THE GLOBAL ATTRACTOR ASSOCIATED WITH THE VISCOUS LAKE EQUATIONS

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Abstract. We consider the motion of an incompressible fluid confined to a shallow basin with varying bottom topography. A two-dimensional shallow water model has been derived from a three-dimensional anisotropic eddy viscosity model and has been shown to be globally well posed in [15]. The dynamical system associated with the shallow water model is studied. We show that this system possesses a global attractor and that the Hausdorff and box-counting dimensions of this attractor are bounded above by a value proportional to the weighted L^2-norm of the wind forcing function. A weighted Sobolev-Lieb-Thirring inequality plays the key role in the obtention of the dimension estimate.

1. Introduction

In this paper we study the asymptotic behavior of the solutions of a two-dimensional shallow water model with eddy viscosity for basins with varying bottom topography. The shallow water model has been derived from a three-dimensional anisotropic eddy viscosity model and has been shown to be globally well-posed in [15]. The derivation exploits two main scaling assumptions. First, one assumes that the ratio of the horizontal fluid velocity to the gravity wave speed is small, while the ratio of the length scale of the top surface height variation to the basin depth is much smaller still. Second, one assumes that the basin is shallow compared with the horizontal length scales of interest. The viscous shallow water model refines the lake system [3] and the great lake system [4]. These systems are derived from three-dimensional Euler flow under the same scaling assumptions. As Levermore and Sammartino [15] point out, the lake and great lake systems neglect several physical phenomena of crucial dynamical importance. The effects of viscous stresses are restored in the viscous lake system.

The viscous shallow water model bears considerable structural resemblance to the two-dimensional incompressible Navier-Stokes system. The study of the attractor associated with the Navier-Stokes equations has motivated a considerable amount of the theory of infinite-dimensional dynamical systems. Consider first the two-dimensional incompressible Navier-Stokes system on a bounded domain with Dirichlet boundary conditions. Invoking a Sobolev-Lieb-Thirring inequality, one may show [5, 8, 20] that the dimension of the global attractor is bounded above by a constant multiple of the Grashof number G, a nondimensional quantity proportional to the L^2-norm of the forcing function. The Sobolev-Lieb-Thirring

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inequalities play an important role in the estimation of the trace of certain linear operators arising in the study of infinite-dimensional dynamical systems and have led to sharp bounds on attractor dimension in terms of the physical data. Lieb and Thirring [16] prove the first such inequality, a powerful generalization of the Sobolev-Gagliardo-Nirenberg inequalities for a finite family of functions which are orthonormal in $L^2(\mathbb{R}^n)$. Systems amenable to dynamical systems methods include reaction-diffusion equations, nonlinear dissipative wave equations, complex Ginzburg-Landau equations, and various fluid models.

Now consider the Navier-Stokes system on the torus $\mathbb{T}^2$. Using an L^∞ estimate of Constantin on collections of functions whose gradients are orthonormal [6], one may improve the previous bound and show that the dimension of the global attractor is bounded above by a value proportional to $G^{2/3}(1+\log G)^{1/3}$ in the space-periodic case [9, 10, 11]. This estimate is consistent up to a logarithmic correction with the predictions of the conventional theory of turbulence due to Constantin, Foias, and Temam [9].

One strives to establish sharp bounds on the attractor dimension, for physical interpretation becomes especially significant once such bounds have been established. Research in this direction has followed two streams of thought. Liu [17] derives a lower bound in terms of the Grashof number when the domain is the torus $\mathbb{T}^2$. A family of external forces is constructed such that

$$\dim(\mathcal{A}) \geqslant \gamma G^{2/3}.$$

Therefore, in the space-periodic case, the best available lower and upper bounds agree up to a logarithmic correction. Alternatively, one may study a flow on the elongated domain $\Omega_\alpha = [0, 2\pi/\alpha] \times [0, 2\pi]$ and investigate the aspect-ratio limit $\alpha \to 0$. In the space-periodic case, a sharp estimate exists. Babin and Vishik [1] choose a specific volume force for which a simple stationary solution can be found. An estimate on the number of unstable modes around the stationary solution yields the lower bound

$$\dim(\mathcal{A}) \geqslant \frac{\gamma_1}{\alpha}.$$

Ziane [21] establishes the sharpness of this lower bound by employing a version of the Sobolev-Lieb-Thirring inequalities for elongated domains to derive the upper bound

$$\dim(\mathcal{A}) \leqslant \frac{\gamma_2}{\alpha}.$$

Doering and Wang [12] show that an application of a Lieb-Thirring inequality with the domain-dependence of the prefactors carefully controlled produces a sharp dependence of the attractor dimension on the length of the channel for certain channel flows. The derivation of a sharp estimate in the case of a general bounded domain with Dirichlet boundary conditions remains an open problem.

Given the structural similarity between the Navier-Stokes equations and the shallow water model, one suspects that a physically significant upper bound may be established for the dimension of the attractor $\mathcal{A}$ of the shallow water system. We initiate the study of this question in the present work. The Hausdorff and box-counting dimensions of $\mathcal{A}$ are shown to be bounded above by a value proportional to the weighted L^2-norm of the wind forcing function. The key technical innovation is the use of a new weighted Sobolev-Lieb-Thirring inequality. This weighted inequality is crucial because the natural function spaces for the shallow water system are the energy spaces with Lebesgue measure weighted by the basin depth function.

Many interesting questions remain open. Is the linear-in-norm bound derived in the present work sharp? Does this bound agree with any qualitative theoretical picture? In particular, how does the attractor dimension scale with the aspect ratio? Illumination of the physical significance of the scaling of an attractor dimension estimate becomes especially meaningful when the estimate is sharp. The use of inequalities akin to the L^∞ estimates of Constantin [6] may lead to an improved dimension estimate. Finally, for simplified geometries one might obtain a sharp result via an argument similar in spirit to the work of Doering and Wang [12] on channel flows.

The paper is organized as follows. In Section 2 we introduce the shallow water model and discuss its mathematical structure. The existence of a global attractor for the shallow water system is established in Section 3. Section 4 contains the derivation of the main attractor dimension estimate. We present the weighted Sobolev-Lieb-Thirring inequality in Section 5.

