MEMORY LOSS FOR NONEQUILIBRIUM OPEN DYNAMICAL SYSTEMS

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ABSTRACT. We introduce a notion of conditional memory loss for nonequilibrium open dynamical systems. We prove that this type of memory loss occurs at an exponential rate for nonequilibrium open systems generated by one-dimensional piecewise-differentiable expanding Lasota-Yorke maps. This result may be viewed as a prototype for time-dependent dynamical systems with holes.

CONTENTS

1. INTRODUCTION

This paper studies memory loss in nonequilibrium open dynamical systems. By *nonequilibrium* we mean that the dynamical model itself may vary with time. By *open* we mean that the phase space contains holes through which trajectories may escape. Memory loss in this setting is an analog of decay of correlations.

The memory loss problem has been studied extensively in the contexts of stochastic differential equations (SDEs), random dynamical systems[1], and autonomous (time-independent) deterministic dynamical systems. An SDE of the form

$$\mathrm{d}x_t = a(x_t)\,\mathrm{d}t + \sum_{i=1}^{n} b_i(x_t) \circ \mathrm{d}W_t^i$$

gives rise to a stochastic flow of diffeomorphisms in which almost every Brownian path defines a time-dependent flow (see *e.g.* [9]). Lyapunov exponents for such flows are known to be well-defined, nonrandom (they do not depend on the realization of the noise), and constant almost everywhere in phase space if the system is ergodic. Ergodic systems for which the greatest Lyapunov exponent $\lambda_{\max}$ is negative exhibit a phenomenon known as *random sinks*. Under suitable conditions, any ensemble of initial conditions will coalesce near a unique equilibrium point that evolves in time [10]. This phenomenon occurs in dissipative systems such as the Navier-Stokes system (see *e.g.* [14, 15]) and in certain coupled oscillator networks modeling neuronal activity [12]. Memory loss also occurs when $\lambda_{\max} > 0$ if one thinks in terms of measures, for in this case initial distributions will track random SRB measures (see [11]). For further information about random dynamical systems, see *e.g.* [1, 2].

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[1]In the context of random dynamical systems, the maps are chosen according to a known distribution.

We now introduce nonequilibrium open dynamical systems. Let X be a Riemannian manifold and let λ denote Riemannian volume (Lebesgue measure) on X. Consider a sequence of maps $(\hat{f}_i : X \to X)_{i=1}^{\infty}$. For $m \in \mathbb{N}$, define $\widehat{F}_m = \hat{f}_m \circ \cdots \circ \hat{f}_1$. We call the sequence $(\widehat{F}_m)_{m=1}^{\infty}$ a *nonequilibrium closed dynamical system*. Unlike the random dynamical systems setting, we do not assume that the $\hat{f}_i$ are drawn from a known distribution. Our setting is meant to model scenarios such as dynamical processes with time-varying parameters or dynamics in time-varying environments.

An open system is produced by introducing holes. For $j \in \mathbb{N}$, let $H_j \subset X$. We call H_j the *hole at time* j. Informally, we create an open system from $(\widehat{F}_m)_{m=1}^{\infty}$ by tracking trajectories until they fall into a hole. Once a trajectory falls into a hole, it is deemed to have escaped. Formally, for $m \in \mathbb{N}$ define the *time-m survivor set S_m* by

$$S_m = X \setminus \bigcup_{i=1}^{m} (\widehat{F}_i)^{-1}(H_i).$$

Let F_m denote the restriction $\widehat{F}_m | S_m$; that is, F_m is defined on points with trajectories that have not fallen into a hole after m iterates. We call the pair $((F_m), (H_j))$ a *nonequilibrium open dynamical system*.

