When Lyapunov exponents fail to exist

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We describe a simple continuous-time flow such that Lyapunov exponents fail to exist at nearly every point in the phase space $\mathbb{R}^2$ despite the fact that the flow admits a unique natural measure. This example illustrates that the existence of Lyapunov exponents is a subtle question for systems that are not conservative.

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I. INTRODUCTION

Scientists often compute Lyapunov exponents without addressing whether or not the exponents actually exist [1–4]. Lyapunov exponents measure the exponential rates of contraction and expansion along orbits of dynamical systems. Given a dynamical system and a randomly chosen point $\boldsymbol{x}$ in the phase space, do Lyapunov exponents exist for the orbit of $\boldsymbol{x}$? In this paper we prove the surprising result that for the simple flow with a 'Figure-8' attractor depicted in Figure 1, Lyapunov exponents do not exist for nearly every trajectory in the phase space.

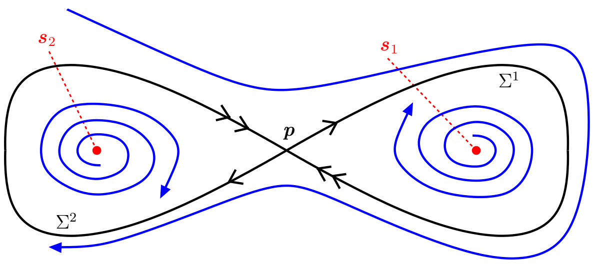

FIG. 1: The Figure-8 attractor. The point $\boldsymbol{p}$ is a saddle with more contraction than expansion, $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are repelling foci, and the curves Σ^1 and Σ^2 are invariant loops. No trajectory that converges to the Figure-8 has Lyapunov exponents except for the trajectories that start on the Figure-8.

One might think that because Lyapunov exponents are asymptotic quantities, it should follow from some ergodic theorem that Lyapunov exponents do exist for the orbit of a randomly chosen point. However, ergodic theorems are stated in terms of invariant measures. The multiplicative ergodic theorem of Oseledec [5] states that for an invariant measure μ, Lyapunov exponents exist for the orbit of μ-almost every (a.e.) point $\boldsymbol{x}$. Therefore, Lyapunov exponents exist for the orbit of a randomly chosen point $\boldsymbol{x}$ ***with respect to*** $\boldsymbol{\mu}$. If the dynamical system is conservative (preserves a measure equivalent to Lebesgue measure), then the multiplicative ergodic theorem does imply that Lyapunov exponents exist for Lebesgue almost every point in the phase space.

What about attractors? Suppose that the dynamical system is dissipative and that it admits an attractor with an open basin of attraction. Since the dynamics inside the basin are dissipative, every invariant measure supported inside the basin must be supported on the attractor (zero away from the attractor) and must be singular with respect to Lebesgue measure (supported on a set of Lebesgue measure zero). The multiplicative ergodic theorem does not say anything about a point $\boldsymbol{x}$ in the basin that is not on the attractor itself.

Let $\boldsymbol{\varphi}(\boldsymbol{x}, t)$ denote a dissipative flow on $\mathbb{R}^n$ that admits an attractor A with open basin of attraction U. Even though any $\boldsymbol{\varphi}$-invariant measure ν supported in U must be supported on A and must be singular with respect to Lebesgue measure, such a measure can nevertheless organize the statistics of large sets of orbits in the basin. We call ν a natural measure if there exists a set $E \subset U$ of positive Lebesgue measure such that for $\boldsymbol{x} \in E$, ν governs the statistics of the orbit of $\boldsymbol{x}$ in the following sense. For every continuous function (or observable) $\psi : U \to \mathbb{R}$, we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \psi(\boldsymbol{\varphi}(\boldsymbol{x}, s))\, ds = \int_U \psi(\boldsymbol{x})\, d\nu(\boldsymbol{x}).$$

That is, the time average of ψ along the orbit of $\boldsymbol{x}$ is equal to the spatial average of ψ with respect to ν. It has been shown that natural measures exist for several classes of dynamical systems. See [6, 7] for expository surveys in this direction. Nevertheless, there exist simple dynamical systems that do not have natural measures and there exist many complicated dynamical systems that are not known to have natural measures.

