is called an ***SRB measure*** if (F, ν) has a positive Lyapunov exponent ν almost everywhere (a.e.) and if ν has absolutely continuous conditional measures on unstable manifolds.

A large class of these more general SRB measures are observable. If ν is an ergodic SRB measure with no zero Lyapunov exponents, then there exists a set S of positive Riemannian volume such that (1) holds for every continuous observable $\varphi : M \to \mathbb{R}$ and for every $x \in S$. Statistical properties of these measures have been studied using transfer operator methods (*e.g.* [8, 45]), convex cones and projective metrics (*e.g.* [23]), and coupling techniques (*e.g.* [10, 46]).

1.2. **Shear-induced chaos.** Identifying mechanisms that produce nonuniform hyperbolicity and proving that nonuniform hyperbolicity is present in concrete models remain major challenges. Recent work has shown that shear is one such mechanism. If a system possesses a substantial amount of intrinsic shear, nonuniform hyperbolicity may be produced when the system is suitably forced. The forcing does not overwhelm the intrinsic dynamics; rather, it acts as an amplifier, engaging the shear to produce nonuniform hyperbolicity. Systems with substantial intrinsic shear may be thought of as excitable systems.

1.2.1. ***Periodically-kicked limit cycles.*** Periodically-kicked limit cycles have received the most attention thus far. We discuss a model of linear shear flow originally studied by Zaslavsky [47]. Consider the following vector field on

the cylinder $\mathbb{S}^1 \times \mathbb{R}$:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1 + \sigma z \tag{2a}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\lambda z. \tag{2b}$$

Here $\sigma \geqslant 0$ measures the strength of the angular velocity gradient and $\lambda > 0$ gives the rate of contraction to the limit cycle γ located at $z = 0$. System (2) has simple dynamics: every trajectory converges to the limit cycle. However, (2) is excitable in a certain parameter regime. The ratio σ/λ measures the amount of intrinsic shear in the system. If this ratio is large, the system is excitable.

Suppose that periodic pulsatile forcing is added to (2b), giving

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1 + \sigma z \tag{3a}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\lambda z + A\Phi(\theta) \sum_{n=0}^{\infty} \delta(t - nT) \tag{3b}$$

Here $A \geqslant 0$ is the amplitude of the forcing, $\Phi : \mathbb{S}^1 \to \mathbb{R}$ is a C^3 function with finitely many nondegenerate critical points, δ is the Dirac delta, and T is the time between kicks (the relaxation time). Figure 1 illustrates the dynamics of (3). At each time nT, the system receives an instantaneous vertical kick with amplitude A and profile Φ. In particular, the limit cycle γ is deformed into a curve such as the sinusoidal wave depicted in Figure 1. After each kick, the system evolves according to (2) for T units of time (until the next kick). If both $A\sigma/\lambda$ and T are large, then shear and contraction combine to produce stretch and fold geometry. Figure 1 illustrates this geometry: the sinusoidal wave representing the kicked limit cycle morphs into the blue curve during the relaxation period.

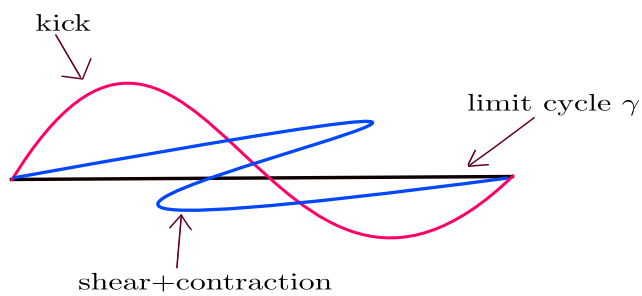


Figure 1. Stretch and fold geometry associated with (3).

Stretch and fold geometry suggests the presence of SRB measures. It has been shown that (3) does produce SRB measures. Wang and Young [41] prove that there exists $C(\Phi) > 0$ such that if $A\sigma/\lambda > C(\Phi)$, then for a set of values of T of positive Lebesgue measure, the time-T map generated by (3) has an attractor that supports a unique ergodic SRB measure ν. The dynamics are genuinely nonuniformly hyperbolic and ν has strong statistical properties, among them a central limit theorem and exponential decay of correlations. Wang and Young prove their theorem by applying the theory of rank one maps.

1.2.2. ***Theory of rank one maps.*** The theory of rank one maps [39, 42, 43] provides checkable conditions that imply the existence of nonuniformly hyperbolic dynamics and SRB measures in parametrized families $\{F_a\}$ of dissipative embeddings in dimension N for any $N \geqslant 2$. We give a descriptive summary of the theory and its applications here and a technical description in Section 4. The term 'rank one' refers to the local character of the embeddings: some instability in one direction and strong contraction in all other directions. Roughly speaking, the theory asserts that under certain checkable conditions, there exists a set Δ of values of a of positive Lebesgue measure such that for $a \in \Delta$, F_a is a genuinely nonuniformly hyperbolic map with an attractor that supports an SRB measure. A comprehensive dynamical profile is given for such F_a; we describe some aspects of this profile now.

The map F_a admits a unique SRB measure ν and ν is mixing. Lebesgue almost every trajectory in the basin of the attractor is asymptotically distributed according to ν and has a positive Lyapunov exponent. Thus the chaos associated with F_a is both observable and sustained in time. The system (F_a, ν) satisfies a central limit theorem, correlations decay at an exponential rate for Hölder observables, and a large deviations principle holds. The source of the nonuniform hyperbolicity is identified and the geometric structure of the attractor is analyzed in detail.

Figure 2 illustrates the progression of ideas that has led to the theory of rank one maps. At its core, the theory is based on theoretical developments concerning one-dimensional maps with critical points. We note in particular the

parameter exclusion technique of Jakobson [16] and the analysis of the quadratic family by Benedicks and Carleson [6]. The analysis of the Hénon family by Benedicks and Carleson [7] provided a breakthrough from one-dimensional maps with critical points (the quadratic family) to two-dimensional diffeomorphisms (small perturbations of the quadratic family). Mora and Viana [26] generalized the work of Benedicks and Carleson to small perturbations of the Hénon family and proved the existence of Hénon-like attractors in parametrized families of diffeomorphisms that generically unfold a quadratic homoclinic tangency.

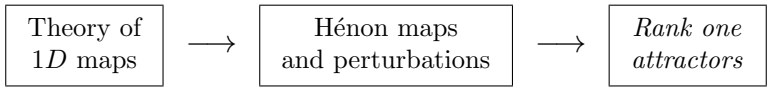
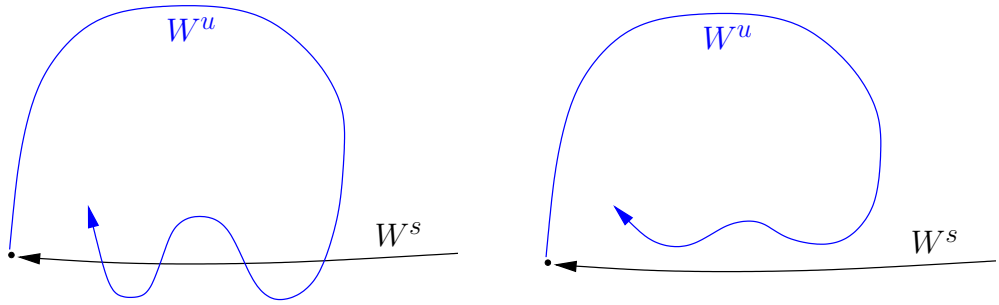


Figure 2. Progression of ideas leading to the theory of rank 1 maps.

The theory of rank one maps has been applied to many concrete models. The dynamical scenario studied most extensively thus far is that of weakly stable structures subjected to periodic pulsatile forcing. Weakly stable equilibria [31], limit cycles in finite-dimensional systems [32, 40, 41], and supercritical Hopf bifurcations in finite-dimensional systems [41] and infinite-dimensional systems [24] have been treated. Guckenheimer, Wechselberger, and Young [14] connect the theory of rank one maps and geometric singular perturbation theory by formulating a general technique for proving the existence of chaotic attractors for 3-dimensional vector fields with two time scales. Lin [21] demonstrates how the theory of rank one maps can be combined with sophisticated computational techniques to analyze the response of concrete nonlinear oscillators of interest in biological applications to periodic pulsatile drives. Electronic circuits have been treated as well [29, 30, 36, 37].

1.2.3. ***Saddle point shear.*** This paper studies flows with homoclinic orbits or heteroclinic cycles in dimension $N \geqslant 2$. A homoclinic orbit is an orbit that converges to a single stationary point of saddle type in both forward and backward time. When a system with a homoclinic orbit is forced with a periodic signal of period T, the stable and unstable manifolds that coincide in the unforced system will typically become distinct. Figure 3 illustrates two of the possibilities for the time-T maps. If the stable and unstable manifolds intersect transversely as in Figure 3a, then homoclinic tangles and horseshoes may be produced. The point of intersection may be a point of tangency between the stable and unstable manifolds, a so-called homoclinic tangency. Rich dynamics emerge as a homoclinic tangency is unfolded. We mention Hénon-like strange attractors [12, 20, 26, 39], the coexistence of infinitely many attracting periodic orbits [27, 28, 33], and nonuniformly hyperbolic horseshoes [34].



(a) W^s and W^u intersect transversely. **(b)** W^s and W^u do not intersect.

Figure 3. Some time-T maps that can occur when a system with a homoclinic loop is subjected to periodic forcing of period T.

We focus on the case in which the stable and unstable manifolds of the forced system do not intersect (Figure 3b). Afraimovich and Shilnikov [2] initiated the study of this case by proving that it is possible to define a flow-induced map on a certain cross-section. Our main results concern the dynamics of this flow-induced map. For an unforced flow in any dimension $N \geqslant 2$ with either a homoclinic loop or a heteroclinic cycle, we formulate checkable hypotheses under which a natural map induced by the flow of the forced system admits an attractor that supports a unique ergodic SRB measure for a set of forcing amplitudes μ of positive Lebesgue measure. For such μ, the flow-induced map is rank one in the sense of Wang and Young and therefore the dynamical profile described in [43] applies. In particular, the dynamics are genuinely nonuniformly hyperbolic, a central limit theorem holds, and correlations decay at an exponential rate. Wang and Ott [38] establish an analogous result in dimension two for a specific parametrized family of forcing functions forcing a flow with a homoclinic loop.

Heteroclinic cycles have been studied extensively in connection with dynamics on networks and systems possessing symmetries; see *e.g.* [3, 15, 17, 18, 19]. Our main results are independent of symmetry considerations. They apply in the presence of symmetries and in the absence of symmetries.

Figure 4 illustrates saddle point shear, the mechanism responsible for the creation of nonuniform hyperbolicity.

2. Statement of results: homoclinic loops

Let $N \geqslant 2$ be an integer. Let $\boldsymbol{\xi} = (\xi_i)_{i=1}^N$ denote the standard coordinates in $\mathbb{R}^N$ and let $\{\boldsymbol{e}_i : 1 \leqslant i \leqslant N\}$ be the standard basis for $\mathbb{R}^N$. We start with a C^4 vector field $\boldsymbol{f} : \mathbb{R}^N \to \mathbb{R}^N$ and the associated autonomous differential equation

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{\xi}) \tag{4}$$

2.1. Local dynamical picture.

We assume the following dynamical picture in a neighborhood of the origin.

(A1) The origin $\mathbf{0}$ is a stationary point of (4) ($\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$). The derivative $D\boldsymbol{f}(\mathbf{0})$ is a diagonal operator with eigenvalues $-\alpha_{N-1} \leqslant -\alpha_{N-2} \leqslant \cdots \leqslant -\alpha_1 < 0 < \beta$ corresponding to eigenvectors $\boldsymbol{e}_1$ to $\boldsymbol{e}_N$, respectively.

(A2) (dissipative saddle) The eigenvalues of $D\boldsymbol{f}(\mathbf{0})$ satisfy $0 < \beta < \alpha_1$.