2. The Shallow Water Model

We consider an incompressible fluid that is confined to a three-dimensional basin by a uniform gravitational field of magnitude g. In terms of the standard Cartesian coordinates with the positive z-axis oriented upward, the basin is defined by its orthogonal projection onto the xy-plane, Ω, and by its bottom. The bottom is defined by $z = -b(\boldsymbol{x})$ for $\boldsymbol{x} = (x, y) \in \Omega$. The domain $\Omega \subset \mathbb{R}^2$ is assumed to be bounded with a smooth boundary $\partial\Omega$. We assume that b is a positive, smooth function over $\overline{\Omega}$. Let the free top surface of the fluid at time t be given by $z = h(\boldsymbol{x}, t)$. We assume that the free top surface never meets the bottom and that the average level of the top surface is $z = 0$. The domain occupied by the fluid at time t, denoted $\Sigma(t)$, is given by

$$\Sigma(t) = \{(\boldsymbol{x}, z) \in \mathbb{R}^3 : \boldsymbol{x} \in \Omega \;\; -b(\boldsymbol{x}) < z < h(\boldsymbol{x}, t)\}.$$

The shallow water model governs the evolution of $\boldsymbol{u}(\boldsymbol{x}, t)$, the horizontal fluid velocity averaged vertically over $\boldsymbol{x} \in \Omega$ at time t, and the top surface height $h(\boldsymbol{x}, t)$. The system of equations is as follows.

$$\begin{gathered}
\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{u} + g \nabla_{\boldsymbol{x}} h = b^{-1} \nabla_{\boldsymbol{x}} \cdot [b\nu(\nabla_{\boldsymbol{x}} \boldsymbol{u} + (\nabla_{\boldsymbol{x}} \boldsymbol{u})^T - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} \boldsymbol{I})] - \eta \boldsymbol{u} + \boldsymbol{f}, \\
\nabla_{\boldsymbol{x}} \cdot (b\boldsymbol{u}) = 0, \\
\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \\
\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad (\text{for } \boldsymbol{x} \in \partial\Omega), \\
\nu \boldsymbol{t} \cdot (\nabla_{\boldsymbol{x}} \boldsymbol{u} + (\nabla_{\boldsymbol{x}} \boldsymbol{u})^T) \cdot \boldsymbol{n} = -\beta \boldsymbol{t} \cdot \boldsymbol{u} \quad (\text{for } \boldsymbol{x} \in \partial\Omega).
\end{gathered}$$

Here $\nu(\boldsymbol{x})$ and $\eta(\boldsymbol{x})$ are a positive eddy viscosity coefficient and a non-negative turbulent drag coefficient defined over Ω, $\boldsymbol{I}$ is the 2×2 identity, $\boldsymbol{f}(\boldsymbol{x}, t)$ is the wind forcing defined over $\Omega \times [0, \infty)$, $\boldsymbol{n}(\boldsymbol{x})$ and $\boldsymbol{t}(\boldsymbol{x})$ are the outward unit normal and a unit tangent to $\partial\Omega$ at $\boldsymbol{x}$ and $\beta(\boldsymbol{x})$ is a non-negative turbulent boundary drag coefficient defined on $\partial\Omega$.

We reformulate the shallow water equations as an abstract evolution equation governing the velocity field $\boldsymbol{u}$. It is natural to work with Sobolev spaces weighted by the function b. The scalar-valued spaces are denoted L^p_b, $W^{s,p}_b$, and H^s_b with norms $\|\cdot\|_{L^p_b}$, $\|\cdot\|_{W^{s,p}_b}$, and $\|\cdot\|_{H^s_b}$, respectively. The vector-valued counterparts are given by $\mathbf{L}_b^p$, $\mathbf{W}_b^{s,p}$, and $\mathbf{H}_b^s$. The inner product between $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{L}_b^2$ is denoted $(\boldsymbol{u}, \boldsymbol{v})$ and is defined by

$$(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega b\boldsymbol{u} \cdot \boldsymbol{v}\, d\boldsymbol{x} = \int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\lambda(\boldsymbol{x}),$$

where λ denotes the two-dimensional Lebesgue measure weighted by b. We define the spaces

$$H = \{\boldsymbol{u} : \boldsymbol{u} \in \mathbf{L}_b^2,\ \nabla_{\boldsymbol{x}} \cdot (b\boldsymbol{u}) = 0,\ \boldsymbol{n} \cdot \boldsymbol{u} = 0 \text{ for } \boldsymbol{x} \in \partial\Omega\},$$
$$V = \{\boldsymbol{u} : \boldsymbol{u} \in \mathbf{H}_b^1,\ \nabla_{\boldsymbol{x}} \cdot (b\boldsymbol{u}) = 0,\ \boldsymbol{n} \cdot \boldsymbol{u} = 0 \text{ for } \boldsymbol{x} \in \partial\Omega\}.$$

When there is no possibility of confusion we write $|\cdot| = \|\cdot\|_{\mathbf{L}_b^2}$ and $\|\cdot\| = \|\cdot\|_{\mathbf{H}_b^1}$.

Assume $\beta(\boldsymbol{x}) \geqslant \kappa(\boldsymbol{x})$ for all $\boldsymbol{x} \in \partial\Omega$, where κ is the curvature of $\partial\Omega$. Suppose that b and ν are smooth, positive functions such that $b\nu \geqslant C > 0$ for some constant C. Under these assumptions, the bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ defined by

$$a(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{2} \int_\Omega b\nu(\nabla_{\boldsymbol{x}}\boldsymbol{u} + (\nabla_{\boldsymbol{x}}\boldsymbol{u})^T - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}\boldsymbol{I}) : (\nabla_{\boldsymbol{x}}\boldsymbol{v} + (\nabla_{\boldsymbol{x}}\boldsymbol{v})^T - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v}\boldsymbol{I})\, d\boldsymbol{x}$$
$$+ \int_\Omega b\nu\eta\boldsymbol{u} \cdot \boldsymbol{v}\, d\boldsymbol{x} + \int_{\partial\Omega} b\nu\beta\boldsymbol{u} \cdot \boldsymbol{v}\, ds$$

is coercive; that is, there exists $\alpha > 0$ such that $a(\boldsymbol{u}, \boldsymbol{u}) \geqslant \alpha\|\boldsymbol{u}\|^2$ for all $\boldsymbol{u} \in V$. By the Lax-Milgram theorem, the operator $A : V \to V'$ defined by