What is the relationship between natural measures and Lyapunov exponents? If $\boldsymbol{\varphi}$ admits no natural measure or at least 2 natural measures, it is reasonable to suspect that Lyapunov exponents do not exist for most points in the basin because the presence of no natural measure or at least 2 natural measures indicates permanent oscillation in the flow. Indeed, this phenomenon is well-known to nonlinear scientists. In Section IV we analyze an explicit example of a flow on $\mathbb{R}^2$ that does not admit a natural measure. We prove that Lyapunov exponents do not exist for the orbit of Lebesgue almost every $\boldsymbol{x} \in \mathbb{R}^2$.

What if the system admits a unique natural measure? We show that even in this case, it is possible for Lyapunov exponents to fail to exist for the orbit of Lebesgue almost every point in the phase space. In Section III, we prove that the flow depicted in Figure 1 admits a unique natural measure but no trajectory that converges to the Figure-8 has Lyapunov exponents except for the trajectories that start on the Figure-8. To our knowledge, this is the first time that the nonexistence of Lyapunov exponents has been rigorously established in a system with a unique natural measure. This result is the main contribution of our paper.

For the examples we study, the following mechanism causes Lyapunov exponents to fail to exist. Let $\boldsymbol{x}$ be a point in the basin of attraction. The finite-time Lyapunov exponent (2) for the direction of the flow perpetually oscillates as $t \to \infty$, causing the infinite-time Lyapunov exponent (3) for the flow direction to fail to converge. Volume along the orbit of $\boldsymbol{x}$ contracts at an exponential rate. These two properties imply that the Lyapunov exponent computed in ***any*** direction at $\boldsymbol{x}$ fails to converge. We present one explicit example that admits a unique natural measure and one explicit example that admits no natural measure. In each example, the Lyapunov exponent fails to exist for every nonzero vector $\boldsymbol{v}$ at ***every*** point in the basin that is not on the attractor.

We focus on two-dimensional flows with homoclinic attractors or heteroclinic attractors. We choose this setting to illustrate that the mechanism described above can appear even in relatively simple systems. The study of homoclinic/heteroclinic phenomena has a rich history. The presence of such orbits often has significant dynamical implications. For example, sensitivity to detuning in networks of coupled oscillators can be caused by the existence of heteroclinic cycles [8]. In general, the mechanism described above must be considered when asking about the existence of Lyapunov exponents in any given system.

The nonexistence of Lyapunov exponents has significant implications. This is especially true when a finite-time Lyapunov exponent fluctuates about zero. Such fluctuations are associated with the loss of shadowability of orbits [9] and with the hypersensitivity of invariant measures to noise [10]. Finite-time Lyapunov exponents can fluctuate on long time scales in high-dimensional systems exhibiting 'chaotic itinerancy' [11].

II. THEORY OF LYAPUNOV EXPONENTS

We now review the theory of Lyapunov exponents. Consider the autonomous differential equation

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}) \tag{1}$$

where $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$. Let $\boldsymbol{\varphi}(\boldsymbol{x}, t)$ be the solution of (1) at time t with initial condition $\boldsymbol{x}$ at time $t = 0$. We refer to $\boldsymbol{\varphi}$ as a flow. Assume there exists a compact region $M \subset \mathbb{R}^n$ such that $\boldsymbol{\varphi}(M, t) \subset M$ for all $t \geqslant 0$. We study the flow on M.

For $\boldsymbol{x} \in M$, $\boldsymbol{v} \in \mathbb{R}^n$, and $t > 0$, define

$$\begin{aligned} \lambda_t(\boldsymbol{x}, \boldsymbol{v}) &= \frac{1}{t} \log \|D\boldsymbol{\varphi}(\boldsymbol{x}, t)\boldsymbol{v}\|, \qquad (2)\\ \lambda^*(\boldsymbol{x}, \boldsymbol{v}) &= \limsup_{t\to\infty} \lambda_t(\boldsymbol{x}, \boldsymbol{v}), \\ \lambda_*(\boldsymbol{x}, \boldsymbol{v}) &= \liminf_{t\to\infty} \lambda_t(\boldsymbol{x}, \boldsymbol{v}), \end{aligned}$$

where D denotes the spatial derivative. The value $\lambda_t(\boldsymbol{x}, \boldsymbol{v})$ is the ***finite-time Lyapunov exponent*** associated with $\boldsymbol{x}$ and $\boldsymbol{v}$ evaluated at time t. If $\lambda_*(\boldsymbol{x}, \boldsymbol{v}) = \lambda^*(\boldsymbol{x}, \boldsymbol{v})$, the common value