(A3) (analytic linearization) There exists a neighborhood U of $\mathbf{0}$ on which $\boldsymbol{f}$ is analytic and on which there exists an analytic coordinate transformation that transforms (4) into the linear equation

$$\frac{\mathrm{d}\boldsymbol{\eta}}{\mathrm{d}t} = D\boldsymbol{f}(\mathbf{0})\boldsymbol{\eta}.$$

We now add time-periodic forcing to the right side of (4). Let $\boldsymbol{p} : \mathbb{R}^N \times \mathbb{S}^1 \to \mathbb{R}^N$ be a C^4 map for which there exists a neighborhood U_2 of $\mathbf{0}$ such that $\boldsymbol{p}$ is analytic on $U_2 \times \mathbb{S}^1$. Adding $\boldsymbol{p}$ to the right side of (4) yields the nonautonomous equation

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{\xi}) + \mu\boldsymbol{p}(\boldsymbol{\xi}, \omega t), \tag{5}$$

where ω is a frequency parameter and μ controls the amplitude of the forcing. We convert (5) into an autonomous system on the augmented phase space $\mathbb{R}^N \times \mathbb{S}^1$, giving

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{\xi}) + \mu\boldsymbol{p}(\boldsymbol{\xi}, \theta) \tag{6a}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega. \tag{6b}$$

2.2. Two small scales and a useful local coordinate system.

Let $\varepsilon_0 > 0$ be such that $\mathcal{U}_{\varepsilon_0} := B(\mathbf{0}, 2\varepsilon_0) \subset U \cap U_2$ and let $\mu_0 > 0$ satisfy $\mu_0 \ll \varepsilon_0$. We focus on forcing amplitudes in the range $[0, \mu_0]$.

When the phase space is augmented with an $\mathbb{S}^1$ factor, the hyperbolic saddle $\mathbf{0}$ becomes the hyperbolic periodic orbit $\boldsymbol{\gamma}_0 := \{\mathbf{0}\} \times \mathbb{S}^1$. This hyperbolic periodic orbit persists for μ sufficiently small. Let $\boldsymbol{\gamma}_\mu$ denote the perturbed orbit. There exists a μ-dependent coordinate system $(\boldsymbol{X}, \theta) = (X_1, \ldots, X_N, \theta)$ defined on $\mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1$ such that for every $\mu \in [0, \mu_0]$, $\boldsymbol{\gamma}_\mu = \{(\boldsymbol{X}, \theta) : \boldsymbol{X} = \mathbf{0}\}$. That is, we have standardized the location of the hyperbolic periodic orbit. Further, the stable and unstable manifolds $W^s(\boldsymbol{\gamma}_\mu)$ and $W^u(\boldsymbol{\gamma}_\mu)$ are locally flat:

$$W^s(\boldsymbol{\gamma}_\mu) \cap (\mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1) \subset \{(\boldsymbol{X}, \theta) : X_N = 0\}$$
$$W^u(\boldsymbol{\gamma}_\mu) \cap (\mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1) \subset \{(\boldsymbol{X}, \theta) : X_i = 0 \text{ for every } 1 \leqslant i \leqslant N-1\}.$$

The existence of this coordinate system is proved in [38, Section 4.2] by performing the following sequence of transformations:

(a) linearize the flow defined by (4) in a neighborhood of $\mathbf{0}$ using (A3);
(b) standardize the location of the hyperbolic periodic orbit;
(c) flatten the local stable and unstable manifolds.

2.3. Global dynamical picture.

Define the μ-dependent sections Γ^1 and Γ^2 as follows:

$$\Gamma^1 = \{(\boldsymbol{X}, \theta) : X_N = \varepsilon_0,\ \ 0 \leqslant X_1 \leqslant K_0\mu,\ \ -K_0\mu \leqslant X_i \leqslant K_0\mu \text{ for } 2 \leqslant i \leqslant N-1\}$$
$$\Gamma^2 = \left\{(\boldsymbol{X}, \theta) : X_1 = \varepsilon_0,\ \ K_1^{-1}\mu \leqslant X_N \leqslant K_1\mu,\ \ -K_2\mu \leqslant X_i \leqslant K_2\mu \text{ for } 2 \leqslant i \leqslant N-1\right\},$$

where $K_0 > 0$ satisfies $K_0\mu_0 \ll \varepsilon_0$ and $K_1 > 0$ and $K_2 > 0$ are suitably chosen. We assume that for $\mu \in (0, \mu_0]$, the flow generated by (6) induces a map from Γ^1 into Γ^2.

(A4) For $\mu \in (0, \mu_0]$, the flow generated by (6) induces a C^3 embedding $\mathcal{G}_\mu : \Gamma^1 \to \Gamma^2$. Writing $\mathcal{G}_\mu(\boldsymbol{X}, \theta) = (\boldsymbol{Y}, \rho)$, $(\boldsymbol{Y}, \rho)$ has the form

$$Y_1 = \varepsilon_0 \tag{7a}$$

$$Y_k = \sum_{i=1}^{N-1} c_{ki} X_i + \mu \Phi_k(X_1, \ldots, X_{N-1}, \theta) \qquad (2 \leqslant k \leqslant N) \tag{7b}$$

$$\rho = \theta + \zeta_1 + \mu \Phi_{N+1}(X_1, \ldots, X_{N-1}, \theta). \tag{7c}$$

Here (c_{ki}) is an invertible matrix of constants, ζ_1 is a constant, and the functions $\Phi_2, \ldots, \Phi_{N+1}$ are C^3 functions from Γ^1 into $\mathbb{R}$. We assume that $\Phi_N > 0$.

Hypothesis (A4) is motivated by bifurcation scenarios involving homoclinic orbits. Suppose that (4) ($\mu = 0$) has a homoclinic solution associated with the saddle $\boldsymbol{X} = \boldsymbol{0}$ that coincides with the positive X_N axis as $t \to -\infty$ and coincides with the positive X_1 axis as $t \to \infty$. (The assumption that the homoclinic orbit coincides with the positive X_1 axis as $t \to \infty$ is not necessary. We proceed in this way to clarify the presentation.) When system (4) is forced with a periodic signal ($\mu > 0$), the stable and unstable manifolds will typically break apart. When this happens, transversal intersections may be formed. It is also possible that the stable and unstable manifolds *do not intersect* for $\mu > 0$. In the latter case, it may be possible to define a flow-induced global map from Γ^1 into Γ^2 for $\mu > 0$ sufficiently small. See [38] for an example in which explicit formulas for the global map are derived.

Assuming (A1)–(A4) hold, for $\mu \in (0, \mu_0]$ the flow generated by (6) induces a map $\mathcal{M}_\mu : \Gamma^1 \to \Gamma^1$ given by the composition $\mathcal{M}_\mu = \mathcal{L}_\mu \circ \mathcal{G}_\mu$, where $\mathcal{G}_\mu$ is from (A4) and $\mathcal{L}_\mu : \Gamma^2 \to \Gamma^1$ is the 'local' map induced by (6). Our primary theorem for systems with homoclinic loops concerns the dynamical properties of the family $\{\mathcal{M}_\mu : 0 < \mu \leqslant \mu_0\}$. Figure 4 illustrates the geometry of $\mathcal{M}_\mu$ when $N = 2$.

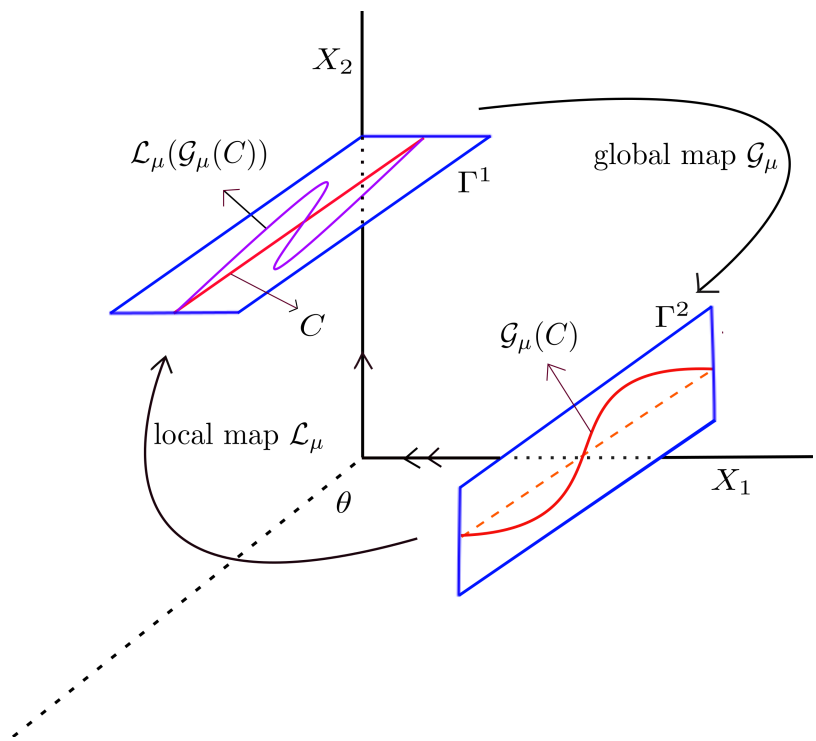


Figure 4. Saddle point shear associated with $\mathcal{M}_\mu$. Start with the flat red curve C on Γ^1. Generically, the flow from Γ^1 to Γ^2 will create ripples, meaning that when $\mathcal{G}_\mu(C)$ is viewed as a function of θ, X_2 varies as θ varies. Since the time it takes to travel from Γ^2 to Γ^1 depends on the X_2 coordinate, shear occurs in the θ direction. The purple curve $\mathcal{L}_\mu(\mathcal{G}_\mu(C))$ illustrates the resulting stretch and fold geometry of $\mathcal{M}_\mu$.

2.4. **Primary theorem for systems with homoclinic loops.**

Theorem 2.1. *Assume that system (6) satisfies (A1)–(A4). Suppose that the C^3 function $\Phi_N(\boldsymbol{0}, \theta) : \mathbb{S}^1 \to \mathbb{R}$ has finitely many nondegenerate critical points. Then there exists $\omega_0 > 0$ such that for any frequency ω satisfying*

$|\omega| \geqslant \omega_0$, there exists a set $\Delta_\omega \subset (0, \mu_0]$ of positive Lebesgue measure with the following property. For every $\mu \in \Delta_\omega$, the flow-induced map $\mathcal{M}_\mu$ admits a strange attractor Ω that supports a unique ergodic SRB measure ν. The orbit of Lebesgue almost every point on Γ^1 has a positive Lyapunov exponent and is asymptotically distributed according to ν. The SRB measure ν is mixing, satisfies the central limit theorem, and exhibits exponential decay of correlations for Hölder-continuous observables.

3. Statement of results: heteroclinic cycles

We start with (4) in two dimensions. Set $N = 2$.

3.1. **Existence of a heteroclinic cycle for the unforced system.** We assume that (4) has a heteroclinic cycle. The heteroclinic cycle consists of Q_0 hyperbolic saddle equilibria $\{\boldsymbol{q}_i : 1 \leqslant i \leqslant Q_0\}$ and connecting orbits $\{\boldsymbol{\varphi}_i : 1 \leqslant i \leqslant Q_0\}$. Let $-\lambda_i < 0 < \beta_i$ denote the eigenvalues of $D\boldsymbol{f}(\boldsymbol{q}_i)$. The connecting orbits satisfy

$$\lim_{t\to-\infty} \boldsymbol{\varphi}_i(t) = \boldsymbol{q}_i, \qquad \lim_{t\to\infty} \boldsymbol{\varphi}_i(t) = \boldsymbol{q}_{i+1}$$

for $1 \leqslant i \leqslant Q_0 - 1$ and

$$\lim_{t\to-\infty} \boldsymbol{\varphi}_{Q_0}(t) = \boldsymbol{q}_{Q_0}, \qquad \lim_{t\to\infty} \boldsymbol{\varphi}_{Q_0}(t) = \boldsymbol{q}_1.$$

We assume that the saddles satisfy the following hypotheses.