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = a(\boldsymbol{u}, \boldsymbol{v}) \quad (\boldsymbol{u}, \boldsymbol{v} \in V)$$

maps V isomorphically onto V'. This operator is a linear unbounded operator on H with dense domain $D(A) = \mathbf{H}_b^2 \cap V$. The inverse operator A^{-1} is self-adjoint and compact by virtue of Rellich's theorem. Thus there exists an orthonormal basis of H and a sequence (λ_j) such that

$$\begin{cases} 0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots,\ \lambda_j \to \infty, \\ A\boldsymbol{w}_j = \lambda_j \boldsymbol{w}_j\ \forall j. \end{cases}$$

We define the trilinear form $(\cdot,\cdot,\cdot)$ on V by

$$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega b\boldsymbol{u} \cdot \nabla_{\boldsymbol{x}}\boldsymbol{v} \cdot \boldsymbol{w}\, d\boldsymbol{x},$$

and the corresponding bilinear operator $B(\cdot,\cdot) : V \times V \to V'$ by

$$\langle B(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w}\rangle = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}).$$

The shallow water system is equivalent to the evolution equation

$$\partial_t \boldsymbol{u} + A\boldsymbol{u} + B(\boldsymbol{u}, \boldsymbol{u}) = \boldsymbol{f}, \tag{2.1}$$

coupled with initial data

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}). \tag{2.2}$$

The shallow water system is shown to be globally well-posed in [15]. The following is established therein.

Theorem 2.1 ([15])**.** *Let Ω be smooth. Suppose that $b(\boldsymbol{x})$, $\nu(\boldsymbol{x})$, and $\eta(\boldsymbol{x})$ are non-negative functions over $\overline{\Omega}$. Suppose that b and ν are smooth, that $b\nu \geqslant C > 0$ for some constant C, and that $\beta(\boldsymbol{x}) \geqslant \kappa(\boldsymbol{x})$ on $\partial\Omega$, where $\kappa(\boldsymbol{x})$ is the curvature of $\partial\Omega$ at $\boldsymbol{x}$. Let $\boldsymbol{f} \in \mathbf{L}_b^2$ and let $T > 0$. If $\boldsymbol{u}_0 \in H$, then there exists a unique*

$$\boldsymbol{u} \in C([0, T], H) \cap L^2([0, T], V)$$

that satisfies (2.1) and (2.2). If $\boldsymbol{u}_0 \in \mathbf{H}^2_b \cap V$, then one has moreover that

$$\boldsymbol{u} \in L^\infty([0,T], \mathbf{H}^2_b) \cap C([0,T], V),$$

and

$$\partial_t \boldsymbol{u} \in L^\infty([0,T], H) \cap L^2([0,T], V).$$

We define the semigroup $S(\cdot)$ of continuous operators on H as follows. For fixed $t \geqslant 0$, $S(t) : H \to H$ is given by $S(t)\boldsymbol{u}_0 = \boldsymbol{u}(t)$.

3. The Attractor

To demonstrate the existence of the global attractor $\mathcal{A}$ associated with $\{S(t) : t \geqslant 0\}$, we show that the semigroup is dissipative and uniformly asymptotically compact. Dissipativity in this context is characterized by the existence of a bounded absorbing set in H. The existence proof relies on standard techniques. We include the argument to fix notation and to establish estimates that are needed for the dimension calculation.

Definition 3.1. Let $\mathcal{C} \subset H$. We say that $\mathcal{C}$ is absorbing in H if for each bounded set $\mathcal{B} \subset H$ there exists $t_1(\mathcal{B})$ such that $S(t)\mathcal{B} \subset \mathcal{C}$ for all $t \geqslant t_1(\mathcal{B})$.

Definition 3.2. The semigroup $S(\cdot)$ is said to be uniformly asymptotically compact if for each bounded set $\mathcal{B} \subset H$ there exists $t_0(\mathcal{B})$ such that

$$\bigcup_{t \geqslant t_0} S(t)\mathcal{B}$$

is relatively compact in H.

We establish the uniform asymptotic compactness of the semigroup by establishing the existence of a bounded absorbing set in V and noting that V embeds compactly into H. One uses energy methods to produce absorbing sets in H and V.

3.1. **Absorbing Set in H.**

We will need the following orthogonality relation.

Lemma 3.3. *For $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in V$ one has*

$$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}),$$

and thus one has the orthogonality relation

$$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0.$$

By Sobolev embeddings there exists a constant c_1 such that

$$|\boldsymbol{u}| \leqslant c_1 \|\boldsymbol{u}\|.$$

Now $\|A^{-1}\|_{L(V',V)} \leqslant \frac{1}{\alpha}$ so one has

$$\|\boldsymbol{u}\| \leqslant \frac{1}{\alpha}\|A\boldsymbol{u}\|_{V'} \leqslant \frac{c_1}{\alpha}|A\boldsymbol{u}|.$$

Set $c_2 = \frac{c_1}{\alpha}$ and $c_3 = c_1^2$. Taking the scalar product of (2.1) with $\boldsymbol{u}$ in H, we obtain

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}|^2 + a(\boldsymbol{u}, \boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{u}).$$

Bounding the right-hand side, we have

$$\begin{aligned}(\boldsymbol{f}, \boldsymbol{u}) &\leqslant |\boldsymbol{f}|\,|\boldsymbol{u}| \\ &\leqslant c_1|\boldsymbol{f}|\,\|\boldsymbol{u}\| \\ &\leqslant \frac{\alpha}{2}\|\boldsymbol{u}\|^2 + \frac{c_1^2}{2\alpha}|\boldsymbol{f}|^2 \quad \text{(Young's inequality).}\end{aligned}$$

We obtain

$$\begin{aligned}\frac{d}{dt}|\boldsymbol{u}|^2 + \alpha\|\boldsymbol{u}\|^2 &\leqslant \frac{c_1^2}{\alpha}|\boldsymbol{f}|^2, \\ \frac{d}{dt}|\boldsymbol{u}|^2 + \frac{\alpha}{c_1^2}|\boldsymbol{u}|^2 &\leqslant \frac{c_1^2}{\alpha}|\boldsymbol{f}|^2, \\ \frac{d}{dt}|\boldsymbol{u}|^2 \leqslant \left(-\frac{\alpha}{c_3}\right)|\boldsymbol{u}|^2 &+ \left(\frac{c_3}{\alpha}|\boldsymbol{f}|^2\right).\end{aligned}$$

An application of the classical Gronwall inequality yields the estimate

$$|\boldsymbol{u}(t)|^2 \leqslant |\boldsymbol{u}_0|^2 \exp\left(-\frac{\alpha}{c_3}t\right) + \frac{c_3^2}{\alpha^2}|\boldsymbol{f}|^2\left[1 - \exp\left(-\frac{\alpha}{c_3}t\right)\right].$$

Taking the upper limit, one obtains

$$\varlimsup_{t\to\infty} |\boldsymbol{u}(t)| \leqslant \frac{c_3}{\alpha}|\boldsymbol{f}| := \rho_0.$$

We conclude that $\mathcal{B}_H(0,\rho)$, the metric ball in H of radius ρ, is absorbing for $\rho > \rho_0$. For fixed $\rho > \rho_0$ and a bounded set $\mathcal{B} \subset H$, there exists $t_1(\mathcal{B},\rho)$ such that $S(t)\mathcal{B} \subset \mathcal{B}_H(0,\rho)$ for all $t \geqslant t_1(\mathcal{B},\rho)$.