$$\lambda(\boldsymbol{x}, \boldsymbol{v}) = \lim_{t\to\infty} \frac{1}{t} \log \|D\boldsymbol{\varphi}(\boldsymbol{x}, t)\boldsymbol{v}\| \tag{3}$$

is the ***Lyapunov exponent*** associated with $\boldsymbol{x}$ and $\boldsymbol{v}$. The quantities $\lambda^*(\boldsymbol{x}, \boldsymbol{v})$ and $\lambda_*(\boldsymbol{x}, \boldsymbol{v})$ are called the upper and lower Lyapunov exponents associated with $\boldsymbol{x}$ and $\boldsymbol{v}$. A point $\boldsymbol{x} \in M$ is said to be ***Lyapunov regular*** if there exist values

$$-\infty \leqslant \lambda_1(\boldsymbol{x}) \leqslant \lambda_2(\boldsymbol{x}) \leqslant \cdots \leqslant \lambda_n(\boldsymbol{x})$$

and linear subspaces $V_k(\boldsymbol{x}) \subset \mathbb{R}^n$ of dimension k satisfying

$$\{\boldsymbol{0}\} = V_0(\boldsymbol{x}) \subset V_1(\boldsymbol{x}) \subset V_2(\boldsymbol{x}) \subset \cdots \subset V_n(\boldsymbol{x}) = \mathbb{R}^n$$

such that $\lambda(\boldsymbol{x}, \boldsymbol{v}) = \lambda_i(\boldsymbol{x})$ for every $1 \leqslant i \leqslant n$ and for every $\boldsymbol{v} \in V_i(\boldsymbol{x})$ except for $\boldsymbol{v} \in V_{i-1}(\boldsymbol{x})$. The values $\lambda_i(\boldsymbol{x})$ are the Lyapunov exponents associated with $\boldsymbol{x}$.

The multiplicative ergodic theorem of Oseledec [5] states that for a $\boldsymbol{\varphi}$-invariant probability measure μ on M, μ-almost every $\boldsymbol{x}$ is Lyapunov regular. On the set of Lyapunov regular points, the values $\lambda_i(\boldsymbol{x})$ are flow-invariant and depend measurably on $\boldsymbol{x}$. The functions λ_i are constant μ-a.e. if μ is ergodic. In this case, we think of the λ_i as constants and we refer to them as the Lyapunov exponents associated with the measure μ.

Lyapunov exponents express the asymptotic regularity of the action of the spatial derivative along orbits. One may ask about the statistical coherence of the orbits themselves. The notion of natural measure addresses this line of inquiry. Let ν be a $\boldsymbol{\varphi}$-invariant probability measure. The point $\boldsymbol{x} \in M$ is said to be ***$\boldsymbol{\nu}$-generic*** if for every continuous function $\psi : M \to \mathbb{R}$, we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \psi(\boldsymbol{\varphi}(\boldsymbol{x}, s))\, ds = \int_M \psi(\boldsymbol{x})\, d\nu(\boldsymbol{x}).$$

That is, the time average of ψ along the orbit of $\boldsymbol{x}$ is equal to the spatial average of ψ with respect to ν. The measure ν is said to be a ***natural measure*** if the set of ν-generic points has positive Lebesgue measure in M.

Natural measures are observable in the sense that with positive probability, the orbit of a randomly chosen point $\boldsymbol{x}$ (in the basin) will be asymptotically distributed according to ν.

The notion of natural measure described above is a pathwise notion. There exist 2 additional commonly-used notions of natural (or SRB) measure. In the first alternative, one tracks the statistics of an ensemble of initial data rather than the statistics of an individual orbit. The second alternative is based on the observation that for some dynamical systems with strong stochastic properties, there exist special invariant measures with absolutely continuous conditional measures on unstable manifolds. See [6, 7] and [12, Section 2] for discussions about the various notions of natural measure.

Natural measures may be thought of as the phases of a system. A change in the number of natural measures can be interpreted as a phase transition. Blank and Bunimovich [12] study this idea in the context of coupled maps.

We now describe 2 flows that exhibit the mechanism described in the introduction. Each flow has a unique attractor and in each case, Lyapunov exponents fail to exist for every point in the basin that is not on the attractor. In the first example, the attractor supports a unique natural measure that describes the asymptotic distribution of the orbit of every point in the phase space except for two unstable equilibria.