(B1) (dissipative saddles) For each $1 \leqslant i \leqslant Q_0$, the eigenvalues of $D\boldsymbol{f}(\boldsymbol{q}_i)$ satisfy $\lambda_i > \beta_i$.

(B2) (analytic linearizations) For each $1 \leqslant i \leqslant Q_0$, there exists a neighborhood of $\boldsymbol{q}_i$ on which $\boldsymbol{f}$ is analytic and on which there exists an analytic coordinate transformation that transforms (4) into its linearization at $\boldsymbol{q}_i$.

As in the homoclinic case, we study system (6). Here we assume that $\boldsymbol{p}$ is C^4 on $\mathbb{R}^2 \times \mathbb{S}^1$ and analytic in a neighborhood of each $\{\boldsymbol{q}_i\} \times \mathbb{S}^1$.

When the phase space is augmented with with an $\mathbb{S}^1$ factor, each hyperbolic saddle $\boldsymbol{q}_i$ becomes a hyperbolic periodic orbit $\boldsymbol{\gamma}_{\boldsymbol{q}_i,0} := \{\boldsymbol{q}_i\} \times \mathbb{S}^1$. This hyperbolic periodic orbit persists for μ sufficiently small. Let $\boldsymbol{\gamma}_{\boldsymbol{q}_i,\mu}$ denote the perturbed orbit. There exists $\varepsilon_0 > 0$ such that for each $1 \leqslant i \leqslant Q_0$, there exists a μ-dependent coordinate system $(\boldsymbol{Z}^{(i)}, \theta) = (Z_1^{(i)}, Z_2^{(i)}, \theta)$ defined on $B(\boldsymbol{q}_i, 2\varepsilon_0) \times \mathbb{S}^1$ such that for every $\mu \in [0, \mu_0]$, $\boldsymbol{\gamma}_{\boldsymbol{q}_i,\mu} = \left\{(\boldsymbol{Z}^{(i)}, \theta) : \boldsymbol{Z}^{(i)} = \boldsymbol{0}\right\}$ and the stable and unstable manifolds are locally flat:

$$W^s(\boldsymbol{\gamma}_{\boldsymbol{q}_i,\mu}) \cap (B(\boldsymbol{q}_i, 2\varepsilon_0) \times \mathbb{S}^1) \subset \left\{(\boldsymbol{Z}^{(i)}, \theta) : Z_2^{(i)} = 0\right\}$$
$$W^u(\boldsymbol{\gamma}_{\boldsymbol{q}_i,\mu}) \cap (B(\boldsymbol{q}_i, 2\varepsilon_0) \times \mathbb{S}^1) \subset \left\{(\boldsymbol{Z}^{(i)}, \theta) : Z_1^{(i)} = 0\right\}.$$

3.2. **Global dynamical picture.** For each $1 \leqslant i \leqslant Q_0$ and $\mu \in (0, \mu_0]$, define the μ-dependent sections S_i and S_i' as follows:

$$S_i = \left\{(\boldsymbol{Z}^{(i)}, \theta) : Z_1^{(i)} = \varepsilon_0,\ \ C_i^{-1}\mu \leqslant Z_2^{(i)} \leqslant C_i\mu\right\}$$
$$S_i' = \left\{(\boldsymbol{Z}^{(i)}, \theta) : Z_2^{(i)} = \varepsilon_0,\ \ 0 \leqslant Z_1^{(i)} \leqslant C_i'\mu\right\}.$$

Here the constants C_i' satisfy $C_i'\mu_0 \ll \varepsilon_0$ and the C_i are suitably chosen. We assume that for each $1 \leqslant i \leqslant Q_0$ and $\mu \in (0, \mu_0]$, the flow generated by (6) induces a map from S_i' into S_{i+1} (we set $S_{Q_0+1} = S_1$).

(B3) For each $1 \leqslant i \leqslant Q_0$ and $\mu \in (0, \mu_0]$, the flow generated by (6) induces a C^3 embedding $\mathcal{G}_{i,\mu} : S_i' \to S_{i+1}$ of the form

$$\mathcal{G}_{i,\mu}(Z_1^{(i)}, \varepsilon_0, \theta) = (\varepsilon_0, b_i Z_1^{(i)} + \mu\Upsilon_i(Z_1^{(i)}, \theta), \theta + \zeta_i + \mu\Psi_i(Z_1^{(i)}, \theta)). \tag{8}$$

Here the constants b_i and ζ_i satisfy $b_i \neq 0$ and $\zeta_i \geqslant 0$ for all $1 \leqslant i \leqslant Q_0$. The functions Υ_i and Ψ_i are C^3. We assume that $\Upsilon_i > 0$ for all $1 \leqslant i \leqslant Q_0$.

Figure 5 illustrates a sample configuration with 4 saddle equilibria.

3.3. **Primary theorem for systems with heteroclinic cycles.** Assuming (B1)–(B3) hold, for $\mu \in (0, \mu_0]$ the flow generated by (6) induces a map $\mathcal{M}_\mu : S_1' \to S_1'$ given by the composition

$$\mathcal{M}_\mu = (\mathcal{L}_{1,\mu} \circ \mathcal{G}_{Q_0,\mu}) \circ (\mathcal{L}_{Q_0,\mu} \circ \mathcal{G}_{Q_0-1,\mu}) \circ \cdots \circ (\mathcal{L}_{3,\mu} \circ \mathcal{G}_{2,\mu}) \circ (\mathcal{L}_{2,\mu} \circ \mathcal{G}_{1,\mu}),$$

where the maps $\mathcal{G}_{i,\mu}$ are from (B3) and $\mathcal{L}_{i,\mu} : S_i \to S_i'$ are the local maps induced by (6). Our primary theorem concerns the dynamics of the family $\{\mathcal{M}_\mu : 0 < \mu \leqslant \mu_0\}$.

Define $\Pi : \mathbb{S}^1 \to \mathbb{R}$ by

$$\Pi(\theta^{(1)}) = \sum_{i=1}^{Q_0} \frac{1}{\beta_{i+1}} \ln\left(\Upsilon_i(0, \theta^{(i)})\right).$$

Here $\beta_{Q_0+1} := \beta_1$. The $\theta^{(i)}$ for $2 \leqslant i \leqslant Q_0$ depend on $\theta^{(1)}$ and arise from a certain singular limit of the family $\{\mathcal{M}_\mu : 0 < \mu \leqslant \mu_0\}$. Our primary theorem assumes that Π is a Morse function.

Theorem 3.1. *Assume that system (6) satisfies (B1)–(B3). Suppose that the C^3 function $\Pi : \mathbb{S}^1 \to \mathbb{R}$ has finitely many nondegenerate critical points. Then there exists $\omega_0 > 0$ such that for any frequency ω satisfying $|\omega| \geqslant \omega_0$, there exists a set $\Delta_\omega \subset (0, \mu_0]$ of positive Lebesgue measure with the following property. For every $\mu \in \Delta_\omega$, the flow-induced map $\mathcal{M}_\mu$ admits a strange attractor Ω that supports a unique ergodic SRB measure ν. The orbit of Lebesgue almost every point on S_1' has a positive Lyapunov exponent and is asymptotically distributed according to ν. The SRB measure ν is mixing, satisfies the central limit theorem, and exhibits exponential decay of correlations for Hölder-continuous observables.*

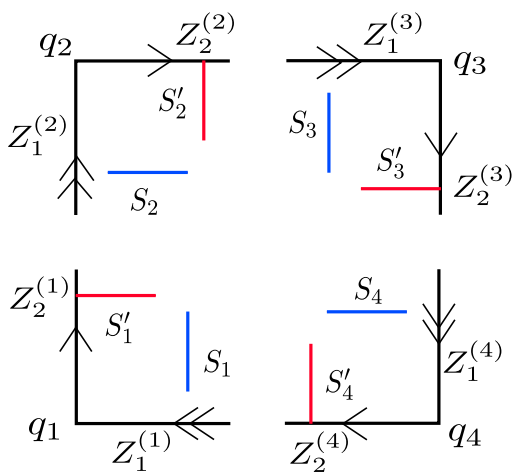


Figure 5. A sample configuration with 4 saddle equilibria. Pictured are the projections of the sections S_i and S_i' onto the plane.

3.4. **Heteroclinic cycles in physical dimension at least two.** Theorem 3.1 generalizes naturally to physical dimension $N \geqslant 2$. Here we describe the key aspects of the generalization.

First, let $-\alpha_{N-1}^{(i)} \leqslant -\alpha_{N-2}^{(i)} \leqslant \cdots \leqslant -\alpha_1^{(i)} < 0 < \beta^{(i)}$ denote the eigenvalues of $D\boldsymbol{f}(\boldsymbol{q}_i)$. We assume the following version of (B1):

(B1)* For every $1 \leqslant i \leqslant Q_0$, we have $\alpha_1^{(i)} > \beta^{(i)}$.

Second, the sections S_i and S_i' are positioned as follows. The coordinate systems $(\boldsymbol{Z}^{(i)}, \theta)$ are now given by $(\boldsymbol{Z}^{(i)}, \theta) = (Z_1^{(i)}, \ldots, Z_N^{(i)}, \theta)$ and satisfy

$$W^s(\boldsymbol{\gamma}_{\boldsymbol{q}_i,\mu}) \cap (B(\boldsymbol{q}_i, 2\varepsilon_0) \times \mathbb{S}^1) \subset \left\{(\boldsymbol{Z}^{(i)}, \theta) : Z_N^{(i)} = 0\right\}$$
$$W^u(\boldsymbol{\gamma}_{\boldsymbol{q}_i,\mu}) \cap (B(\boldsymbol{q}_i, 2\varepsilon_0) \times \mathbb{S}^1) \subset \left\{(\boldsymbol{Z}^{(i)}, \theta) : Z_1^{(i)} = \cdots = Z_{N-1}^{(i)} = 0\right\}.$$

Working in $(\boldsymbol{Z}^{(i)}, \theta)$ coordinates, for each i let H_i denote the hyperplane in $\mathbb{R}^N$ that is orthogonal to the corresponding incoming connecting orbit ($\mu = 0$) and at distance ε_0 from saddle $\boldsymbol{q}_i$. Section S_i is positioned such that the projection of S_i onto $\mathbb{R}^N$ is a subset of H_i. Further, the projection of S_i onto the $Z_N^{(i)}$ direction is the interval $[(C_N^{(i)})^{-1}\mu, C_N^{(i)}\mu]$ for $C_N^{(i)} > 0$ suitably chosen. Section S_i' is positioned such that the projection of S_i' onto $\mathbb{R}^N$ is contained in the hyperplane that is orthogonal to the corresponding outgoing connecting orbit and at distance ε_0 from saddle $\boldsymbol{q}_i$.

4. Theory of rank one maps

Let D denote the closed unit disk in $\mathbb{R}^{n-1}$ and let $M = \mathbb{S}^1 \times D$. We consider a family of maps $F_{\boldsymbol{a},b} : M \to M$, where $\boldsymbol{a} = (a_1, \dots, a_k) \in \mathcal{V}$ is a vector of parameters and $b \in B_0$ is a scalar parameter. Here $\mathcal{V} = \mathcal{V}_1 \times \cdots \times \mathcal{V}_k \subset \mathbb{R}^k$ is a product of intervals and $B_0 \subset \mathbb{R} \setminus \{0\}$ is a subset of $\mathbb{R}$ with an accumulation point at 0. Points in M are denoted by (x, y) with $x \in \mathbb{S}^1$ and $y \in D$. Rank one theory postulates the following.

(H1) Regularity conditions.