3.2. **Absorbing Set in V.** We need the following continuity property of the trilinear form $(\cdot,\cdot,\cdot)$.

Lemma 3.4. *There exists a constant k such that for $\boldsymbol{u} \in V$, $\boldsymbol{v} \in D(A)$, and $\boldsymbol{w} \in H$ one has*

$$|(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leqslant k|\boldsymbol{u}|^{\frac{1}{2}}\|\boldsymbol{u}\|^{\frac{1}{2}}\|\boldsymbol{v}\|^{\frac{1}{2}}|A\boldsymbol{v}|^{\frac{1}{2}}|\boldsymbol{w}|. \tag{3.1}$$

Proof. The proof is based on two key facts. The first is an interpolation inequality known as Ladyzhenskaya's inequality.

Lemma 3.5. *For $\boldsymbol{u} \in \mathbf{H}_b^1(\Omega)$ one has*

$$\|\boldsymbol{u}\|_{\mathbf{L}_b^4} \leqslant c_4|\boldsymbol{u}|^{\frac{1}{2}}\|\boldsymbol{u}\|^{\frac{1}{2}}.$$

We also need an elliptic regularity estimate for the strong Stokes problem associated with the shallow water system. It is shown in [15] that for $\boldsymbol{f} \in \mathbf{L}_b^p$, $p \in (1,\infty)$, the strong Stokes problem admits a unique solution $\boldsymbol{u} \in \mathbf{W}_b^{2,p}$ satisfying

$$\|\boldsymbol{u}\|_{\mathbf{W}_b^{2,p}} \leqslant c(\|\boldsymbol{f}\|_{\mathbf{L}_b^p} + \|\boldsymbol{u}\|_{\mathbf{L}_b^p}). \tag{3.2}$$

Notice that A^{-1}, the operator mapping $\mathbf{L}_b^2$ data to the solution of the strong Stokes problem, may be extended as a linear continuous operator from $\mathbf{L}_b^p(\Omega)$ into $\mathbf{W}_b^{2,p}(\Omega)$

for all $p \in (1, \infty)$. For $\boldsymbol{u} \in V$, $\boldsymbol{v} \in D(A)$, and $\boldsymbol{w} \in H$, one has

$$\begin{aligned}
\left| \int_\Omega \boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{v} \cdot \boldsymbol{w} \, d\lambda \right| &\leqslant \sum_{i,j=1}^{2} \int_\Omega |u_i (D_i v_j) w_j| \, d\lambda \\
&\leqslant \sum_{i,j=1}^{2} \|u_i\|_{L_b^4} \|D_i v_j\|_{L_b^4} |w_j| \\
&\leqslant \sum_{i,j=1}^{2} c_4^2 |u_i|^{\frac{1}{2}} \|u_i\|^{\frac{1}{2}} |D_i v_j|^{\frac{1}{2}} \|D_i v_j\|^{\frac{1}{2}} |w_j| \\
&\leqslant c_4^2 \left(\sum_{i=1}^{2} |u_i| \, \|u_i\| \right)^{\frac{1}{2}} \left(\sum_{i,j=1}^{2} |D_i v_j| \, \|D_i v_j\| \right)^{\frac{1}{2}} \left(\sum_{j=1}^{2} |w_j|^2 \right)^{\frac{1}{2}} \\
&\leqslant c_4^2 |\boldsymbol{u}|^{\frac{1}{2}} \|\boldsymbol{u}\|^{\frac{1}{2}} \|\boldsymbol{v}\|^{\frac{1}{2}} \|\boldsymbol{v}\|^{\frac{1}{2}}_{\mathbf{H}_b^2(\Omega)} |\boldsymbol{w}| \\
&\leqslant c_4^2 c^{\frac{1}{2}} (1 + c_2)^{\frac{1}{2}} |\boldsymbol{u}|^{\frac{1}{2}} \|\boldsymbol{u}\|^{\frac{1}{2}} \|\boldsymbol{v}\|^{\frac{1}{2}} |A\boldsymbol{v}|^{\frac{1}{2}} |\boldsymbol{w}|.
\end{aligned}$$

Setting $k = c_4^2 c^{\frac{1}{2}} (1 + c_2)^{\frac{1}{2}}$, the lemma is established. □

We are now in position to establish the existence of an absorbing set in V. Taking the scalar product of (2.1) with $A\boldsymbol{u}$ gives

$$\frac{1}{2} \frac{d}{dt} a(\boldsymbol{u}, \boldsymbol{u}) + |A\boldsymbol{u}|^2 = (\boldsymbol{f}, A\boldsymbol{u}) - (\boldsymbol{u}, \boldsymbol{u}, A\boldsymbol{u}).$$

Applying the continuity estimate (3.1), we obtain

$$\begin{aligned}
|(\boldsymbol{u}, \boldsymbol{u}, A\boldsymbol{u})| &\leqslant (|A\boldsymbol{u}|^{\frac{3}{2}})(k |\boldsymbol{u}|^{\frac{1}{2}} \|\boldsymbol{u}\|) \\
&\leqslant \frac{3}{8} |A\boldsymbol{u}|^2 + 2k^4 |\boldsymbol{u}|^2 \|\boldsymbol{u}\|^4 \quad \text{(Young's inequality)}.
\end{aligned}$$

Bounding the scalar product $(\boldsymbol{f}, A\boldsymbol{u})$, one has

$$(\boldsymbol{f}, A\boldsymbol{u}) \leqslant |\boldsymbol{f}| \, |A\boldsymbol{u}| \leqslant \frac{|A\boldsymbol{u}|^2}{4} + |\boldsymbol{f}|^2.$$