III. EXAMPLE 1: A UNIQUE NATURAL INVARIANT MEASURE EXISTS

Let $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defining the flow depicted in Figure 1 and let $\boldsymbol{\varphi}(\boldsymbol{x}, t)$ denote the flow generated by $\boldsymbol{f}$. The equilibrium point $\boldsymbol{p}$ is a saddle with eigenvalues $-\alpha$ and γ satisfying $\alpha > \gamma > 0$. The saddle is dissipative because $-\alpha + \gamma < 0$. The stable and unstable manifolds of $\boldsymbol{p}$ coincide and form the homoclinic loops Σ^1 and Σ^2. The set $\mathcal{A} = \{\boldsymbol{p}\} \cup \Sigma^1 \cup \Sigma^2$ is the Figure-8 attractor.

For $\boldsymbol{x} \in \mathbb{R}^2 \setminus \{\boldsymbol{s}_1, \boldsymbol{s}_2\}$, the orbit $\boldsymbol{\varphi}(\boldsymbol{x}, \cdot)$ spends all of its time near $\boldsymbol{p}$ in the limit. The δ-measure $\delta_{\boldsymbol{p}}$ is therefore the unique natural measure for the flow $\boldsymbol{\varphi}$. The orbit of every point in the phase space (except for the two unstable foci) is asymptotically distributed according to $\delta_{\boldsymbol{p}}$.

Let $\mathcal{B}$ be all of $\mathbb{R}^2$ except for $\mathcal{A}$, $\boldsymbol{s}_1$, and $\boldsymbol{s}_2$.We prove that the Lyapunov exponent $\lambda(\boldsymbol{x}, \boldsymbol{v})$ fails to exist for all $\boldsymbol{v} \neq \boldsymbol{0}$ and for all $\boldsymbol{x} \in \mathcal{B}$. The proof consists of two steps. First, we show that the ***flow Lyapunov exponent*** $\lambda(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x}))$ does not exist because $\lambda_*(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})) < 0$ and $\lambda^*(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})) = 0$. Second, we show that volume contracts asymptotically at a definite exponential rate along the orbit of $\boldsymbol{x}$.

We give the argument for $\boldsymbol{x}$ is located inside Σ^1. The arguments for points located inside Σ^2 and outside the Figure-8 are similar. For simplicity, we assume we can choose coordinates such that $\boldsymbol{\varphi}$ has the following properties. In the rectangle $R = \{(x, y) \in \mathbb{R}^2 : |x| \leqslant 1 \text{ and } |y| \leqslant 1\}$, the differential equation $\frac{dx}{dt} = \boldsymbol{f}(\boldsymbol{x})$ has the linear form $\frac{dx}{dt} = A\boldsymbol{x}$ with

$$A = \begin{pmatrix} -\alpha & 0 \\ 0 & \gamma \end{pmatrix}.$$

See Figure 2. Loop Σ^1 is located in the first quadrant and contains the segments $\{(0, y) : 0 < y \leqslant 1\}$ and $\{(x, 0) : 0 < x \leqslant 1\}$. Fix $0 < \xi \ll 1$ and define transversals $S_1 = \{(1, y) : 0 < y \leqslant \xi\}$ and $S_2 = \{(x, 1) : 0 < x \leqslant \xi\}$. The flow maps S_2 into S_1. This map is given by $(x, 1) \mapsto (1, ax)$ for some $0 < a \leqslant 1$.

Completion of the argument assuming the flow Lyapunov exponent does not exist. Fix $\boldsymbol{x}_0$ inside Σ^1 ($\boldsymbol{x}_0 \neq \boldsymbol{s}_1$). Assume that the flow Lyapunov exponent does not exist and let $\boldsymbol{v}$ be any nonzero vector not parallel to $\boldsymbol{f}(\boldsymbol{x}_0)$. Let $\Phi(\boldsymbol{x}_0, t)$ be the matrix solution of the variational equation

$$\frac{d\boldsymbol{w}}{dt} = D\boldsymbol{f}(\boldsymbol{\varphi}(\boldsymbol{x}_0, t))\boldsymbol{w} \tag{4}$$

with initial data

$$\Phi(\boldsymbol{x}_0, 0) = \begin{pmatrix} \boldsymbol{f}(\boldsymbol{x}_0) & \boldsymbol{v} \end{pmatrix}$$

The determinant $\det(\Phi(\boldsymbol{x}_0, t))$ satisfies

$$\det(\Phi(\boldsymbol{x}_0, t)) = e^{\int_0^t \operatorname{tr}(D\boldsymbol{f}(\boldsymbol{\varphi}(\boldsymbol{x}_0, s)))\, ds} \det(\Phi(\boldsymbol{x}_0, 0)).$$