We define a notion of memory loss for nonequilibrium open systems that is both statistical and conditional in nature. Let φ_0 and ψ_0 be two initial probability densities defined on X. Let φ_t and ψ_t denote the evolved densities under the action of the nonequilibrium open system. Since mass is allowed to escape through the holes, φ_t and ψ_t will not be probability densities in general: we expect $\|\varphi_t\|_{L^1(\lambda)} < 1$ and $\|\psi_t\|_{L^1(\lambda)} < 1$. We say that a nonequilibrium open system exhibits *conditional memory loss in the statistical sense* if for all initial densities φ_0 and ψ_0 chosen from a suitable class, we have

$$\lim_{t\to\infty} \left\| \frac{\varphi_t}{\|\varphi_t\|_{L^1(\lambda)}} - \frac{\psi_t}{\|\psi_t\|_{L^1(\lambda)}} \right\|_{L^1(\lambda)} = 0.$$

Ideally one explicitly estimates the rate of convergence as well.

In this paper we establish conditional memory loss in the statistical sense for a class of nonequilibrium open systems generated by one-dimensional piecewise-differentiable expanding Lasota-Yorke maps. We work in this setting because it is simple enough to allow for a clear development of ideas yet complicated enough in that it has some of the features of more realistic settings. Using convex cones and a projective metric known as the Hilbert metric, we show that memory loss occurs at an exponential rate and we explicitly estimate this rate.

Our work is a synthesis of two areas: nonequilibrium closed systems (no holes, dynamical model changes in time) and equilibrium open systems (fixed hole, iterates of a single map). Memory loss for nonequilibrium closed systems has been established for expanding maps and $1D$ piecewise-differentiable expanding maps [16], a class of piecewise-differentiable expanding maps in higher dimension studied by Saussol [8], topologically transitive Anosov diffeomorphisms on compact two-dimensional Riemannian manifolds [17], and certain dispersing billiards with moving scatterers [18]. When studying equilibrium open systems, one is often interested in conditionally invariant measures, escape rates, and related statistical properties. See [7] for an overview of this area and *e.g.* [4, 5, 6] for analyses of specific models.

We conclude the introduction with a comment about techniques. When studying memory loss or the related problem of decay to equilibrium/decay of correlations, one may employ a number of techniques, including spectral methods, coupling methods, and the use of convex cones and the Hilbert metric. We believe the latter two are especially useful in nonequilibrium contexts.

2. Setting and statement of results

2.1. **Underlying closed dynamical systems.** Let $[0, 1]$ be the phase space on which our dynamical processes act. Let λ denote Lebesgue measure on $[0, 1]$.

Definition 2.1. For $s < 1$, let $\mathcal{M}(s, K_2)$ denote the set of maps $\hat{g} : [0, 1] \to [0, 1]$ that satisfy the following hypotheses:

(a) there exists a finite partition $\mathcal{A}(\hat{g})$ of $[0, 1]$ into subintervals such that for each $J \in \mathcal{A}(\hat{g})$, $\hat{g}$ is $\mathcal{C}^2$ on J and extends to a $\mathcal{C}^2$ function on $\overline{J}$;

(b) $\max_{J\in\mathcal{A}(\hat{g})}\sup_{x\in J}|\hat{g}'(x)|^{-1}\leqslant s$;
(c) $\max_{J\in\mathcal{A}(\hat{g})}\sup_{x\in J}|\hat{g}''(x)|\leqslant K_2$.

We now define δ-perturbations within $\mathcal{M}(s,K_2)$. Let $\hat{g}\in\mathcal{M}(s,K_2)$. Let $\Omega(\hat{g})=\{0=x_1,\dots,x_k=1\}$ be the set of partition points associated with $\mathcal{A}(\hat{g})$ and define $d_\Omega(\hat{g})=\min_{1\leqslant i\leqslant k-1}x_{i+1}-x_i$.