- **(a)** For each $b \in B_0$, the function $(x, y, \boldsymbol{a}) \mapsto F_{\boldsymbol{a},b}(x, y)$ is C^3.
- **(b)** Each map $F_{\boldsymbol{a},b}$ is an embedding of M into itself.
- **(c)** There exists $K_D > 0$ independent of $\boldsymbol{a}$ and b such that for all $\boldsymbol{a} \in \mathcal{V}$, $b \in B_0$, and z, $z' \in M$, we have
$$\frac{|\det DF_{\boldsymbol{a},b}(z)|}{|\det DF_{\boldsymbol{a},b}(z')|} \leqslant K_D.$$

(H2) Existence of a singular limit. For $\boldsymbol{a} \in \mathcal{V}$, there exists a map $F_{\boldsymbol{a},0} : M \to \mathbb{S}^1 \times \{0\}$ such that the following holds. For every $(x, y) \in M$ and $\boldsymbol{a} \in \mathcal{V}$, we have
$$\lim_{b \to 0} F_{\boldsymbol{a},b}(x, y) = F_{\boldsymbol{a},0}(x, y)$$
Identifying $\mathbb{S}^1 \times \{0\}$ with $\mathbb{S}^1$, we refer to $F_{\boldsymbol{a},0}$ and the restriction $f_{\boldsymbol{a}} : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $f_{\boldsymbol{a}}(x) = F_{\boldsymbol{a},0}(x, 0)$ as the ***singular limit*** of $F_{\boldsymbol{a},b}$.

(H3) C^3 convergence to the singular limit. We select a special index $j \in \{1, \dots, k\}$. Fix $a_i \in \mathcal{V}_i$ for $i \neq j$. For every such choice of parameters a_i, the maps $(x, y, a_j) \mapsto F_{\boldsymbol{a},b}(x, y)$ converge in the C^3 topology to $(x, y, a_j) \mapsto F_{\boldsymbol{a},0}(x, y)$ on $M \times \mathcal{V}_j$ as $b \to 0$.

(H4) Existence of a sufficiently expanding map within the singular limit. There exists $\boldsymbol{a}^* = (a_1^*, \dots, a_k^*) \in \mathcal{V}$ such that $f_{\boldsymbol{a}^*} \in \mathcal{E}$, where $\mathcal{E}$ is the set of Misiurewicz-type maps defined in Definition 4.1 below.

(H5) Parameter transversality. Let $C_{\boldsymbol{a}^*}$ denote the critical set of $f_{\boldsymbol{a}^*}$. For $a_j \in \mathcal{V}_j$, define the vector $\tilde{\boldsymbol{a}}_j \in \mathcal{V}$ by $\tilde{\boldsymbol{a}}_j = (a_1^*, \dots, a_{j-1}^*, a_j, a_{j+1}^*, \dots, a_k^*)$. We say that the family $\{f_{\boldsymbol{a}}\}$ satisfies the ***parameter transversality*** condition with respect to parameter a_j if the following holds. For each $x \in C_{\boldsymbol{a}^*}$, let $p = f_{\boldsymbol{a}^*}(x)$ and let $x(\tilde{\boldsymbol{a}}_j)$ and $p(\tilde{\boldsymbol{a}}_j)$ denote the continuations of x and p, respectively, as the parameter a_j varies around a_j^*. The point $p(\tilde{\boldsymbol{a}}_j)$ is the unique point such that $p(\tilde{\boldsymbol{a}}_j)$ and p have identical symbolic itineraries under $f_{\tilde{\boldsymbol{a}}_j}$ and $f_{\boldsymbol{a}^*}$, respectively. We have
$$\frac{d}{da_j} f_{\tilde{\boldsymbol{a}}_j}(x(\tilde{\boldsymbol{a}}_j))\bigg|_{a_j = a_j^*} \neq \frac{d}{da_j} p(\tilde{\boldsymbol{a}}_j)\bigg|_{a_j = a_j^*}.$$

(H6) Nondegeneracy at 'turns'. For each $x \in C_{\boldsymbol{a}^*}$, there exists $1 \leqslant m \leqslant n-1$ such that
$$\frac{\partial}{\partial y_m} F_{\boldsymbol{a}^*,0}(x, y)\bigg|_{y=0} \neq 0.$$

(H7) Conditions for mixing.

- **(a)** We have $e^{\frac{1}{3}\lambda_0} > 2$, where λ_0 is defined within Definition 4.1.
- **(b)** Let $J_1, \dots, J_r$ be the intervals of monotonicity of $f_{\boldsymbol{a}^*}$. Let $Q = (q_{im})$ be the matrix of 'allowed transitions' defined by
$$q_{im} = \begin{cases} 1, & \text{if } f_{\boldsymbol{a}^*}(J_i) \supset J_m, \\ 0, & \text{otherwise.} \end{cases}$$
There exists $N > 0$ such that $Q^N > 0$.

We now define the family $\mathcal{E}$.

Definition 4.1. We say that $f \in C^2(\mathbb{S}^1, \mathbb{R})$ is a Misiurewicz map and we write $f \in \mathcal{E}$ if the following hold for some neighborhood U of the critical set $C = C(f) = \{x \in \mathbb{S}^1 : f'(x) = 0\}$.

(1) **(Outside of U)** There exist $\lambda_0 > 0$, $M_0 \in \mathbb{Z}^+$, and $0 < d_0 \leqslant 1$ such that
  - (a) for all $m \geqslant M_0$, if $f^i(x) \notin U$ for $0 \leqslant i \leqslant m-1$, then $|(f^m)'(x)| \geqslant e^{\lambda_0 m}$,
  - (b) for any $m \in \mathbb{Z}^+$, if $f^i(x) \notin U$ for $0 \leqslant i \leqslant m-1$ and $f^m(x) \in U$, then $|(f^m)'(x)| \geqslant d_0 e^{\lambda_0 m}$.

(2) **(Critical orbits)** For all $c \in C$ and $i > 0$, $f^i(c) \notin U$.

(3) **(Inside U)**

(a) We have $f''(x) \neq 0$ for all $x \in U$, and
(b) for all $x \in U \setminus C$, there exists $p_0(x) > 0$ such that $f^i(x) \notin U$ for all $i < p_0(x)$ and $|(f^{p_0(x)})'(x)| \geqslant d_0^{-1} e^{\frac{1}{3}\lambda_0 p_0(x)}$.

The theory of rank one maps states that given a family $\{F_{\boldsymbol{a},b}\}$ satisfying (H1)–(H6), a measure-theoretically significant subset of this family consists of maps admitting attractors with strong chaotic and stochastic properties. We formulate the precise results and we then describe the properties that the attractors possess.

Theorem 4.2 ([39, 42, 43])**.** *Suppose the family $\{F_{\boldsymbol{a},b}\}$ satisfies (H1), (H2), (H4), and (H6). The following holds for all $1 \leqslant j \leqslant k$ such that the parameter a_j satisfies (H3) and (H5). For all sufficiently small $b \in B_0$, there exists a subset $\Delta_j \subset \mathcal{V}_j$ of positive Lebesgue measure such that for $a_j \in \Delta_j$, $F_{\tilde{\boldsymbol{a}}_j,b}$ admits a strange attractor Ω with properties (P1), (P2), and (P3).*

Theorem 4.3 ([39, 40, 42, 43])**.** *In the sense of Theorem 4.2,*

$$\text{(H1)–(H7)} \Longrightarrow \text{(P1)–(P4)}.$$

Remark 4.4. The proof of Theorem 4.2 for the special case $n = 2$ appears in [39]. The additional component (H7) $\Rightarrow$ (P4) in Theorem 4.3 is proved in [40]. For general n, Wang and Young [42] prove the existence of an SRB measure for $F_{\tilde{\boldsymbol{a}}_j,b}$ if $a_j \in \Delta_j$. The complete proofs of (P1)–(P3) (and (P4) assuming (H7)) for $F_{\tilde{\boldsymbol{a}}_j,b}$ with $a_j \in \Delta_j$ appear in [43] for general n.

We now describe (P1)–(P4) precisely. Write $F = F_{\tilde{\boldsymbol{a}}_j,b}$.

(P1) Positive Lyapunov exponent. Let U denote the basin of attraction of the attractor Ω. This means that U is an open set satisfying $F(\overline{U}) \subset U$ and

$$\Omega = \bigcap_{m=0}^{\infty} F^m(\overline{U}).$$

For almost every $z \in U$ with respect to Lebesgue measure, the orbit of z has a positive Lyapunov exponent. That is,

$$\lim_{m\to\infty} \frac{1}{m} \log \|DF^m(z)\| > 0.$$

(P2) Existence of SRB measures and basin property.

(a) The map F admits at least one and at most finitely many ergodic SRB measures each one of which has no zero Lyapunov exponents. Let $\nu_1, \cdots, \nu_r$ denote these measures.

(b) For Lebesgue-a.e. $z \in U$, there exists $j(z) \in \{1, \ldots, r\}$ such that for every continuous function $\varphi : U \to \mathbb{R}$,

$$\frac{1}{m} \sum_{i=0}^{m-1} \varphi(F^i(x,y)) \to \int \varphi \, \mathrm{d}\nu_{j(z)}.$$

(P3) Statistical properties of dynamical observations.

(a) For every ergodic SRB measure ν and every Hölder continuous function $\varphi : \Omega \to \mathbb{R}$, the sequence $\{\varphi \circ F^i : i \in \mathbb{Z}^+\}$ obeys a central limit theorem. That is, if $\int \varphi \, \mathrm{d}\nu = 0$, then the sequence

$$\frac{1}{\sqrt{m}} \sum_{i=0}^{m-1} \varphi \circ F^i$$

converges in distribution (with respect to ν) to the normal distribution. The variance of the limiting normal distribution is strictly positive unless $\varphi = \psi \circ F - \psi$ for some $\psi \in L^2(\nu)$.

(b) Suppose that for some $L \geqslant 1$, F^L has an SRB measure ν that is mixing. Then given a Hölder exponent η, there exists $\tau = \tau(\eta) < 1$ such that for all Hölder φ, $\psi : \Omega \to \mathbb{R}$ with Hölder exponent η, there exists $K = K(\varphi, \psi)$ such that for all $m \in \mathbb{N}$,

$$\left| \int (\varphi \circ F^{mL}) \psi \, \mathrm{d}\nu - \int \varphi \, \mathrm{d}\nu \int \psi \, \mathrm{d}\nu \right| \leqslant K(\varphi, \psi) \tau^m.$$

(P4) Uniqueness of SRB measures and ergodic properties.

(a) The map F admits a unique (and therefore ergodic) SRB measure ν, and

(b) the dynamical system (F, ν) is mixing, or, equivalently, isomorphic to a Bernoulli shift.

5. Proof of Theorem 2.1

The proof of Theorem 2.1 requires careful study of the family of flow-induced maps $\{\mathcal{M}_\mu : 0 < \mu \leqslant \mu_0\}$. We will prove that the theory of rank one maps applies to this family. In Section 5.2 we compute $\mathcal{L}_\mu$ in a C^3-controlled manner. The $\mu \to 0$ singular limit of the family $\{\mathcal{M}_\mu : 0 < \mu \leqslant \mu_0\}$ is computed in Section 5.3. Here we must introduce auxiliary parameters because the direct $\mu \to 0$ limit does not exist. Finally, in Section 5.4 we prove that $\{\mathcal{M}_\mu : 0 < \mu \leqslant \mu_0\}$ satisfies the hypotheses of the theory of rank one maps.

5.1. Preliminaries.

5.1.1. ***The parameter*** p**.** Let $p = \ln(\mu^{-1})$. We regard p as the fundamental parameter associated with (6). Notice that $\mu \in (0, \mu_0]$ corresponds to $p \in [\ln(\mu_0^{-1}), \infty)$.