Since

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \operatorname{tr}(D\boldsymbol{f}(\boldsymbol{\varphi}(\boldsymbol{x}_0, s)))\, ds = \operatorname{tr}(A) = \gamma - \alpha,$$

it follows that

$$\det(\Phi(\boldsymbol{x}_0, t)) \approx e^{(\gamma-\alpha)t} \det(\Phi(\boldsymbol{x}_0, 0))$$

for large values of t. Consequently, $\lambda(\boldsymbol{x}_0, \boldsymbol{v})$ does not exist because $\lambda(\boldsymbol{x}_0, \boldsymbol{f}(\boldsymbol{x}_0))$ does not exist.

Proof that the flow Lyapunov exponent does not exist. The structure of the local flow from S_1 to S_2 plays the central role in the proof that $\lambda_*(\boldsymbol{x}_0, \boldsymbol{f}(\boldsymbol{x}_0)) < 0$. Let $\boldsymbol{y} = (1, y) \in S_1$. The trajectory $\boldsymbol{\varphi}(\boldsymbol{y}, \cdot)$ is given by

$$\boldsymbol{\varphi}(\boldsymbol{y}, t) = \begin{pmatrix} e^{-\alpha t} \\ y e^{\gamma t} \end{pmatrix}$$

until it crosses S_2. Let $s = s(y)$ be the first time the orbit $\boldsymbol{\varphi}(\boldsymbol{y}, \cdot)$ meets S_2. We have

$$s(y) = \frac{1}{\gamma} \log(y^{-1}), \quad \varphi_1(\boldsymbol{y}, s(y)) = y^{\frac{\alpha}{\gamma}}.$$

Let $\tau = \tau(y)$ denote the time t satisfying $0 < t < s(y)$ that minimizes $\|\boldsymbol{\varphi}(\boldsymbol{y}, t)\|$. We have

$$\tau(y) = \frac{1}{\gamma + \alpha} \log(y^{-1}) + K_1(\alpha, \gamma),$$

where $K_1(\alpha, \gamma)$ is a constant. Evaluating $\|\boldsymbol{\varphi}(\boldsymbol{y}, \tau(y))\|$, we obtain

$$\|\boldsymbol{\varphi}(\boldsymbol{y}, \tau(y))\| = K_2(\alpha, \gamma) y^{\frac{\alpha}{\gamma+\alpha}},$$

where $K_2(\alpha, \gamma)$ is a constant. The analysis of the local flow is complete.

Since $\frac{d\boldsymbol{\varphi}}{dt}$ satisfies (4), we have

$$\boldsymbol{f}(\boldsymbol{\varphi}(\boldsymbol{x}_0, t)) = D\boldsymbol{\varphi}(\boldsymbol{x}_0, t)\boldsymbol{f}(\boldsymbol{x}_0). \tag{5}$$

Using (5), we have

$$\lambda_*(\boldsymbol{x}_0, \boldsymbol{f}(\boldsymbol{x}_0)) = \liminf_{t\to\infty} \frac{1}{t} \log \|\boldsymbol{f}(\boldsymbol{\varphi}(\boldsymbol{x}_0, t))\|.$$

Define sequences $(\boldsymbol{y}_n) \subset S_1$ and $(\boldsymbol{z}_n) \subset S_2$ as follows. Let $\hat{t}$ denote the time at which the orbit $\boldsymbol{\varphi}(\boldsymbol{x}_0, \cdot)$ first crosses S_1. Let $y_0 = \varphi_2(\boldsymbol{x}_0, \hat{t})$ and $\boldsymbol{y}_0 = (1, y_0)$. Define $\boldsymbol{z}_0 = \boldsymbol{\varphi}(\boldsymbol{y}_0, s(y_0))$. We have $\boldsymbol{z}_0 = (z_0, 1) = (y_0^\beta, 1)$, where $\beta = \frac{\alpha}{\gamma}$. For $n \geqslant 1$, let $\boldsymbol{y}_n = (1, y_n)$ and $\boldsymbol{z}_n = (z_n, 1)$ denote the n^{th} intersections of the trajectory $\boldsymbol{\varphi}(\boldsymbol{z}_0, \cdot)$ with S_1 and S_2, respectively. Computing y_n and z_n, we have

$$y_n = a^{\frac{1-\beta^n}{1-\beta}} y_0^{\beta^n} \text{ and } z_n = a^{\frac{\beta-\beta^{n+1}}{1-\beta}} y_0^{\beta^{n+1}}.$$

Figure 2 illustrates the flow from $\boldsymbol{y}_n$ to $\boldsymbol{z}_n$.