Collecting these estimates, we obtain

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} a(\boldsymbol{u}, \boldsymbol{u}) + \frac{3}{8} |A\boldsymbol{u}|^2 &\leqslant |\boldsymbol{f}|^2 + 2k^4 |\boldsymbol{u}|^2 \|\boldsymbol{u}\|^4 \\
&\leqslant |\boldsymbol{f}|^2 + 2k^4 |\boldsymbol{u}|^2 \|\boldsymbol{u}\|^2 \left(\frac{a(\boldsymbol{u}, \boldsymbol{u})}{\alpha} \right),
\end{aligned}$$

and we conclude that

$$\frac{d}{dt} a(\boldsymbol{u}, \boldsymbol{u}) \leqslant 2|\boldsymbol{f}|^2 + c_5 |\boldsymbol{u}|^2 \|\boldsymbol{u}\|^2 a(\boldsymbol{u}, \boldsymbol{u}),$$

where $c_5 = \frac{4k^4}{\alpha}$. In order to control $\|\boldsymbol{u}(t)\|$ as $t \to \infty$ we invoke the uniform Gronwall lemma.

Lemma 3.6 (Uniform Gronwall)**.** *Let g, h, and y be three positive locally integrable functions on $[t_1, \infty)$ such that y is absolutely continuous on $[t_1, \infty)$ and which satisfy*

$$\frac{dy}{dt} \leqslant gy + h,$$
$$\int_t^{t+r} g(s)\,ds \leqslant a_1, \quad \int_t^{t+r} h(s)\,ds \leqslant a_2, \quad \int_t^{t+r} y(s)\,ds \leqslant a_3$$

for $t \geqslant t_1$, where r, a_1, a_2, and a_3 are positive constants. Then

$$y(t+r) \leqslant \left(\frac{a_3}{r} + a_2\right)\exp(a_1) \quad (t \geqslant t_1).$$

Fix $\rho > \rho_0$ and $r > 0$. Let $\mathcal{B} \subset H$ be a bounded subset of H. As we have seen, there exists $t_1(\mathcal{B}, \rho)$ such that $S(t)\mathcal{B} \subset \mathcal{B}_H(0, \rho)$ for all $t \geqslant t_1(\mathcal{B}, \rho)$. We apply the uniform Gronwall lemma with

$$\begin{cases} y = a(\boldsymbol{u}, \boldsymbol{u}) \\ g = c_5|\boldsymbol{u}|^2\|\boldsymbol{u}\|^2 \\ h = 2|\boldsymbol{f}|^2 \end{cases}$$

by producing constants a_1, a_2, and a_3 valid for $t \geqslant t_1(\mathcal{B}, \rho)$. One must first bound the integral of $\|\boldsymbol{u}\|^2$ over time intervals $[t, t+r]$ with $t \geqslant t_1(\mathcal{B}, \rho)$. Recall the inequality

$$\frac{d}{dt}|\boldsymbol{u}|^2 + \alpha\|\boldsymbol{u}\|^2 \leqslant \frac{c_1^2}{\alpha}|\boldsymbol{f}|^2.$$

Integrating in time, we obtain

$$\begin{aligned} \int_t^{t+r} \frac{d}{ds}|\boldsymbol{u}|^2\,ds + \int_t^{t+r} \alpha\|\boldsymbol{u}\|^2\,ds &\leqslant \frac{c_1^2}{\alpha}|\boldsymbol{f}|^2 r, \\ \int_t^{t+r} \|\boldsymbol{u}\|^2\,ds &\leqslant \frac{c_1^2}{\alpha^2}|\boldsymbol{f}|^2 r + \frac{|\boldsymbol{u}(t)|^2}{\alpha} \\ &\leqslant \frac{c_1^2}{\alpha^2}|\boldsymbol{f}|^2 r + \frac{\rho^2}{\alpha}. \end{aligned}$$

The constants a_1, a_2, and a_3 are defined as follows:

$$\begin{aligned} \int_t^{t+r} g(s)\,ds &\leqslant c_5\rho^2 \int_t^{t+r} \|\boldsymbol{u}\|^2\,ds \\ &\leqslant c_5\rho^2\left(\frac{c_1^2}{\alpha^2}|\boldsymbol{f}|^2 r + \frac{\rho^2}{\alpha}\right) := a_1, \\ \int_t^{t+r} h(s)\,ds &= \int_t^{t+r} 2|\boldsymbol{f}|^2\,ds = 2|\boldsymbol{f}|^2 r := a_2, \\ \int_t^{t+r} a(\boldsymbol{u}, \boldsymbol{u})\,ds &\leqslant \int_t^{t+r} M\|\boldsymbol{u}(s)\|^2\,ds \quad (a(\boldsymbol{u}, \boldsymbol{u}) \leqslant M\|\boldsymbol{u}\|^2) \\ &\leqslant M\left(\frac{c_1^2}{\alpha^2}|\boldsymbol{f}|^2 r + \frac{\rho^2}{\alpha}\right) := a_3. \end{aligned}$$

The uniform Gronwall lemma yields the bound

$$\alpha\|\boldsymbol{u}\|^2 \leqslant a(\boldsymbol{u}(t), \boldsymbol{u}(t)) \leqslant \left(\frac{a_3}{r} + a_2\right)\exp(a_1),$$

valid for every $t \geqslant t_1(\mathcal{B}, \rho) + r$. We conclude that the ball in V of radius

$$\left[\left(\frac{a_3}{r} + a_2\right) \frac{\exp(a_1)}{\alpha}\right]^{\frac{1}{2}}$$

is absorbing.

4. Upper Bound on the Attractor Dimension

4.1. **Uniform Lyapunov Exponents.** Fix $T = 1$. According to the ergodic theory of dynamical systems, the attractor $\mathcal{A}$ is the support of a measure μ that is invariant under the action of $S(T)$. The multiplicative ergodic theorem of Oseledec implies the existence of classical Lyapunov exponents for μ-almost every $\boldsymbol{u} \in \mathcal{A}$. Because the classical Lyapunov exponents may fail to exist, we employ the concept of uniform Lyapunov exponents (see [5, 20]).