5.1.2. ***Magnified local coordinates.*** We make the coordinate change $(\boldsymbol{X}, \theta) \mapsto (\mu \boldsymbol{x}, \theta)$ on $\mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1$. This stabilizes Γ^1 and Γ^2:

$$\Gamma^1 = \left\{(\boldsymbol{x}, \theta) : x_N = \varepsilon_0 \mu^{-1},\ \ 0 \leqslant x_1 \leqslant K_0,\ \ -K_0 \leqslant x_i \leqslant K_0 \text{ for } 2 \leqslant i \leqslant N-1\right\}$$
$$\Gamma^2 = \left\{(\boldsymbol{x}, \theta) : x_1 = \varepsilon_0 \mu^{-1},\ \ K_1^{-1} \leqslant x_N \leqslant K_1,\ \ -K_2 \leqslant x_i \leqslant K_2 \text{ for } 2 \leqslant i \leqslant N-1\right\}.$$

5.2. Computation of $\mathcal{L}_\mu$.

We compute $\mathcal{L}_\mu$ in a C^3-controlled manner. We begin by computing a normal form for (6) that is valid on $\mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1 \times [0, \mu_0]$.

Proposition 5.1 ([38])**.** *In terms of $(\boldsymbol{X}, \theta)$ coordinates, system (6) on $\mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1 \times [0, \mu_0]$ has the following form:*

$$\frac{\mathrm{d}X_i}{\mathrm{d}t} = (-\alpha_i + \mu G_i(\boldsymbol{X}, \theta; \mu))X_i \qquad (1 \leqslant i \leqslant N-1) \tag{9a}$$
$$\frac{\mathrm{d}X_N}{\mathrm{d}t} = (\beta + \mu G_N(\boldsymbol{X}, \theta; \mu))X_N \tag{9b}$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega. \tag{9c}$$

There exists $K_3 > 0$ such that for each $1 \leqslant k \leqslant N$, G_k is analytic on $\mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1 \times [0, \mu_0]$ and satisfies

$$\|G_k\|_{C^3(\mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1 \times [0, \mu_0])} \leqslant K_3.$$

The rescaling $\boldsymbol{X} = \mu \boldsymbol{x}$ transforms (9) into

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = (-\alpha_i + \mu g_i(\boldsymbol{x}, \theta; \mu))x_i \qquad (1 \leqslant i \leqslant N-1) \tag{10a}$$
$$\frac{\mathrm{d}x_N}{\mathrm{d}t} = (\beta + \mu g_N(\boldsymbol{x}, \theta; \mu))x_N \tag{10b}$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega, \tag{10c}$$

where $g_k(\boldsymbol{x}, \theta; \mu) = G_k(\mu \boldsymbol{x}, \theta; \mu)$ for $1 \leqslant k \leqslant N$. System (10) is valid on $\mathcal{D}(\boldsymbol{x}, \theta, \mu) := \mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1 \times [0, \mu_0]$.

5.2.1. ***On the time-t map induced by*** (10)**.** Let $V(\Gamma^2)$ be a small open neighborhood of Γ^2 in $\mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1$. We study the time-$t$ map induced by (10) assuming that all solutions beginning in $V(\Gamma^2)$ remain inside $\mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1$ up to time t.

Let $\boldsymbol{q}_0 = (\boldsymbol{x}_0, \theta_0) \in V(\Gamma^2)$ and let $\boldsymbol{q}(t, \boldsymbol{q}_0; \mu) = (\boldsymbol{x}(t, \boldsymbol{q}_0; \mu), \theta(t, \boldsymbol{q}_0; \mu))$ denote the solution of (10) with $\boldsymbol{q}(0, \boldsymbol{q}_0; \mu) = \boldsymbol{q}_0$. Integrating (10), we have

$$x_i(t, \boldsymbol{q}_0; \mu) = x_{i,0} \exp\left(\int_0^t \big(-\alpha_i + \mu g_i(\boldsymbol{q}(s, \boldsymbol{q}_0; \mu); \mu)\big)\,\mathrm{d}s\right) \qquad (1 \leqslant i \leqslant N-1) \tag{11a}$$
$$x_N(t, \boldsymbol{q}_0; \mu) = x_{N,0} \exp\left(\int_0^t \big(\beta + \mu g_N(\boldsymbol{q}(s, \boldsymbol{q}_0; \mu); \mu)\big)\,\mathrm{d}s\right) \tag{11b}$$
$$\theta(t, \boldsymbol{q}_0; \mu) = \theta_0 + \omega t. \tag{11c}$$

We introduce the functions $w_k = w_k(t, \boldsymbol{q}_0; \mu)$ for $1 \leqslant k \leqslant N$ by formulating (11) as

$$x_i(t, \boldsymbol{q}_0; \mu) = x_{i,0} \exp\big(t(-\alpha_i + w_i(t, \boldsymbol{q}_0; \mu))\big) \qquad (1 \leqslant i \leqslant N-1) \tag{12a}$$
$$x_N(t, \boldsymbol{q}_0; \mu) = x_{N,0} \exp\big(t(\beta + w_N(t, \boldsymbol{q}_0; \mu))\big) \tag{12b}$$
$$\theta(t, \boldsymbol{q}_0; \mu) = \theta_0 + \omega t, \tag{12c}$$

where

$$w_k(t, \boldsymbol{q}_0; \mu) = \frac{1}{t} \int_0^t \mu g_k(\boldsymbol{q}(s, \boldsymbol{q}_0; \mu); \mu) \, \mathrm{d}s. \tag{13}$$

The following proposition establishes C^3 control of the w_k on the domain

$$\mathcal{D}(t, \boldsymbol{q}_0, p) := \left\{ (t, \boldsymbol{q}_0, p) : t \in [1, T^*], \;\; \boldsymbol{q}_0 \in V(\Gamma^2), \;\; p \in [\ln(\mu_0^{-1}), \infty) \right\},$$

where T^* is chosen so that all solutions of (10) that start in $V(\Gamma^2)$ remain in $\mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1$ up to time T^*. We view the w_k as functions of t, $\boldsymbol{q}_0$, and p (not μ) for the following estimate.

Proposition 5.2. *There exists $K_4 > 0$ such that the following holds. For any $T^* > 1$ such that all solutions of (10) that start in $V(\Gamma^2)$ remain in $\mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1$ up to time T^*, we have*

$$\|w_k\|_{C^3(\mathcal{D}(t, \boldsymbol{q}_0, p))} \leqslant K_4 \mu \qquad (1 \leqslant k \leqslant N).$$

Proof of Proposition 5.2. Proposition 5.2 is proved in dimension two in [38, Proposition 5.5]; the argument generalizes naturally to N physical dimensions.

■

5.2.2. ***The stopping time*** $T(\boldsymbol{q}_0, p)$**.** Let $\boldsymbol{q}_0 = (\boldsymbol{x}_0, \theta_0) \in \Gamma^2$ and let $T(\boldsymbol{q}_0, p)$ be the time at which the solution to (10) starting from $\boldsymbol{q}_0$ reaches Γ^1. This stopping time is determined implicitly by (12b):

$$\varepsilon_0 \mu^{-1} = x_N(T(\boldsymbol{q}_0, p), \boldsymbol{q}_0; \mu) = x_{N,0} \exp\big(T(\boldsymbol{q}_0, p) \cdot (\beta + w_N(T(\boldsymbol{q}_0, p), \boldsymbol{q}_0; \mu))\big).$$

Solving for T, we have

$$T(\boldsymbol{q}_0, p) = \frac{1}{\beta + w_N(T(\boldsymbol{q}_0, p), \boldsymbol{q}_0; \mu)} \ln\left(\frac{\varepsilon_0 \mu^{-1}}{x_{N,0}}\right).$$

The following proposition provides a precise C^3 control of T.

Proposition 5.3. *There exists $K_5 > 0$ such that T, viewed as a function of $\boldsymbol{q}_0$ and p, satisfies*

$$\left\| T - \frac{1}{\beta} \ln(\varepsilon_0 \mu^{-1}) \right\|_{C^3(\Gamma^2 \times [\ln(\mu_0^{-1}), \infty))} \leqslant K_5.$$

Proof of Proposition 5.3. A version of Proposition 5.3 is proved in dimension two in [38, Proposition 5.7]; this proof may be adapted to the current setting and extended to N physical dimensions.

■

5.2.3. ***A*** C^3***-controlled formula for*** $\mathcal{L}_\mu$**.** Let $\boldsymbol{q}_0 = (\boldsymbol{y}, \rho) \in \Gamma^2$ and define $(\boldsymbol{z}, \hat{\theta}) = \mathcal{L}_\mu(\boldsymbol{y}, \rho) := \boldsymbol{q}(T(\boldsymbol{q}_0, p), \boldsymbol{q}_0; p)$. We have

$$z_N = \varepsilon_0 \mu^{-1} \tag{14a}$$

$$z_i = y_i \left(\frac{\varepsilon_0 \mu^{-1}}{y_N}\right)^{\frac{-\alpha_i + w_i}{\beta + w_N}} \qquad (1 \leqslant i \leqslant N-1) \tag{14b}$$

$$\hat{\theta} = \rho + \frac{\omega}{\beta + w_N} \ln\left(\frac{\varepsilon_0 \mu^{-1}}{y_N}\right). \tag{14c}$$

5.3. **The singular limit of** $\{\mathcal{M}_\mu : 0 < \mu \leqslant \mu_0\}$**.** We begin by computing $\mathcal{M}_\mu$. Referring to (A4), the global map $\mathcal{G}_\mu : \Gamma^1 \to \Gamma^2$ is given in rescaled coordinates by $\mathcal{G}_\mu(\boldsymbol{x}, \theta) = (\boldsymbol{y}, \rho)$, where

$$y_1 = \varepsilon_0 \mu^{-1} \tag{15a}$$

$$y_i = \sum_{j=1}^{N-1} c_{ij} x_j + \phi_i(x_1, \ldots, x_{N-1}, \theta) \qquad (2 \leqslant i \leqslant N) \tag{15b}$$

$$\rho = \theta + \zeta_1 + \mu \phi_{N+1}(x_1, \ldots, x_{N-1}, \theta), \tag{15c}$$

where $\phi_i(x_1,\dots,x_{N-1},\theta)=\Phi_i(\mu x_1,\dots,\mu x_{N-1},\theta)$ for $2\leqslant i\leqslant N+1$. Now let $(\boldsymbol{x},\theta)\in\Gamma^1$. Using (14) and (15), the flow-induced map $\mathcal{M}_\mu$ is given by $\mathcal{M}_\mu(\boldsymbol{x},\theta)=(\boldsymbol{z},\hat\theta)$, where

$$z_N=\varepsilon_0\mu^{-1} \tag{16a}$$

$$z_1=\varepsilon_0\mu^{-1}\left(\frac{\varepsilon_0\mu^{-1}}{\sum_{j=1}^{N-1}c_{Nj}x_j+\phi_N}\right)^{\frac{-\alpha_1+w_1}{\beta+w_N}} \tag{16b}$$

$$z_i=\left(\sum_{j=1}^{N-1}c_{ij}x_j+\phi_i\right)\left(\frac{\varepsilon_0\mu^{-1}}{\sum_{j=1}^{N-1}c_{Nj}x_j+\phi_N}\right)^{\frac{-\alpha_i+w_i}{\beta+w_N}}\qquad(2\leqslant i\leqslant N-1) \tag{16c}$$

$$\hat\theta=\theta+\zeta_1+\mu\phi_{N+1}+\frac{\omega}{\beta+w_N}\ln\left(\frac{\varepsilon_0\mu^{-1}}{\sum_{j=1}^{N-1}c_{Nj}x_j+\phi_N}\right). \tag{16d}$$

We compute the singular limit of $\left\{\mathcal{M}_{\mu(p)}:p\in[\ln(\mu_0^{-1}),\infty)\right\}$ by deriving an auxiliary parameter a from p. This is necessary because the term

$$\frac{\omega}{\beta+w_N}\ln(\varepsilon_0\mu^{-1})$$

in (16d) does not converge as $\mu\to0$. Define $\kappa:(0,\infty)\to\mathbb{R}$ by

$$\kappa(s)=\frac{\omega}{\beta}\ln(s^{-1}).$$

Let $(\mu_n)_{n=1}^\infty$ be any strictly decreasing sequence such that $\mu_n\in(0,\mu_0]$ for all $n\in\mathbb{N}$, $\mu_n\to0$ as $n\to\infty$, and $\kappa(\mu_n)\in2\pi\mathbb{Z}$ for all $n\in\mathbb{N}$. For $a\in\mathbb{S}^1$ (here $\mathbb{S}^1$ is identified with $[0,2\pi)$), define

$$\mu_{a,n}=\kappa^{-1}(\kappa(\mu_n)+a),\qquad p(a,n)=\ln(\mu_{a,n}^{-1}).$$

We now view the family of flow-induced maps as a two-parameter family of embeddings: $\left\{\mathcal{M}_{\mu(p(a,n))}:a\in\mathbb{S}^1,\ \ n\in\mathbb{N}\right\}$. The parameter n measures the amount of dissipation associated with $\mathcal{M}_{\mu(p(a,n))}$. The following proposition establishes C^3 convergence to a singular limit as $n\to\infty$.