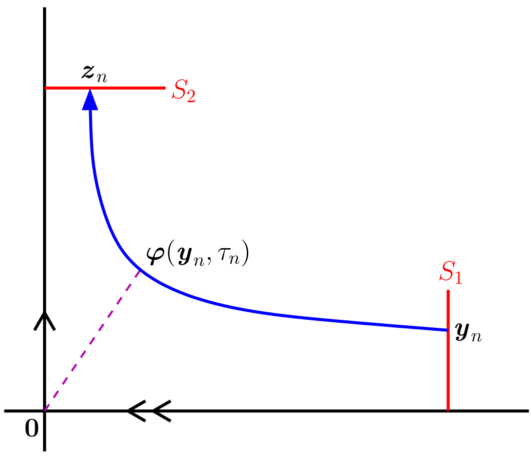

FIG. 2: The flow from $\boldsymbol{y}_n$ to $\boldsymbol{z}_n$.

Set $\tau_n = \tau(y_n)$ and $s_n = s(y_n)$. Let q_n be the time at which the orbit $\boldsymbol{\varphi}(\boldsymbol{z}_n, \cdot)$ first crosses S_1. Define the sequence of times (T_n) by setting $T_0 = \hat{t} + \tau_0$ and

$$T_n = \hat{t} + \sum_{j=0}^{n-1}(s_j + q_j) + \tau_n$$

for $n \geqslant 1$. Calculating the evolution of $\boldsymbol{f}(\boldsymbol{x}_0)$ along the sequence (T_n), we obtain

$$\lim_{n\to\infty} \frac{1}{T_n} \log \|\boldsymbol{f}(\boldsymbol{\varphi}(\boldsymbol{x}_0, T_n))\| = \lim_{n\to\infty} \frac{1}{T_n} \log \|\boldsymbol{\varphi}(\boldsymbol{x}_0, T_n)\| = \frac{\gamma - \alpha}{2} < 0.$$

Therefore, $\lambda_*(\boldsymbol{x}_0, \boldsymbol{f}(\boldsymbol{x}_0)) < 0$.

Now choose $\boldsymbol{\zeta} \in \Sigma^1$. Let $(\boldsymbol{\zeta}_n)$ be a sequence of points on the orbit of $\boldsymbol{x}_0$ such that $\boldsymbol{\zeta}_n \to \boldsymbol{\zeta}$ as $n \to \infty$ and let t_n be such that $\boldsymbol{\zeta}_n = \boldsymbol{\varphi}(\boldsymbol{x}_0, t_n)$. Since $\boldsymbol{f}(\boldsymbol{\zeta}_n) \to \boldsymbol{f}(\boldsymbol{\zeta})$ as $n \to \infty$, we conclude that

$$\lim_{n\to\infty} \frac{1}{t_n} \log \|\boldsymbol{f}(\boldsymbol{\varphi}(\boldsymbol{x}_0, t_n))\| = 0$$

and therefore $\lambda^*(\boldsymbol{x}_0, \boldsymbol{f}(\boldsymbol{x}_0)) = 0$.

The flow $\boldsymbol{\varphi}(\boldsymbol{x}, t)$ analyzed in Example 1 is not generic in the space of smooth flows on $\mathbb{R}^2$ because the stable and unstable manifolds of the hyperbolic saddle coincide. Nevertheless, homoclinic phenomena of this type commonly occur in parametrized families of flows on $\mathbb{R}^2$ and often are a source of rich dynamical structures as the parameters are varied [13].

IV. EXAMPLE 2: NO NATURAL INVARIANT MEASURES EXIST

We analyze a flow with four dissipative saddles. This flow admits no natural invariant measures and Lyapunov exponents fail to exist for Lebesgue almost every point in the phase space of the flow. Example 2 is pedagogical in nature. We include it as a simple example in the spirit of the work of Barreira and Schmeling [14] on nonexistence of Lyapunov exponents in abstract dynamical systems. We thereby hope to bring this work to the attention of the physics community.