Definition 2.2. We say that $\hat{f}\in\mathcal{M}(s,K_2)$ is a δ-perturbation of $\hat{g}\in\mathcal{M}(s,K_2)$ if

(a) $\delta<\frac{1}{4}d_\Omega(\hat{g})$;
(b) $\Omega(\hat{f})=\{0=y_1,\dots,y_k=1\}$ where $|y_i-x_i|<\delta$ for every $1\leqslant i\leqslant k$;
(c) if $\xi_{\hat{f}\hat{g}}$ maps each interval $[x_i,x_{i+1}]$ affinely onto $[y_i,y_{i+1}]$, then on every $J\in\mathcal{A}(\hat{g})$, we have

$$\left\|\hat{f}\circ\xi_{\hat{f}\hat{g}}-\hat{g}\right\|_{\mathcal{C}^2(J)}<\delta.$$

Let $\mathcal{N}(\hat{g},\delta;s,K_2)$ denote the set of δ-perturbations of $\hat{g}$.

Remark 2.3. The set

$$\left\{\mathcal{N}(\hat{g},\delta;s,K_2):\hat{g}\in\mathcal{M}(s,K_2),\ \ \delta<\frac{1}{4}d_\Omega(\hat{g})\right\}$$

is a basis for a topology on $\mathcal{M}(s,K_2)$.

Iterates of a single map $\hat{g}\in\mathcal{M}(s,K_2)$ do not necessarily exhibit memory loss. Indeed, memory loss is equivalent to measure-theoretic mixing in this context, and a single map $\hat{g}\in\mathcal{M}(s,K_2)$ may not even be ergodic. For this reason, we consider suitable subsets of $\mathcal{M}(s,K_2)$.

Definition 2.4 (class $\mathcal{E}$)**.** Let $\zeta_1\in(0,1)$ and $\zeta_2\in(1,\infty)$. We say that $\hat{g}:[0,1]\to[0,1]$ belongs to $\mathcal{E}(\zeta_1,\zeta_2)$ if the following hold.

(a) For every partition $\mathcal{Q}$ of $[0,1]$ into subintervals of equal length, there exists a time $E(\mathcal{Q},\zeta_1,\zeta_2)$ such that for every $J_1,J_2\in\mathcal{Q}$, we have

$$\zeta_1<\frac{\lambda(J_1\cap\hat{g}^{-i}(J_2))}{\lambda(J_1)\lambda(J_2)}<\zeta_2 \tag{1}$$

for every $i\geqslant E(\mathcal{Q},\zeta_1,\zeta_2)$.

(b) For every $x_j\in\Omega(\hat{g})$ and every $i\in\mathbb{N}$, we have

$$\operatorname{dist}\left(\lim_{z\to x_j^-}\hat{g}^i(z),\Omega(\hat{g})\setminus\{0,1\}\right)>0,\qquad \operatorname{dist}\left(\lim_{z\to x_j^+}\hat{g}^i(z),\Omega(\hat{g})\setminus\{0,1\}\right)>0.$$

For $x_j=0$ ($x_j=1$), only the limit from the right (left) is considered.

Definition 2.4(a) is a mixing condition; notice that $\mathcal{Q}$ is not the dynamical partition for $\hat{g}$ in general. We use the boundary control asserted in Definition 2.4(b) to obtain uniform Lasota-Yorke estimates (see Proposition 3.1).

2.2. **Nonequilibrium open dynamical systems and the main result.** Start with a 'base map' $\hat{g}\in\mathcal{M}(s,K_2)$. Let $\delta>0$ be small and consider a sequence of maps $(\hat{f}_i)_{i=1}^\infty$ in $\mathcal{N}(\hat{g},\delta;s,K_2)$. For $m\in\mathbb{N}$, let $\widehat{F}_m=\hat{f}_m\circ\cdots\circ\hat{f}_1$. We call the sequence $(\widehat{F}_m)_{m=1}^\infty$ a *nonequilibrium closed dynamical system.*

We now introduce holes. For $j\in\mathbb{N}$, let $H_j\subset[0,1]$ denote the *hole at time j*. We assume that H_j consists of at most L pairwise-disjoint open subintervals $H_{j,k}$ of $[0,1]$. For $m\in\mathbb{N}$, define

$$S_m=[0,1]\setminus\bigcup_{i=1}^m(\widehat{F}_i)^{-1}(H_i).$$

We call S_m the *time-m survivor set*. Let F_m denote the restriction $\widehat{F}_m|S_m$. We call the pair $((F_m),(H_j))$ a *nonequilibrium open dynamical system.*