Definition 4.1. The semigroup $\{S(t)\}$ is said to be uniformly quasidifferentiable on $\mathcal{A}$ if for $t \geqslant 0$ and $\boldsymbol{u} \in \mathcal{A}$ there exists a bounded linear operator $L(t, \boldsymbol{u}) : H \to H$, the quasidifferential, such that

$$\frac{|S(t)\boldsymbol{v} - S(t)\boldsymbol{u} - L(t, \boldsymbol{u})(\boldsymbol{v} - \boldsymbol{u})|}{|\boldsymbol{v} - \boldsymbol{u}|} \leqslant \gamma(t, |\boldsymbol{v} - \boldsymbol{u}|) \text{ for } \boldsymbol{v} \in \mathcal{A}$$

where $\gamma(t, s) \to 0$ as $s \to 0$.

Proposition 4.2. *The semigroup $\{S(t)\}$ associated with the shallow water model is uniformly quasidifferentiable on $\mathcal{A}$. Moreover, the quasidifferential $L(t, \boldsymbol{u}(t))$ uniquely solves the linear variational equation*

$$\begin{cases} \partial_t \boldsymbol{\xi} = F'(\boldsymbol{u}(t))\boldsymbol{\xi} \\ \boldsymbol{\xi}(\boldsymbol{x}, 0) = \boldsymbol{v}(\boldsymbol{x}) \end{cases} \tag{4.1}$$

where F' denotes the Fréchet derivative of F. One has the uniform bound

$$\sup_{\boldsymbol{u} \in \mathcal{A}} \|L(T, \boldsymbol{u})\|_{\mathcal{L}(H,H)} < \infty.$$

Proof. The result follows from the implicit function theorem and is analogous to the corresponding result for the semigroup associated with the two-dimensional Navier-Stokes system. See Theorem 7.1.1 of [2] or Chapter 13 of [18]. □

This proposition implies that the uniform Lyapunov exponents, denoted μ_j, are well-defined. We relate these exponents to the evolution of the volume element. Fix $\boldsymbol{u}_0 \in \mathcal{A}$. Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ be m elements of H and let $\boldsymbol{\xi}_i$ denote the solution of the variational equation with initial data $\boldsymbol{v}_i$. The volume element satisfies the evolution equation

$$\|\boldsymbol{\xi}_1(t) \wedge \cdots \wedge \boldsymbol{\xi}_m(t)\|_{\bigwedge_H^m} = \|\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_m\|_{\bigwedge_H^m} \exp\left(\int_0^t \operatorname{Tr}\left(F'(\boldsymbol{u}(\tau)) \circ Q_m(\tau)\right) d\tau\right)$$

where $Q_m(t) = Q_m(t, \boldsymbol{u}_0; \boldsymbol{v}_1, \ldots, \boldsymbol{v}_m)$ is the orthogonal projector onto the space spanned by $\boldsymbol{\xi}_1(t), \ldots, \boldsymbol{\xi}_m(t)$. We introduce the quantities

$$q_m(t) = \sup_{\boldsymbol{u}_0 \in \mathcal{A}} \sup_{\substack{\boldsymbol{v}_i \in H \\ |\boldsymbol{v}_i| \leqslant 1 \\ i=1,\ldots,m}} \left(\frac{1}{t} \int_0^t \operatorname{Tr}\left(F'(S(\tau)\boldsymbol{u}_0) \circ Q_m(\tau)\right) d\tau\right),$$

$$q_m = \overline{\lim_{t \to \infty}}\, q_m(t).$$

The uniform Lyapunov exponents satisfy

$$\mu_1 + \cdots + \mu_m \leqslant q_m.$$

For the shallow water model we will establish the bound

$$q_m \leqslant \psi(m) := -\gamma_1 m^2 + \gamma_2$$

for some $\gamma_1 > 0$, $\gamma_2 > 0$. Applying Theorem III.3.2 of [5], one concludes that the Hausdorff and box dimensions of $\mathcal{A}$ are bounded above by

$$N + \frac{\psi(N)}{\psi(N) - \psi(N+1)}$$

where N is the smallest integer such that $\psi(N+1) < 0$ and $\psi(N) \geqslant 0$.

4.2. **The Estimate.** The variational equation (4.1) is equivalent to

$$\frac{d\boldsymbol{\xi}}{dt} + A\boldsymbol{\xi} + B(\boldsymbol{u}, \boldsymbol{\xi}) + B(\boldsymbol{\xi}, \boldsymbol{u}) = 0.$$

Fix $\tau > 0$. Let $\{\boldsymbol{\varphi}_j(\tau) : j = 1, \ldots, m\}$ be an orthonormal basis of $Q_m(\tau)H$. One has

$$\begin{aligned}
\operatorname{Tr}\left(F'(S(\tau)\boldsymbol{u}_0) \circ Q_m(\tau)\right) &= \sum_{j=1}^{m} (F'(\boldsymbol{u}(\tau))\boldsymbol{\varphi}_j(\tau), \boldsymbol{\varphi}_j(\tau)) \\
&= -\sum_{j=1}^{m} (A\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_j) - \sum_{j=1}^{m} (\boldsymbol{\varphi}_j, \boldsymbol{u}, \boldsymbol{\varphi}_j).
\end{aligned}$$

Notice that the first term has the good sign. Gaining control of the second term is the key to the estimate. Now

$$\sum_{j=1}^{m} (\boldsymbol{\varphi}_j, \boldsymbol{u}, \boldsymbol{\varphi}_j) = \int_\Omega \sum_{j=1}^{m} \sum_{i,k=1}^{2} \varphi_{ji}(\boldsymbol{x}) D_i u_k(\boldsymbol{x}) \varphi_{jk}(\boldsymbol{x})\, d\lambda(\boldsymbol{x}),$$

whence for almost every $\boldsymbol{x} \in \Omega$ we have

$$\left| \sum_{j=1}^{m} \sum_{i,k=1}^{2} \varphi_{ji}(\boldsymbol{x}) D_i u_k(\boldsymbol{x}) \varphi_{jk}(\boldsymbol{x}) \right| \leqslant |\nabla \boldsymbol{u}(\boldsymbol{x})| \rho(\boldsymbol{x}),$$

where

$$|\nabla \boldsymbol{u}(\boldsymbol{x})| = \left(\sum_{i,k=1}^{2} |D_i u_k(\boldsymbol{x})|^2 \right)^{\frac{1}{2}},$$

$$\rho(\boldsymbol{x}) = \sum_{i=1}^{2} \sum_{j=1}^{m} (\varphi_{ji}(\boldsymbol{x}))^2.$$