Proposition 5.4. *We have*

$$\lim_{n\to\infty}\left\|\mathcal{M}_{\mu(p(a,n))}-(\mathbf{0},\mathcal{F}_a)\right\|_{C^3(\Gamma^1\times[0,2\pi))}=0,$$

where $\mathcal{F}_a:\Gamma^1\to\mathbb{S}^1$ is given by

$$\mathcal{F}_a(\boldsymbol{x},\theta)=\theta+a-\frac{\omega}{\beta}\ln\left(\sum_{j=1}^{N-1}c_{Nj}x_j+\phi_N\right)+\frac{\omega}{\beta}\ln(\varepsilon_0)+\zeta_1. \tag{17}$$

Proof of Proposition 5.4. We first address the term

$$\frac{\omega}{\beta+w_N}\ln(\mu^{-1})$$

in (16d). Decomposing, we have

$$\frac{\omega}{\beta+w_N}\ln(\mu^{-1})=\frac{\omega}{\beta}\ln(\mu^{-1})-\frac{\omega w_N}{\beta(\beta+w_N)}\ln(\mu^{-1}). \tag{18}$$

Since $\mu=\mu(p(a,n))$, the first term on the right side of (18) is equal to a. The asserted C^3 convergence now follows from (A2), Proposition 5.2, and Proposition 5.3. ■

5.4. **Verification of the hypotheses of the theory of rank one maps.** We show that the family of flow-induced maps $\left\{\mathcal{M}_{\mu(p(a,n))}:a\in\mathbb{S}^1,\ \ n\in\mathbb{N}\right\}$ satisfies (H1)–(H7). We establish the distortion bound (H1)(c) by studying the families of local maps and global maps separately. Since the matrix (c_{ij}) is invertible, direct computation using (15) implies that there exists a distortion constant $D_1>0$ such that for every $\mu\in(0,\mu_0]$ and $(\boldsymbol{x},\theta),(\boldsymbol{x}',\theta')\in\Gamma^1$, we have

$$\frac{|\det D\mathcal{G}_\mu(\boldsymbol{x},\theta)|}{|\det D\mathcal{G}_\mu(\boldsymbol{x}',\theta')|}\leqslant D_1. \tag{19}$$

Now let $(\boldsymbol{y},\rho)\in\Gamma^2$. Expanding $\det(D\mathcal{L}_\mu(\boldsymbol{y},\rho))$ via permutations, it follows from (14), Proposition 5.2, and Proposition 5.3 that the leading order term of $\det(D\mathcal{L}_\mu(\boldsymbol{y},\rho))$ arises from the following combination of derivatives:

$$\partial_{y_N}z_1\cdot\partial_\rho\hat\theta\cdot\prod_{i=2}^{N-1}\partial_{y_i}z_i.$$

It follows by direct computation that there exists $D_2 > 0$ such that for every $\mu \in (0, \mu_0]$ and $(\boldsymbol{y}, \rho), (\boldsymbol{y}', \rho') \in \Gamma^2$, we have

$$\frac{|\det D\mathcal{L}_\mu(\boldsymbol{y}, \rho)|}{|\det D\mathcal{L}_\mu(\boldsymbol{y}', \rho')|} \leqslant D_2. \tag{20}$$

Bounds (19) and (20) imply (H1)(c) with $K_D = D_1 D_2$. Hypotheses (H2) and (H3) follow from Proposition 5.4.

Hypotheses (H4), (H5), and (H7) concern the family of circle maps $\left\{h_a : \mathbb{S}^1 \to \mathbb{S}^1,\ \ a \in \mathbb{S}^1\right\}$ defined by

$$h_a(\theta) := \mathcal{F}_a(\mathbf{0}, \theta) = \theta + a - \frac{\omega}{\beta} \ln(\phi_N(\mathbf{0}, \theta)) + \frac{\omega}{\beta} \ln(\varepsilon_0) + \zeta_1.$$

Since $\phi_N(\mathbf{0}, \cdot)$ has finitely many nondegenerate critical points, (H4), (H5), and (H7) follow from [41, Proposition 2.1] if $|\omega|$ is sufficiently large.

Finally, the nondegeneracy condition (H6) follows by direct computation using (17) and the fact that $c_{Nk} \neq 0$ for some $1 \leqslant k \leqslant N-1$.

6. Proof of Theorem 3.1

The proof of Theorem 3.1 closely follows the proof of Theorem 2.1 given in Section 5. We therefore present only the modifications of the argument given in Section 5 that are needed for the heteroclinic setting.

In magnified coordinates, for each $1 \leqslant i \leqslant Q_0$ the global map $\mathcal{G}_{i,\mu}$ is given by

$$(z_1^{(i)}, z_2^{(i)} = \varepsilon_0 \mu^{-1}, \theta) \mapsto (y_1, y_2, \gamma),$$

where

$$y_1 = \varepsilon_0 \mu^{-1} \tag{21a}$$

$$y_2 = b_i z_1^{(i)} + \Upsilon_i(\mu z_1^{(i)}, \theta) \tag{21b}$$

$$\gamma = \theta + \zeta_i + \mu \Psi_i(\mu z_1^{(i)}, \theta). \tag{21c}$$

The local map $\mathcal{L}_{i,\mu}$ is given by

$$(z_1^{(i)}, z_2^{(i)}, \theta) \mapsto (x_1, x_2, \rho),$$

where

$$x_1 = \varepsilon_0 \mu^{-1} \left(\frac{\varepsilon_0 \mu^{-1}}{z_2^{(i)}} \right)^{\frac{-\lambda_i + w_1^{(i)}}{\beta_i + w_2^{(i)}}} \tag{22a}$$

$$x_2 = \varepsilon_0 \mu^{-1} \tag{22b}$$

$$\rho = \theta + \left(\frac{\omega}{\beta_i + w_2^{(i)}} \right) \ln \left(\frac{\varepsilon_0 \mu^{-1}}{z_2^{(i)}} \right). \tag{22c}$$

Here $w_1^{(i)}$ and $w_2^{(i)}$ are defined as in (13).

We use (21c) and (22c) to compute the angular component of the flow-induced map

$$\mathcal{M}_\mu = (\mathcal{L}_{1,\mu} \circ \mathcal{G}_{Q_0,\mu}) \circ (\mathcal{L}_{Q_0,\mu} \circ \mathcal{G}_{Q_0-1,\mu}) \circ \cdots \circ (\mathcal{L}_{2,\mu} \circ \mathcal{G}_{1,\mu}).$$

Let $(x_1^{(1)}, x_2^{(1)} = \varepsilon_0 \mu^{-1}, \theta^{(1)}) \in S_1'$. For $1 \leqslant i \leqslant Q_0$, define

$$(x_1^{(i)}, x_2^{(i)} = \varepsilon_0 \mu^{-1}, \theta^{(i)}) = (\mathcal{L}_{i,\mu} \circ \mathcal{G}_{i-1,\mu}) \circ \cdots \circ (\mathcal{L}_{2,\mu} \circ \mathcal{G}_{1,\mu})\, (x_1^{(1)}, x_2^{(1)}, \theta^{(1)}).$$

The flow-induced map $\mathcal{M}_\mu$ is given by $(x_1^{(1)}, x_2^{(1)} = \varepsilon_0 \mu^{-1}, \theta^{(1)}) \mapsto (z_1, z_2 = \varepsilon_0 \mu^{-1}, \hat{\theta})$, where $\hat{\theta}$ is computed using (21c) and (22c):

$$\hat{\theta} = \theta^{(1)} + \sum_{i=1}^{Q_0} \zeta_i + \mu \Psi_i(\mu x_1^{(i)}, \theta^{(i)}) + \left(\frac{\omega}{\beta_{i+1} + w_2^{(i+1)}} \right) \ln \left(\frac{\varepsilon_0 \mu^{-1}}{b_i x_1^{(i)} + \Upsilon_i(\mu x_1^{(i)}, \theta^{(i)})} \right). \tag{23}$$

As in the homoclinic case, we compute the singular limit of $\left\{\mathcal{M}_{\mu(p)} : p \in [\ln(\mu_0^{-1}), \infty)\right\}$ by deriving an auxiliary parameter a from p. Define $\kappa : (0, \infty) \to \mathbb{R}$ by

$$\kappa(s) = \sum_{i=1}^{Q_0} \frac{\omega}{\beta_{i+1}} \ln(s^{-1}).$$

Let $(\mu_n)_{n=1}^{\infty}$ be any strictly decreasing sequence such that $\mu_n \in (0, \mu_0]$ for all $n \in \mathbb{N}$, $\mu_n \to 0$ as $n \to \infty$, and $\kappa(\mu_n) \in 2\pi\mathbb{Z}$ for all $n \in \mathbb{N}$. For $a \in \mathbb{S}^1$, define

$$\mu_{a,n} = \kappa^{-1}(\kappa(\mu_n) + a), \qquad p(a,n) = \ln(\mu_{a,n}^{-1}).$$

We view the family of flow-induced maps as a two-parameter family of embeddings: $\{\mathcal{M}_{\mu(p(a,n))} : a \in \mathbb{S}^1,\ \ n \in \mathbb{N}\}$. The following proposition establishes C^3 convergence to a singular limit as $n \to \infty$.

Proposition 6.1. *We have*

$$\lim_{n\to\infty} \left\|\mathcal{M}_{\mu(p(a,n))} - (0, \mathcal{F}_a)\right\|_{C^3(S_1' \times [0,2\pi))} = 0,$$

where $\mathcal{F}_a : S_1' \to \mathbb{S}^1$ *is given by*

$$\mathcal{F}_a(x_1^{(1)}, \theta^{(1)}) = \theta^{(1)} + a + \left(\sum_{i=1}^{Q_0} \zeta_i + \frac{\omega}{\beta_{i+1}} \ln(\varepsilon_0)\right) - \sum_{i=1}^{Q_0} \frac{\omega}{\beta_{i+1}} \ln\left(b_i x_1^{(i)} + \Upsilon_i(0, \theta^{(i)})\right). \tag{24}$$

Proof of Proposition 6.1. The proof of Proposition 6.1 uses (23) and follows the line of reasoning developed in the proof of Proposition 5.4. ■

We finish the proof of Theorem 3.1 by showing that the family of flow-induced maps $\{\mathcal{M}_{\mu(p(a,n))} : a \in \mathbb{S}^1,\ \ n \in \mathbb{N}\}$ satisfies (H1)–(H7). The distortion bound (H1)(c) follows from the fact that the distortion of each local and global map is bounded. Hypotheses (H2) and (H3) follow from Proposition 6.1. Hypotheses (H4), (H5), and (H7) concern the family of circle maps $\{h_a : \mathbb{S}^1 \to \mathbb{S}^1,\ \ a \in \mathbb{S}^1\}$ defined by setting $x_1^{(1)} = 0$ in (24):

$$h_a(\theta^{(1)}) := \mathcal{F}_a(0, \theta^{(1)}) = \theta^{(1)} + a + \left(\sum_{i=1}^{Q_0} \zeta_i + \frac{\omega}{\beta_{i+1}} \ln(\varepsilon_0)\right) - \sum_{i=1}^{Q_0} \frac{\omega}{\beta_{i+1}} \ln\left(\Upsilon_i(0, \theta^{(i)})\right).$$

Since

$$\sum_{i=1}^{Q_0} \frac{1}{\beta_{i+1}} \ln\left(\Upsilon_i(0, \theta^{(i)})\right)$$

is a Morse function by hypothesis, (H4), (H5), and (H7) follow from [41, Proposition 2.1] if $|\omega|$ is sufficiently large. Finally, the nondegeneracy condition (H6) follows by direct computation using (24) and the fact that $b_1 \neq 0$.