2.2.1. ***Densities and transfer operators.*** Let $\mathrm{BV}([0,1],\mathbb{R})$ denote the space of real-valued functions of bounded variation on $[0,1]$. Let $\mathrm{Var}(\cdot)$ denote the variation seminorm on $\mathrm{BV}([0,1],\mathbb{R})$; for $\varphi \in \mathrm{BV}([0,1],\mathbb{R})$, we have

$$\mathrm{Var}(\varphi) = \sup_{0=x_0<\cdots<x_n=1} \sum_{i=1}^{n} |\varphi(x_i) - \varphi(x_{i-1})|,$$

or equivalently,

$$\mathrm{Var}(\varphi) = \sup_{\substack{\psi \in C^1(I) \\ \|\psi\|_\infty \leqslant 1 \\ \psi(0)=\psi(1)=0}} \int_0^1 \varphi \psi' \, \mathrm{d}\lambda.$$

The evolution of probability densities in $\mathrm{BV}([0,1],\mathbb{R})$ under the action of a nonequilibrium open dynamical system $((F_m),(H_j))$ is described by the family $(\mathcal{L}_{F_m})$ of transfer operators defined by

$$\mathcal{L}_{F_m}(\varphi)(x) = \sum_{z : F_m(z)=x} \frac{\varphi(z)}{|F_m'(z)|}$$

($\mathcal{L}_{F_m}(\varphi)(x) = 0$ if $F_m^{-1}(x) = \emptyset$). Of course, we expect to see $\|\mathcal{L}_{F_m}(\varphi)\|_{L^1(\lambda)} < \|\varphi\|_{L^1(\lambda)}$ in general, since mass will escape through the holes. We define operators $\mathcal{R}_{F_m}$ by renormalizing:

$$\mathcal{R}_{F_m}(\varphi) = \frac{\mathcal{L}_{F_m}(\varphi)}{\|\mathcal{L}_{F_m}(\varphi)\|_{L^1(\lambda)}}.$$

Notice that $\mathcal{R}_{F_m}$ is not linear. We are interested in the action of the sequence $(\mathcal{R}_{F_m})$ on the space

$$\mathcal{D} = \left\{ \varphi \in \mathrm{BV}([0,1],\mathbb{R}) : \varphi \geqslant 0, \ \ \|\varphi\|_{L^1(\lambda)} = 1 \right\}.$$

2.2.2. ***Main theorem.***

Theorem 2.5. *Let $\hat{g} \in \mathcal{M}(s, K_2) \cap \mathcal{E}(\zeta_1, \zeta_2)$ and let $L \in \mathbb{N}$. There exist $\delta_0 > 0$, $\varepsilon_0 > 0$, and $\Lambda < 1$ such that the following holds. Let $(\hat{f}_i)_{i=1}^{\infty}$ be any sequence of maps in $\mathcal{N}(\hat{g}, \delta_0; s, K_2)$ and let $(H_j)_{j=1}^{\infty}$ be any sequence of holes such that H_j consists of at most L pairwise-disjoint open intervals and $\lambda(H_j) \leqslant \varepsilon_0$ for every $j \in \mathbb{N}$. The resultant nonequilibrium open dynamical system $((F_m),(H_j))$ exhibits conditional memory loss in the following sense. There exists a convex cone $\mathcal{C}_a \subset \mathrm{BV}([0,1],\mathbb{R})$ and a constant $C_1 > 0$ such that for every $\varphi, \psi \in \mathcal{D} \cap \mathcal{C}_a$, we have*

$$\|\mathcal{R}_{F_m}(\varphi) - \mathcal{R}_{F_m}(\psi)\|_{L^1(\lambda)} \leqslant C_1 \Lambda^m \tag{2}$$

for all $m \in \mathbb{N}$.