Thus

$$\begin{aligned}
\left| \sum_{j=1}^{m} (\boldsymbol{\varphi}_j, \boldsymbol{u}, \boldsymbol{\varphi}_j) \right| &\leqslant \int_\Omega |\nabla \boldsymbol{u}(\boldsymbol{x})| \rho(\boldsymbol{x})\, d\lambda(\boldsymbol{x}) \\
&\leqslant |\rho|\, |\nabla \boldsymbol{u}| \quad \text{(Hölder)} \\
&\leqslant |\rho|\, \|\boldsymbol{u}\|.
\end{aligned}$$

At this point we have established the estimate

$$\operatorname{Tr}\left(F'(\boldsymbol{u}(\tau))\circ Q_m(\tau)\right) \leqslant -\sum_{j=1}^{m}(A\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_j) + |\rho|\,\|\boldsymbol{u}\|.$$

Applying the weighted Sobolev-Lieb-Thirring inequality (5.2), there exists d_1 independent of the family $\{\boldsymbol{\varphi}_j\}$ and of m such that

$$\int_\Omega \rho(\boldsymbol{x})^2\,d\lambda(\boldsymbol{x}) \leqslant d_1\left(\sum_{j=1}^{m} a(\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_j)\right).$$

Set $\omega = 1/c_1^2$. By the variational principle and the spectral estimate (5.1), there exists d_2 such that

$$\sum_{j=1}^{m} a(\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_j) \geqslant d_2\omega m^2.$$

Substituting, we have

$$\begin{aligned}
\operatorname{Tr}\left(F'(\boldsymbol{u}(\tau))\circ Q_m(\tau)\right) &\leqslant -\sum_{j=1}^{m}(A\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_j) + \|\boldsymbol{u}\|\left(d_1\sum_{j=1}^{m} a(\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_j)\right)^{\frac{1}{2}}\\
&\leqslant -\sum_{j=1}^{m}(A\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_j) + \frac{\|\boldsymbol{u}\|^2 d_1}{2} + \frac{1}{2}\sum_{j=1}^{m} a(\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_j)\\
&= -\frac{1}{2}\sum_{j=1}^{m} a(\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_j) + \frac{\|\boldsymbol{u}\|^2 d_1}{2}\\
&\leqslant -\frac{d_2\omega m^2}{2} + \frac{\|\boldsymbol{u}\|^2 d_1}{2}
\end{aligned}$$

and therefore

$$\frac{1}{t}\int_0^t \operatorname{Tr}\left(F'(\boldsymbol{u}(\tau))\circ Q_m(\tau)\right)d\tau \leqslant -\frac{d_2\omega m^2}{2} + \frac{d_1}{2}\left(\frac{1}{t}\int_0^t \|\boldsymbol{u}(\tau)\|^2\,d\tau\right).$$

Define

$$\epsilon := \alpha\omega \varlimsup_{t\to\infty}\sup_{\boldsymbol{u}_0\in\mathcal{A}}\frac{1}{t}\int_0^t \|\boldsymbol{u}(s)\|^2\,ds.$$

Integrating the estimate

$$\frac{d}{dt}|\boldsymbol{u}|^2 + \alpha\|\boldsymbol{u}\|^2 \leqslant \frac{1}{\alpha\omega}|\boldsymbol{f}|^2$$

in time, one has

$$\frac{1}{t}|\boldsymbol{u}(t)|^2 + \frac{\alpha}{t}\int_0^t \|\boldsymbol{u}(s)\|^2\,ds \leqslant \frac{1}{t}|\boldsymbol{u}_0|^2 + \frac{1}{\alpha\omega}|\boldsymbol{f}|^2.$$

It follows that

$$\epsilon \leqslant G^2\alpha\omega^2,$$

where

$$G := \frac{|\boldsymbol{f}|}{\alpha\omega}.$$

We conclude that

$$q_m(t) \leqslant -\frac{d_2 \omega m^2}{2} + \frac{d_1}{2} \sup_{\boldsymbol{u}_0 \in \mathcal{A}} \frac{1}{t} \int_0^t \|\boldsymbol{u}\|^2 \, d\tau,$$

and thus

$$q_m = \overline{\lim_{t \to \infty}} \, q_m(t) \leqslant -\gamma_1 m^2 + \gamma_2,$$

where

$$\begin{cases} \gamma_1 = \dfrac{d_2 \omega}{2}, \\ \gamma_2 = \dfrac{d_1 \epsilon}{2 \alpha \omega}. \end{cases}$$

Applying the aforementioned Theorem III.3.2 of [5], one sees that the Hausdorff and box dimensions of $\mathcal{A}$ are bounded above by

$$\left(\frac{\gamma_2}{\gamma_1} \right)^{\frac{1}{2}}.$$

Notice that

$$\left(\frac{\gamma_2}{\gamma_1} \right)^{\frac{1}{2}} \leqslant \left(\frac{d_1}{d_2} \right)^{\frac{1}{2}} G.$$

5. The Weighted Lieb-Thirring Inequality and the Spectral Estimate

We prove the spectral estimate for the operator A and outline the proof of the weighted Sobolev-Lieb-Thirring inequality.

Proposition 5.1. *There exists a constant κ_1 such that the eigenvalues λ_j of the operator A satisfy*

$$\lambda_j \geqslant \kappa_1 j.$$

Proof. The argument follows the proof of Theorem 4.11 of [7]. Recall that $(\boldsymbol{w}_j)$ denotes the sequence of eigenfunctions of A corresponding to the sequence (λ_j) of eigenvalues of A. Let $\alpha_1, \ldots, \alpha_j \in \mathbb{R}$ and let

$$\boldsymbol{w} = \sum_{k=1}^{j} \alpha_k \boldsymbol{w}_k.$$

Interpolating between $\mathbf{L}_b^2(\Omega)$ and $\mathbf{H}_b^2(\Omega)$, one has

$$\|\boldsymbol{w}\|_{\mathbf{L}_b^\infty(\Omega)} \leqslant k_1 |\boldsymbol{w}|_{\mathbf{L}_b^2(\Omega)}^{1/2} \|\boldsymbol{w}\|_{\mathbf{H}_b^2(\Omega)}^{1/2}.$$

The Agmon-Douglis-Nirenberg elliptic regularity estimate (3.2) gives

$$\|\boldsymbol{w}\|_{\mathbf{H}_b^2(\Omega)} \leqslant c(1 + c_2) |A\boldsymbol{w}|$$

and hence

$$\|\boldsymbol{w}\|_{\mathbf{L}_b^\infty(\Omega)} \leqslant k_2 |\boldsymbol{w}|^{\frac{1}{2}} |A\boldsymbol{w}|^{\frac{1}{2}}. \tag{5.1}$$