7. On the verification of (A4) and (B3)

When computing the global maps for a concrete system, one may proceed in several ways. First, one may use numerical techniques such as those used by Tucker to analyze the Lorenz equations [35]. Second, one may use Melnikov integral techniques. Such techniques were used in [38] in dimension 2 to prove the existence of rank one dynamics for a periodically-forced damped nonlinear Duffing oscillator. Melnikov theory has been extended to dimension $N \geqslant 2$; see *e.g.* [5, 13, 44]. Here we develop integrals of Melnikov type that yield sufficient conditions for (A4) and (B3).

7.1. A useful normal form.

We begin with the homoclinic case. Suppose that (4) ($\mu = 0$) has a homoclinic orbit $\boldsymbol{\varphi}$ that coincides with the positive X_N axis as $t \to -\infty$ and coincides with the positive X_1 axis as $t \to \infty$.

We introduce a coordinate system $(s, \boldsymbol{v})$ valid in a neighborhood of $\boldsymbol{\varphi}$. Let s parametrize the homoclinic orbit ($s \mapsto \boldsymbol{\varphi}(s)$) such that $\boldsymbol{\varphi}'(s) = \boldsymbol{f}(\boldsymbol{\varphi}(s))$ for every $s \in \mathbb{R}$ (prime denotes differentiation with respect to s). For each $s \in \mathbb{R}$, let $\{\boldsymbol{e}_i(s) : 1 \leqslant i \leqslant N\}$ be an orthonormal basis for $T_{\boldsymbol{\varphi}(s)}\mathbb{R}^N$ such that the following hold.

(a) $s \mapsto \boldsymbol{e}_i(s)$ is C^5 for every $1 \leqslant i \leqslant N$.
(b) $\boldsymbol{e}_N(s)$ is a positive multiple of $\boldsymbol{\varphi}'(s)$ for all $s \in \mathbb{R}$.
(c) $\boldsymbol{e}_1(s), \cdots, \boldsymbol{e}_{N-1}(s)$ are orthogonal to $\boldsymbol{\varphi}$ for all $s \in \mathbb{R}$.
(d) In the $s \to \infty$ regime ($s \gg 0$ and $\boldsymbol{\varphi}(s) \in \mathcal{U}_{\varepsilon_0}$), $\boldsymbol{e}_1(s)$ is orthogonal to the stable manifold of $\mathbf{0}$ and points into the positive ξ_N half space. The set $\{\boldsymbol{e}_2(s), \cdots, \boldsymbol{e}_N(s)\}$ is an orthonormal basis for the tangent space to the stable manifold of $\mathbf{0}$ at $\boldsymbol{\varphi}(s)$.

For $\boldsymbol{\xi} \in \mathbb{R}^N$ sufficiently close to $\boldsymbol{\varphi}$, there exists unique $s \in \mathbb{R}$ and $\boldsymbol{v} = (v_1, \ldots, v_{N-1})$ such that

$$\boldsymbol{\xi} = \boldsymbol{\varphi}(s) + \sum_{i=1}^{N-1} v_i \boldsymbol{e}_i(s). \tag{25}$$

Define

$$\mathsf{E}(s) = \begin{pmatrix} \boldsymbol{e}_1(s)^\mathsf{T} \\ \boldsymbol{e}_2(s)^\mathsf{T} \\ \vdots \\ \boldsymbol{e}_N(s)^\mathsf{T} \end{pmatrix}.$$

Differentiating $\mathsf{E}(s)$ with respect to s, we have $\mathsf{E}'(s) = \mathsf{K}(s)\mathsf{E}(s)$, where $\mathsf{K}(s) = (k_{ji}(s))$ is a skew-symmetric matrix of generalized curvatures defined by $k_{ji}(s) = \left\langle \boldsymbol{e}_j'(s), \boldsymbol{e}_i(s) \right\rangle$. For $1 \leqslant i \leqslant N$, define the vector $\boldsymbol{k}_i(s) = (k_{1i}(s), \cdots, k_{N-1,i}(s))^\mathsf{T}$.

Differentiating (25) with respect to t, we have

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \sum_{i=1}^{N-1} \frac{\mathrm{d}v_i}{\mathrm{d}t} \boldsymbol{e}_i(s) + \frac{\mathrm{d}s}{\mathrm{d}t} \left(\boldsymbol{\varphi}'(s) + \sum_{j=1}^{N-1} v_j \boldsymbol{e}_j'(s) \right) = \boldsymbol{f}(\boldsymbol{\xi}) + \mu \boldsymbol{p}(\boldsymbol{\xi}, \theta). \tag{26}$$

Taking the inner product of (26) with respect to $\boldsymbol{e}_i(s)$ for $1 \leqslant i \leqslant N-1$ yields

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = \langle \boldsymbol{f}(\boldsymbol{\xi}), \boldsymbol{e}_i(s) \rangle + \mu \langle \boldsymbol{p}(\boldsymbol{\xi}, \theta), \boldsymbol{e}_i(s) \rangle - \frac{\mathrm{d}s}{\mathrm{d}t} \langle \boldsymbol{v}, \boldsymbol{k}_i(s) \rangle;$$

with respect to $\boldsymbol{e}_N(s)$ yields

$$\frac{\mathrm{d}s}{\mathrm{d}t} \left(\langle \boldsymbol{v}, \boldsymbol{k}_N(s) \rangle + \|\boldsymbol{\varphi}'(s)\| \right) = \langle \boldsymbol{f}(\boldsymbol{\xi}), \boldsymbol{e}_N(s) \rangle + \mu \langle \boldsymbol{p}(\boldsymbol{\xi}, \theta), \boldsymbol{e}_N(s) \rangle.$$

The following system is therefore valid in a sufficiently narrow tubular neighborhood of $\boldsymbol{\varphi}$:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\langle \boldsymbol{f}(\boldsymbol{\xi}), \boldsymbol{e}_N(s) \rangle + \mu \langle \boldsymbol{p}(\boldsymbol{\xi}, \theta), \boldsymbol{e}_N(s) \rangle}{\langle \boldsymbol{v}, \boldsymbol{k}_N(s) \rangle + \|\boldsymbol{\varphi}'(s)\|} \tag{27a}$$

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = \langle \boldsymbol{f}(\boldsymbol{\xi}), \boldsymbol{e}_i(s) \rangle + \mu \langle \boldsymbol{p}(\boldsymbol{\xi}, \theta), \boldsymbol{e}_i(s) \rangle - \frac{\mathrm{d}s}{\mathrm{d}t} \langle \boldsymbol{v}, \boldsymbol{k}_i(s) \rangle. \tag{27b}$$

Assuming the size of the tubular neighborhood is of order μ, we obtain a useful normal form valid inside the neighborhood:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 1 + \mathcal{O}_{s,\boldsymbol{v},\theta;\mu}(\mu) \tag{28a}$$

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \mathsf{A}(s)\boldsymbol{v} + \mu \boldsymbol{\phi}(s, \theta) + \mathcal{O}_{s,\boldsymbol{v},\theta;\mu}(\mu^2) \tag{28b}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega. \tag{28c}$$

Here the ith row of the matrix $\mathsf{A}(s)$ is given by $(\boldsymbol{\psi}_i^{(1)}(s) - \boldsymbol{k}_i(s))^\mathsf{T}$, where $\boldsymbol{\psi}_i^{(1)}(s)$ is given by

$$\boldsymbol{\psi}_i^{(1)}(s) = \left. \frac{\partial \langle \boldsymbol{f}(\boldsymbol{\xi}), \boldsymbol{e}_i(s) \rangle}{\partial \boldsymbol{v}} \right|_{\boldsymbol{v}=\boldsymbol{0}};$$

$\boldsymbol{\phi}(s, \theta)$ is the vector with components $\phi_i(s, \theta) := \langle \boldsymbol{p}(\boldsymbol{\varphi}(s), \theta), \boldsymbol{e}_i(s) \rangle$. See [32] for a more detailed derivation of (28).

7.2. **Integrals in the spirit of Melnikov.** We define Melnikov integrals by integrating (28). Let $s = -L^-$ be the value of s that corresponds to Γ^1. Let Ξ be the fundamental solution matrix to

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}s} = \mathsf{A}(s)\boldsymbol{v}$$

with $\Xi(-L^-) = \mathsf{Id}$. Integrating (28) with initial data $s = -L^-$, $\boldsymbol{v} = \boldsymbol{0}$, $\theta = \theta_0$ and then discarding terms of order μ^2, we obtain

$$\boldsymbol{v}(s) \approx \mu \Xi(s) \int_{-L^-}^{s} \Xi^{-1}(\zeta) \boldsymbol{\phi}(\zeta, \theta_0 + \omega\zeta + \omega L^-) \, \mathrm{d}\zeta.$$

In particular,

$$\boldsymbol{v}(L^+) \approx \mu \Xi(L^+) \int_{-L^-}^{L^+} \Xi^{-1}(\zeta) \boldsymbol{\phi}(\zeta, \theta_0 + \omega\zeta + \omega L^-) \, \mathrm{d}\zeta, \tag{29}$$

where L^+ is the value of s that corresponds to Γ^2. We define the Melnikov integral $\mathrm{Mel}(\theta_0)$ by projecting the right side of (29) onto $\boldsymbol{e}_1(L^+)$:

$$\mathrm{Mel}(\theta_0) = \left\langle \Xi(L^+) \int_{-L^-}^{L^+} \Xi^{-1}(\zeta) \boldsymbol{\phi}(\zeta, \theta_0 + \omega\zeta + \omega L^-) \, \mathrm{d}\zeta, \boldsymbol{e}_1(L^+) \right\rangle.$$

The function $\mathrm{Mel} : \mathbb{S}^1 \to \mathbb{R}$ measures the separation between the stable and unstable manifolds. We formulate sufficient conditions for (A4) in terms of Mel. Assume that the forcing function $\boldsymbol{p}$ has the form

$$\boldsymbol{p}(\boldsymbol{\xi}, \theta) = \rho \boldsymbol{p}_1(\boldsymbol{\xi}) + \boldsymbol{p}_2(\theta), \tag{30}$$

where ρ is a parameter. In this case, Mel splits as follows. Write $\boldsymbol{\phi}(s, \theta) = \rho \boldsymbol{\phi}^{(1)}(s) + \boldsymbol{\phi}^{(2)}(s, \theta)$, where $\boldsymbol{\phi}^{(1)}(s)$ is the vector with components $\phi_i^{(1)}(s) = \langle \boldsymbol{p}_1(\boldsymbol{\varphi}(s)), \boldsymbol{e}_i(s)\rangle$ and $\boldsymbol{\phi}^{(2)}(s, \theta)$ is the vector with components $\phi_i^{(2)}(s, \theta) = \langle \boldsymbol{p}_2(\theta), \boldsymbol{e}_i(s)\rangle$. The Melnikov function then decomposes as

$$\mathrm{Mel}(\theta_0) = \rho\, \mathrm{Mel}_1 + \mathrm{Mel}_2(\theta_0),$$

where Mel_1 is the number given by

$$\mathrm{Mel}_1 = \left\langle \Xi(L^+) \int_{-L^-}^{L^+} \Xi^{-1}(\zeta) \boldsymbol{\phi}^{(1)}(\zeta)\, \mathrm{d}\zeta, \boldsymbol{e}_1(L^+) \right\rangle$$

and $\mathrm{Mel}_2 : \mathbb{S}^1 \to \mathbb{R}$ is given by

$$\mathrm{Mel}_2(\theta_0) = \left\langle \Xi(L^+) \int_{-L^-}^{L^+} \Xi^{-1}(\zeta) \boldsymbol{\phi}^{(2)}(\zeta, \theta_0 + \omega\zeta + \omega L^-)\, \mathrm{d}\zeta, \boldsymbol{e}_1(L^+) \right\rangle.$$

Roughly speaking, $\rho\, \mathrm{Mel}_1$ gives the distance from the unstable manifold to the stable manifold while Mel_2 describes fluctuations of the unstable manifold.