Remark 2.6. See Section 3.2 and (14) for the definition of $\mathcal{C}_a$.

Remark 2.7. When proving Theorem 2.5, we use Definition 2.4(a) with respect to $\hat{g}$ for a suitably fine equipartition $\mathcal{Q}$ and (1) and Definition 2.4(b) with respect to $\hat{g}$ up to a suitably large time $T \geqslant E(\mathcal{Q}, \zeta_1, \zeta_2; \hat{g})$ (see Section 3.2 on parameter selection.) In particular, given $\hat{g} \in \mathcal{M}(s, K_2)$, a finite amount of information is needed to determine if Theorem 2.5 applies.

3. Proof of Theorem 2.5

3.1. **A Lasota-Yorke inequality.** We begin by stating a Lasota-Yorke inequality for the open systems we consider. The following estimate essentially appears in [13] (see [19] for a variant). We include the proof for the sake of completeness.

We introduce several useful partitions of $[0,1]$. Let $\mathcal{Z}_1^{(n)} = \mathcal{Z}_1^{(n)}(\hat{f}_1, \ldots, \hat{f}_n)$ denote the dynamical partition for $\widehat{F}_n$. Let $\mathcal{Z}_2^{(n)}$ be the coarsest refinement of $\mathcal{Z}_1^{(n)}$ such that every element of $\mathcal{Z}_1^{(n)}$ is divided into subintervals of equal length and we have $\lambda(J) \leqslant 1/K_2 n s$ for every $J \in \mathcal{Z}_2^{(n)}$. For $J \in \mathcal{Z}_2^{(n)}$, we have

$$\mathrm{Var}(|\widehat{F}_n'|^{-1}, J) = \int_J \left| \frac{\widehat{F}_n''(x)}{(\widehat{F}_n'(x))^2} \right| \mathrm{d}x \leqslant K_2 n s^{n+1} \lambda(J) \leqslant s^n. \tag{3}$$

Let $\mathcal{Z}_3^{(n)}$ be the coarsest refinement of $\mathcal{Z}_2^{(n)}$ such that for every $J \in \mathcal{Z}_3^{(n)}$, we have $J \subset S_n$ or $J \cap S_n = \emptyset$.

Proposition 3.1. *Consider the space of maps $\mathcal{M}(s, K_2)$ and let $(H_j)_{j=1}^\infty$ be any sequence of holes such that H_j consists of at most L pairwise-disjoint open intervals for every $j \in \mathbb{N}$. Let $\theta \in (s, 1)$ and let $N_1 \in \mathbb{N}$ be such that*

$$\theta^{N_1} > 6s^{N_1}(LN_1 + 1). \tag{4}$$

For every sequence $(\hat{f}_i)_{i=1}^\infty$ in $\mathcal{M}(s, K_2)$, every $k \in \mathbb{N}$, and every nonnegative $\varphi \in \mathrm{BV}([0,1], \mathbb{R})$, we have

$$\operatorname{Var}\left(\mathcal{L}_{F_{kN_1}}(\varphi), [0,1]\right) \leqslant \theta^{kN_1} \operatorname{Var}(\varphi, [0,1]) + \left((1-\theta^{N_1})^{-1} \cdot 5s^{N_1}(LN_1+1) \sup_{\widetilde{Z} \in \mathcal{Z}_2^{(N_1)}} \lambda(\widetilde{Z})^{-1}\right) \|\varphi\|_{L^1(\lambda)}. \tag{5}$$

Proof of Proposition 3.1. Computing $\mathcal{L}_{F_n}(\varphi)$, we have

$$\begin{aligned} \mathcal{L}_{F_n}(\varphi) &= \sum_{\substack{Z \in \mathcal{Z}_3^{(n)} \\ Z \subset S_n}} \mathcal{L}_{F_n}(\varphi \mathbf{1}_Z) \\ &= \sum_{\substack{Z \in \mathcal{Z}_3^{(n)} \\ Z \subset S_n}} (\varphi \mathbf{1}_Z \cdot |F_n'|^{-1}) \circ (F_n|Z)^{-1}. \end{aligned}$$