Bounding $|A\boldsymbol{w}|^2$, we have

$$\begin{aligned} |A\boldsymbol{w}|^2 &= \sum_{k=1}^{j} \lambda_k^2 \alpha_k^2 \\ &\leqslant \lambda_j^2 \sum_{k=1}^{j} \alpha_k^2. \end{aligned}$$

Applying this bound to (5.1), one has

$$\begin{aligned}\|\boldsymbol{w}\|_{\mathbf{L}_b^\infty(\Omega)} &\leqslant k_2 \left(\sum_{k=1}^{j} \alpha_k^2\right)^{\frac{1}{4}} \lambda_j^{\frac{1}{2}} \left(\sum_{k=1}^{j} \alpha_k^2\right)^{\frac{1}{4}} \\ &= k_2 \lambda_j^{\frac{1}{2}} \left(\sum_{k=1}^{j} \alpha_k^2\right)^{\frac{1}{2}}.\end{aligned}$$

We have established that $|\boldsymbol{w}(\boldsymbol{x})|^2 \leqslant k_3 \lambda_j \sum_{k=1}^{j} \alpha_k^2$ for almost every $\boldsymbol{x} \in \Omega$. In fact, this holds for all $\boldsymbol{x} \in \Omega$ by Sobolev embeddings. Let $1 \leqslant i \leqslant 2$. One has

$$\left|\sum_{k=1}^{j} \alpha_k w_k^{(i)}(\boldsymbol{x})\right|^2 \leqslant |\boldsymbol{w}(\boldsymbol{x})|^2 \leqslant k_3 \lambda_j \sum_{k=1}^{j} \alpha_k^2.$$

Setting $\alpha_k = w_k^{(i)}(\boldsymbol{x})$, we obtain

$$\sum_{k=1}^{j} |w_k^{(i)}(\boldsymbol{x})|^2 \leqslant k_3 \lambda_j.$$

Summing over i,

$$\sum_{k=1}^{j} |\boldsymbol{w}_k(\boldsymbol{x})|^2 \leqslant 2 k_3 \lambda_j$$

for each $\boldsymbol{x} \in \Omega$. Integration over Ω yields the spectral estimate. □

Proposition 5.2 (Weighted Lieb-Thirring Inequality)**.** *Let* $\{\boldsymbol{\varphi}_j \in V, j = 1, \ldots, m\}$ *be an orthonormal set in* H. *For almost every* $\boldsymbol{x} \in \Omega$ *set*

$$\rho(\boldsymbol{x}) = \sum_{j=1}^{m} |\boldsymbol{\varphi}_j(\boldsymbol{x})|^2.$$

For p *satisfying* $1 < p \leqslant 2$ *one has*

$$\left(\int_\Omega \rho(\boldsymbol{x})^{\frac{p}{p-1}} \, d\lambda(\boldsymbol{x})\right)^{p-1} \leqslant \kappa_2 \left(\sum_{j=1}^{m} a(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_j)\right)$$

where κ_2 *is independent of the family* $\{\boldsymbol{\varphi}_j\}$ *and of* m.

Proof. One checks that the arguments given in [16] and the appendix of [20] may be adapted to the case in which the weighted measure λ replaces the Lebesgue measure. We proceed initially by assuming that the operator A satisfies the following hypotheses.

- **(H1)** There exists a constant κ_1 such that the eigenvalues λ_j of the operator A satisfy $\lambda_j \geqslant \kappa_1 j$.
- **(H2)** For each $r > 0$, the operator $(A + r)^{-1} \in \mathcal{L}(V', V)$ extends as a linear continuous operator from $\mathbf{L}_b^s(\Omega)$ into $V \cap \mathbf{W}_b^{2,s}(\Omega)$ for $1 < s < \infty$. This operator considered as an operator on $\mathbf{L}_b^2(\Omega)$ is positive.
- **(H3)** The eigenfunctions $\boldsymbol{w}_j$ of A are uniformly bounded in $\mathbf{L}_b^\infty$.

Hypothesis $H3$ is very strong as it is not true in general and often very difficult to verify when true. Donnelly [13] shows that if an n-dimensional compact Riemannian manifold M admits an isometric circle action, and if the metric is generic, then one has eigenfunctions of the Laplacian corresponding to the eigenvalue γ_k satisfying

$$\|\phi_k\|_\infty \geqslant C\gamma_k^{\frac{n-1}{8}}\|\phi\|_2.$$

Let $p > 2$ and let $f \in L_b^p(\Omega)$. The form

$$a(\boldsymbol{u}, \boldsymbol{v}) + \int_\Omega (f + \alpha)\boldsymbol{u} \cdot \boldsymbol{v}\, d\lambda$$

is bilinear, continuous, and coercive on V for an appropriate choice of the translate α. Therefore, H has an orthonormal basis consisting of eigenfunctions of the Schrödinger-type operator $A_f = A + f$. Let $(\mu_j(f))$ denote the increasing sequence of eigenvalues of $A + f$. Using the Birman-Schwinger inequality [19], one obtains an estimate on the negative part of the spectrum of $A + f$ in terms of a phase space integral involving f. For $0 < \beta \leqslant 1$, there exists $\gamma_1 = \gamma_1(\beta)$ such that

$$\sum_{\mu_j < 0} |\mu_j| \leqslant \gamma_1 \left[\int_\Omega (f_-(\boldsymbol{x}))^{\beta+1} d\lambda\right]^{\frac{1}{\beta}}.$$

This spectral estimate makes crucial use of (H3). The weighted Sobolev-Lieb-Thirring inequality now follows by setting $f = -\alpha\rho^{1/(p-1)}$ for an appropriate value of α and studying the unbounded operator A_f^m on $\bigwedge^m H$ defined by

$$A_f^m(\boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_m) = (A_f\boldsymbol{u}_1 \wedge \boldsymbol{u}_2 \wedge \cdots \wedge \boldsymbol{u}_m) + \cdots + (\boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_{m-1} \wedge A_f\boldsymbol{u}_m).$$

The general weighted Sobolev-Lieb-Thirring inequality reduces to the case of the negative Laplacian with periodic boundary conditions, an operator for which (H1)-(H3) hold. $\square$

Remark 5.3. See [14] for other interesting generalizations of the Sobolev-Lieb-Thirring inequalities.

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