Let $S = \{(x, y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant \pi \text{ and } 0 \leqslant y \leqslant \pi\}$. Consider the following system of differential equations defined on S.

$$\begin{cases} \dfrac{dx}{dt} = \cos(y)\sin(x) - a\cos(x)\sin(x) \\ \dfrac{dy}{dt} = -\cos(x)\sin(y) - a\cos(y)\sin(y) \end{cases} \tag{6}$$

Here $a \in (0, 1)$. Markley [15, page 202] attributes the initial study of (6) to Anosov. System (6) generates the flow $\boldsymbol{\varphi}$ pictured in Figure 3. Let $\boldsymbol{f}(\boldsymbol{x})$ denote the right side of (6).

The corners $\boldsymbol{p}_1 = (0, 0)$, $\boldsymbol{p}_2 = (\pi, 0)$, $\boldsymbol{p}_3 = (\pi, \pi)$, and $\boldsymbol{p}_4 = (0, \pi)$ are saddle equilibria. The eigenvalues of the linearizations of (6) at each of the four corners are $\xi_1 = 1 - a$ and $\xi_2 = -1 - a$. Notice that $\xi_1 > 0$, $\xi_2 < 0$, and $\xi_1 + \xi_2 = -2a < 0$. The corners are therefore dissipative saddles. The fifth and final equilibrium point $\boldsymbol{s} = (\frac{\pi}{2}, \frac{\pi}{2})$ is an unstable focus. Let $V : S \to \mathbb{R}$ be defined by $V(x, y) = \sin(x)\sin(y)$. Differentiating V along trajectories of (6), we have

$$\frac{dV(x(t), y(t))}{dt} = -a\sin(x(t))\sin(y(t)) \times [\cos^2(x(t)) + \cos^2(y(t))].$$

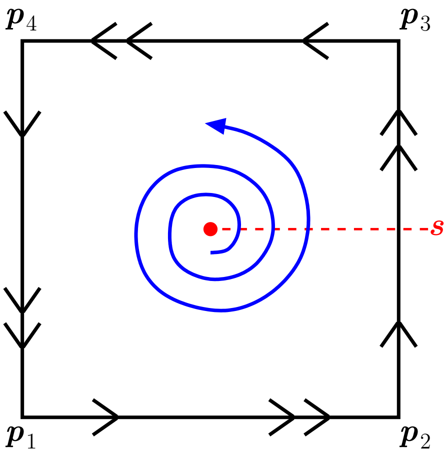

FIG. 3: The flow on the square $S = [0, \pi] \times [0, \pi]$ generated by (6).

Notice that $\frac{dV(x(t),y(t))}{dt} \leqslant 0$ with equality if and only if $(x(t), y(t))$ is on the boundary ∂S of S or $(x(t), y(t)) = (\frac{\pi}{2}, \frac{\pi}{2})$. Every nonstationary trajectory therefore converges to ∂S as $t \to \infty$.

Let $\mathcal{C}$ denote the interior of S excluding $\boldsymbol{s}$ and let $\boldsymbol{z}_0 \in \mathcal{C}$. The point $\boldsymbol{z}_0$ is not generic with respect to any measure because the orbit $\boldsymbol{\varphi}(\boldsymbol{z}_0, \cdot)$ eventually oscillates between small neighborhoods of the corners. Therefore, no natural invariant measure exists. The work of Gaunersdorfer [16] implies that as $t \to \infty$, the set of limit points of the temporal average

$$\frac{1}{t} \int_0^t \boldsymbol{\varphi}(\boldsymbol{z}_0, s)\, ds$$

form a polygon in S.

For every nonzero vector $\boldsymbol{v}$, the Lyapunov exponent $\lambda(\boldsymbol{z}_0, \boldsymbol{v})$ does not exist. One sees this by arguing as in the Figure-8 case.

Figure 4 provides numerical evidence that the finite-time flow Lyapunov exponent associated with any trajectory in $\mathcal{C}$ perpetually oscillates with a definite asymptotic amplitude and therefore does not converge. We have plotted the finite-time flow Lyapunov exponent $\lambda_t(\boldsymbol{x}_0, \boldsymbol{f}(\boldsymbol{x}_0))$ for $200 \leqslant t \leqslant 500$. Here $a = 0.03$ and $\boldsymbol{x}_0$ is such that $\boldsymbol{\varphi}(\boldsymbol{x}_0, 200) = (\frac{\pi+3}{2}, \frac{\pi+3}{2})$. Adapting the Figure-8 analysis to the square flow, we have $\lambda^*(\boldsymbol{x}_0, \boldsymbol{f}(\boldsymbol{x}_0)) = 0$ and

$$\lambda_*(\boldsymbol{x}_0, \boldsymbol{f}(\boldsymbol{x}_0)) = \frac{\xi_1 + \xi_2}{2} = -a.$$