Therefore

$$\operatorname{Var}(\mathcal{L}_{F_n}(\varphi), [0,1]) \leqslant \sum_{\substack{Z \in \mathcal{Z}_3^{(n)} \\ Z \subset S_n}} \operatorname{Var}\left((\varphi \mathbf{1}_Z \cdot |F_n'|^{-1}) \circ (F_n|Z)^{-1}, [0,1]\right). \tag{6}$$

We estimate each term in the sum on the right side of (6). For $Z \subset \mathcal{Z}_3^{(n)}$ such that $Z \subset S_n$, let $\widetilde{Z} \in \mathcal{Z}_2^{(n)}$ be such that $Z \subset \widetilde{Z}$. For any such Z, we have

$$\begin{aligned} \operatorname{Var}\left((\varphi \mathbf{1}_Z \cdot |F_n'|^{-1}) \circ (F_n|Z)^{-1}, [0,1]\right) &\leqslant \operatorname{Var}\left(\varphi |F_n'|^{-1}, \widetilde{Z}\right) + 2 \sup_{\widetilde{Z}} \varphi |F_n'|^{-1} \\ &\leqslant 3 \operatorname{Var}\left(\varphi |F_n'|^{-1}, \widetilde{Z}\right) + 2 \inf_{\widetilde{Z}} \varphi |F_n'|^{-1} \\ &\leqslant 3 \left[s^n \operatorname{Var}(\varphi, \widetilde{Z}) + (\sup_{\widetilde{Z}} \varphi) \operatorname{Var}\left(|F_n'|^{-1}, \widetilde{Z}\right) \right] + 2 \inf_{\widetilde{Z}} \varphi |F_n'|^{-1} \\ &\leqslant 3 \left[s^n \operatorname{Var}(\varphi, \widetilde{Z}) + s^n (\sup_{\widetilde{Z}} \varphi) \right] + 2s^n \inf_{\widetilde{Z}} \varphi \\ &\leqslant 6s^n \operatorname{Var}(\varphi, \widetilde{Z}) + 5s^n \inf_{\widetilde{Z}} \varphi. \end{aligned} \tag{7}$$

Next observe that for every $\widetilde{Z} \in \mathcal{Z}_2^{(n)}$ we have

$$\#\left\{Z \in \mathcal{Z}_3^{(n)} : Z \subset S_n \text{ and } Z \subset \widetilde{Z}\right\} \leqslant Ln + 1 \tag{8}$$

since $\widehat{F}_i^{-1}(\widehat{F}_i(\widetilde{Z}) \cap H_i)$ consists of at most L intervals for every $1 \leqslant i \leqslant n$. Estimates (6), (7), and (8) imply

$$\operatorname{Var}(\mathcal{L}_{F_n}(\varphi), [0,1]) \leqslant (Ln+1)\left(6s^n \operatorname{Var}(\varphi, [0,1]) + 5s^n (\sup_{\widetilde{Z} \in \mathcal{Z}_2^{(n)}} \lambda(\widetilde{Z})^{-1}) \|\varphi\|_{L^1(\lambda)}\right). \tag{9}$$

We choose $N_1 \in \mathbb{N}$ such that

$$\theta^{N_1} > 6s^{N_1}(LN_1 + 1)$$

(see (4)), yielding

$$\operatorname{Var}(\mathcal{L}_{F_{N_1}}(\varphi), [0,1]) \leqslant \theta^{N_1} \operatorname{Var}(\varphi, [0,1]) + 5s^{N_1}(LN_1+1)(\sup_{\widetilde{Z} \in \mathcal{Z}_2^{(N_1)}} \lambda(\widetilde{Z})^{-1}) \|\varphi\|_{L^1(\lambda)}. \tag{10}$$

We obtain the Lasota-Yorke estimate (5) by iterating (10). ∎

3.2. **Parameter selection.** We prove Theorem 2.5 by studying the action of $\{\mathcal{L}_{F_m}\}$ on a suitable convex cone $\mathcal{C}_a$ of functions inside $\mathrm{BV}([0,1],\mathbb{R})$. We choose $\mathcal{Q}$ (recall Definition 2.4(a)) and introduce σ, T, and a such that (P1)–(P3) are simultaneously satisfied. We then choose δ_0 and ε_0.