Sufficient condition for (A4). If $\mathrm{Mel}_1 \neq 0$, then there exists $\rho^* > 0$ such that (A4) holds for every $\rho > \rho^*$ if $\mathrm{Mel}_1 > 0$ and (A4) holds for every $\rho < -\rho^*$ if $\mathrm{Mel}_1 < 0$.

Sufficient conditions for (B3). Each connecting orbit $\boldsymbol{\varphi}_i$ corresponds to a number $\mathrm{Mel}_{1,i}$ and a function $\mathrm{Mel}_{2,i}$. If

(a) $\mathrm{Mel}_{1,i} \neq 0$ for all $1 \leqslant i \leqslant Q_0$;
(b) all of the values $\mathrm{Mel}_{1,i}$ have the same sign,

then there exists $\rho^* > 0$ such that (B3) holds for every $\rho > \rho^*$ if the $\mathrm{Mel}_{1,i}$ are positive and (B3) holds for every $\rho < -\rho^*$ if the $\mathrm{Mel}_{1,i}$ are negative.

8. Discussion

Forcing of type (30) that consists of separate spatial and periodic components is natural in many contexts. Such forcing has been studied for systems that admit heteroclinic cycles 'at infinity' [25]. Further, such forcing is biologically significant for predator-prey models. We mention one such model here.

The asymmetric May-Leonard flow is the flow on the nonnegative octant of $\mathbb{R}^3$ generated by

$$\begin{cases} \dot{x}_1 = x_1(1 - x_1 - a_1 x_2 - b_1 x_3) \\ \dot{x}_2 = x_2(1 - b_2 x_1 - x_2 - a_2 x_3) \\ \dot{x}_3 = x_3(1 - a_3 x_1 - b_3 x_2 - x_3). \end{cases} \tag{31}$$

System (31) models the Lotka-Volterra dynamics of three competing species with equal intrinsic growth rates and differing competition coefficients. Assuming $0 < a_i < 1 < b_i < 2$ for $1 \leqslant i \leqslant 3$, (31) admits a heteroclinic cycle with saddles $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ (see Figure 2 in [1]). The asymptotic stability of this cycle is studied in [11]. If the competition coefficients also satisfy

$$\frac{b_1 - 1}{1 - a_2} > 1, \qquad \frac{b_2 - 1}{1 - a_3} > 1, \qquad \frac{b_3 - 1}{1 - a_1} > 1,$$

then (B1)* is satisfied. We are now in position to apply the Melnikov analysis of Section 7 when forcing of type (30) is applied to (31).

Generalizing the results of this work to the PDE setting will be a major challenge. As a stepping stone, one must remove the assumption that the linearizations of the unforced flow at the saddles are diagonalizable operators.

References

[1] V. S. Afraimovich, S.-B. Hsu, and H.-E. Lin, *Chaotic behavior of three competing species of May-Leonard model under small periodic perturbations*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 11 (2001), pp. 435–447.
[2] V. S. Afraimovich and L. P. Shilnikov, *The ring principle in problems of interaction between two self-oscillating systems*, Journal of Applied Mathematics and Mechanics, 41 (1977), pp. 618–627.
[3] M. Aguiar, P. Ashwin, A. Dias, and M. Field, *Dynamics of coupled cell networks: synchrony, heteroclinic cycles and inflation*, J. Nonlinear Sci., 21 (2011), pp. 271–323.

[4] L. Barreira and Y. Pesin, *Nonuniform hyperbolicity*, vol. 115 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2007. Dynamics of systems with nonzero Lyapunov exponents.

[5] F. Battelli and C. Lazzari, *Exponential dichotomies, heteroclinic orbits, and Melnikov functions*, J. Differential Equations, 86 (1990), pp. 342–366.

[6] M. Benedicks and L. Carleson, *On iterations of* $1 - ax^2$ *on* $(-1, 1)$, Ann. of Math. (2), 122 (1985), pp. 1–25.

[7] ———, *The dynamics of the Hénon map*, Ann. of Math. (2), 133 (1991), pp. 73–169.

[8] M. Blank, G. Keller, and C. Liverani, *Ruelle-Perron-Frobenius spectrum for Anosov maps*, Nonlinearity, 15 (2002), pp. 1905–1973.

[9] C. Bonatti, L. J. Díaz, and M. Viana, *Dynamics beyond uniform hyperbolicity*, vol. 102 of Encyclopaedia of Mathematical Sciences, Springer-Verlag, Berlin, 2005. A global geometric and probabilistic perspective, Mathematical Physics, III.

[10] N. Chernov, *Advanced statistical properties of dispersing billiards*, J. Stat. Phys., 122 (2006), pp. 1061–1094.

[11] C.-W. Chi, S.-B. Hsu, and L.-I. Wu, *On the asymmetric May-Leonard model of three competing species*, SIAM J. Appl. Math., 58 (1998), pp. 211–226 (electronic).

[12] E. Colli, *Infinitely many coexisting strange attractors*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 15 (1998), pp. 539–579.

[13] J. Gruendler, *The existence of homoclinic orbits and the method of Melnikov for systems in* $\mathbf{R}^n$, SIAM J. Math. Anal., 16 (1985), pp. 907–931.

[14] J. Guckenheimer, M. Wechselberger, and L.-S. Young, *Chaotic attractors of relaxation oscillators*, Nonlinearity, 19 (2006), pp. 701–720.

[15] A. J. Homburg, A. C. Jukes, J. Knobloch, and J. S. W. Lamb, *Bifurcation from codimension one relative homoclinic cycles*, Trans. Amer. Math. Soc., 363 (2011), pp. 5663–5701.

[16] M. V. Jakobson, *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps*, Comm. Math. Phys., 81 (1981), pp. 39–88.

[17] V. Kirk, E. Lane, C. M. Postlethwaite, A. M. Rucklidge, and M. Silber, *A mechanism for switching near a heteroclinic network*, Dyn. Syst., 25 (2010), pp. 323–349.

[18] M. Krupa and I. Melbourne, *Asymptotic stability of heteroclinic cycles in systems with symmetry*, Ergodic Theory Dynam. Systems, 15 (1995), pp. 121–147.

[19] ———, *Asymptotic stability of heteroclinic cycles in systems with symmetry. II*, Proc. Roy. Soc. Edinburgh Sect. A, 134 (2004), pp. 1177–1197.

[20] B. Leal, *High dimension diffeomorphisms exhibiting infinitely many strange attractors*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 25 (2008), pp. 587–607.

[21] K. K. Lin, *Entrainment and chaos in a pulse-driven Hodgkin-Huxley oscillator*, SIAM J. Appl. Dyn. Syst., 5 (2006), pp. 179–204 (electronic).

[22] K. K. Lin and L.-S. Young, *Shear-induced chaos*, Nonlinearity, 21 (2008), pp. 899–922.

[23] C. Liverani, *Decay of correlations*, Ann. of Math. (2), 142 (1995), pp. 239–301.

[24] K. Lu, Q. Wang, and L.-S. Young, *Strange attractors for periodically-forced parabolic equations*, Memoirs of the American Mathematical Society, 224 (2013).

[25] M. Messias, *Periodic perturbation of quadratic systems with two infinite heteroclinic cycles*, Discrete Contin. Dyn. Syst., 32 (2012), pp. 1881–1899.

[26] L. Mora and M. Viana, *Abundance of strange attractors*, Acta Math., 171 (1993), pp. 1–71.

[27] S. E. Newhouse, *Diffeomorphisms with infinitely many sinks*, Topology, 13 (1974), pp. 9–18.

[28] ———, *The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms*, Inst. Hautes Études Sci. Publ. Math., (1979), pp. 101–151.

[29] A. Oksasoglu, D. Ma, and Q. Wang, *Rank one chaos in switch-controlled Murali-Lakshmanan-Chua circuit*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 16 (2006), pp. 3207–3234.

[30] A. Oksasoglu and Q. Wang, *A new class of chaotic attractors in Murali-Lakshmanan-Chua circuit*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 16 (2006), pp. 2659–2670.

[31] W. Ott, *Strange attractors in periodically-kicked degenerate Hopf bifurcations*, Comm. Math. Phys., 281 (2008), pp. 775–791.

[32] W. Ott and M. Stenlund, *From limit cycles to strange attractors*, Communications in Mathematical Physics, 296 (2010), pp. 215–249.

[33] J. Palis and M. Viana, *High dimension diffeomorphisms displaying infinitely many periodic attractors*, Ann. of Math. (2), 140 (1994), pp. 207–250.

[34] J. Palis and J.-C. Yoccoz, *Non-uniformly hyperbolic horseshoes arising from bifurcations of Poincaré heteroclinic cycles*, Publ. Math. Inst. Hautes Études Sci., (2009), pp. 1–217.

[35] W. Tucker, *A rigorous ODE solver and Smale's 14th problem*, Found. Comput. Math., 2 (2002), pp. 53–117.

[36] Q. Wang and A. Oksasoglu, *Strange attractors in periodically kicked Chua's circuit*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 15 (2005), pp. 83–98.

[37] ———, *Rank one chaos: theory and applications*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 18 (2008), pp. 1261–1319.

[38] Q. Wang and W. Ott, *Dissipative homoclinic loops of two-dimensional maps and strange attractors with one direction of instability*, Comm. Pure Appl. Math., 64 (2011), pp. 1439–1496.

[39] Q. Wang and L.-S. Young, *Strange attractors with one direction of instability*, Comm. Math. Phys., 218 (2001), pp. 1–97.

[40] ———, *From invariant curves to strange attractors*, Comm. Math. Phys., 225 (2002), pp. 275–304.

[41] ———, *Strange attractors in periodically-kicked limit cycles and Hopf bifurcations*, Comm. Math. Phys., 240 (2003), pp. 509–529.

[42] ———, *Toward a theory of rank one attractors*, Ann. of Math. (2), 167 (2008), pp. 349–480.

[43] ———, *Dynamical profile of a class of rank-one attractors*, Ergodic Theory Dynam. Systems, 33 (2013), pp. 1221–1264.

[44] K. Yagasaki, *The method of Melnikov for perturbations of multi-degree-of-freedom Hamiltonian systems*, Nonlinearity, 12 (1999), pp. 799–822.

[45] L.-S. Young, *Statistical properties of dynamical systems with some hyperbolicity*, Ann. of Math. (2), 147 (1998), pp. 585–650.

[46] ———, *Recurrence times and rates of mixing*, Israel J. Math., 110 (1999), pp. 153–188.

[47] G. M. Zaslavsky, *The simplest case of a strange attractor*, Phys. Lett. A, 69 (1978/79), pp. 145–147.

651 Philip G Hoffman Hall, Department of Mathematics, University of Houston, Houston, TX 77204
URL, Anushaya Mohapatra: `http://www.math.uh.edu/∼mohapatr/`

651 Philip G Hoffman Hall, Department of Mathematics, University of Houston, Houston, TX 77204
URL, William Ott: `http://www.math.uh.edu/∼ott/`