Figure 4 is consistent with this analytical fact. We use the coordinate transformation

$$z = \tan\left(x - \frac{\pi}{2}\right)$$
$$w = \tan\left(y - \frac{\pi}{2}\right)$$

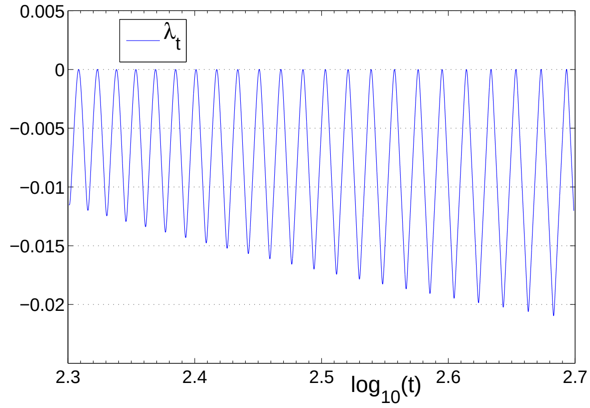

FIG. 4: The finite-time flow Lyapunov exponent associated with an arbitrarily chosen trajectory in $\mathcal{C}$. Here $a = 0.03$. The local minima converge to the limiting value $-a = -0.03$. Observe that the finite-time flow Lyapunov exponent function appears to converge to a periodic sawtooth function of $\log(t)$.

to perform the numerical integration. This change of variable circumvents the problems associated with integrating vector fields near equilibria.

Example 2 extends the analysis of Gaunersdorfer [16] in the sense that for $\boldsymbol{z}_0 \in \mathcal{C}$, we have explicitly computed the set of limit points of the finite-time Lyapunov exponent $\lambda_t(\boldsymbol{z}_0, \boldsymbol{f}(\boldsymbol{z}_0))$. This set is precisely the interval $[-a, 0]$.

V. DISCUSSION AND ACKNOWLEDGMENTS

We return to the question that motivates this paper. Do Lyapunov exponents exist for a randomly chosen point in the phase space? Examples 1 and 2 show that in the context of attractors, there exist flows for which Lyapunov exponents do not exist at every point in the basin that is not on the attractor. Example 1 shows that this can happen even if the flow admits a unique natural measure. The relationship between natural measures and Lyapunov exponents is subtle and complex.

Mathematicians have established the existence of natural invariant measures for many classes of chaotic systems. See [6, 7] for excellent expository surveys of this research. Example 1 demonstrates that even if a unique natural measure exists, Lyapunov exponents may fail to exist at every point in the basin that is not on the attractor. However, if a system admits a natural measure with certain nice properties, then Lyapunov exponents will exist for a large set of points. Tsujii [17] proves that an ergodic invariant measure μ with no zero Lyapunov exponents and at least one positive Lyapunov exponent has absolutely continuous conditional measures on unstable manifolds (such a measure is a natural measure) if and only if there exists a set R with positive Lebesgue measure such that for $\boldsymbol{x} \in R$, $\boldsymbol{x}$ is μ-generic and the Lyapunov exponents of $\boldsymbol{x}$ coincide with those of μ. Since the measure $\delta_{\boldsymbol{p}}$ in Example 1 is natural but not smooth along the unstable manifold, the result of Tsujii implies that the Lyapunov exponents of Lebesgue-a.e. point in the basin of $\mathcal{A}$ cannot be equal to $-\alpha$ and γ. Tsujii's theorem leaves the question of the existence of Lyapunov exponents unresolved in this case.

In the context of abstract dynamical systems, Barreira and Schmeling [14] show that Lyapunov exponents often do not exist. For a general class of dynamical systems that includes subshifts of finite type, conformal repellers, and conformal horseshoes, they prove that the set of points at which the Birkhoff ergodic average and the Lyapunov exponents simultaneously do not exist has ***full*** topological entropy and ***full*** Hausdorff dimension. This irregular set is maximally large from the point of view of dimension theory.

In light of the examples in this paper and the work of Tsujii, Barreira, and Schmeling, it is clear that the existence problem for Lyapunov exponents remains a major challenge.

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