(P1) $0 < \sigma < 1$

(P2) $T \in \mathbb{N}$: choose such that T is a positive integer multiple of N_1, $T \geqslant E(\mathcal{Q}, \zeta_1, \zeta_2)$, and $\theta^T < 1$. In view of (5), define

$$C_{\mathrm{LY}} = 2(1-\theta^{N_1})^{-1} \cdot 5s^{N_1}(LN_1+1) \sup_{\widetilde{Z} \in \mathcal{Z}_2^{(N_1)}(\hat{g},\ldots,\hat{g})} \lambda(\widetilde{Z})^{-1}.$$

(P3) $a > 0$: the aperture of the cone $\mathcal{C}_a$. We choose a such that

$$\zeta_1 - \zeta_2 a \cdot \mathrm{diam}(\mathcal{Q}) > 0,$$
$$\frac{a\theta^T + C_{\mathrm{LY}}}{\zeta_1 - \zeta_2 a \cdot \mathrm{diam}(\mathcal{Q})} \leqslant \sigma a.$$

To see that (P1)–(P3) may be satisfied simultaneously, proceed in the following order:

(a) Choose T sufficiently large so that $\theta^T/(\zeta_1/2) < \sigma$.

(b) Choose a sufficiently large so that

$$\frac{a\theta^T + C_{\mathrm{LY}}}{\zeta_1/2} \leqslant \sigma a.$$

(c) Choose $\mathrm{diam}(\mathcal{Q})$ sufficiently small so that $\zeta_2 a \cdot \mathrm{diam}(\mathcal{Q}) \leqslant \zeta_1/2$.

(d) Increase T (if necessary) so that $T \geqslant E(\mathcal{Q}, \zeta_1, \zeta_2)$.

We now choose δ_0 and ε_0. First, let δ_0 be sufficiently small so that

$$(1-\theta^{N_1})^{-1} \cdot 5s^{N_1}(LN_1+1) \sup_{\hat{f}_1,\ldots,\hat{f}_{N_1} \in \mathcal{N}(\hat{g},\delta_0;s,K_2)} \sup_{\widetilde{Z} \in \mathcal{Z}_2^{(N_1)}} \lambda(\widetilde{Z})^{-1} \leqslant C_{\mathrm{LY}}, \tag{11}$$

and for every sequence $(\hat{f}_k)_{k=1}^T$ in $\mathcal{N}(\hat{g}, \delta_0; s, K_2)$ we have

$$\zeta_1 < \frac{\lambda\big(J_1 \cap (\widehat{F}_T)^{-1}(J_2)\big)}{\lambda(J_1)\lambda(J_2)} < \zeta_2 \tag{12}$$

for all $J_1, J_2 \in \mathcal{Q}$. Here (11) is valid for δ_0 sufficiently small because $\hat{g}$ satisfies Definition 2.4(b) and (12) holds for δ_0 sufficiently small because for every $J_1, J_2 \in \mathcal{Q}$ and for every $\eta > 0$, there exists $\delta > 0$ such that

$$\left| \frac{\lambda\big(J_1 \cap (\widehat{F}_T)^{-1}(J_2)\big)}{\lambda(J_1)\lambda(J_2)} - \frac{\lambda\big(J_1 \cap \hat{g}^{-T}(J_2)\big)}{\lambda(J_1)\lambda(J_2)} \right| < \eta$$

for every $(\hat{f}_k)_{k=1}^T$ in $\mathcal{N}(\hat{g}, \delta_0; s, K_2)$. Second, let ε_0 be sufficiently small so that for every sequence $(\hat{f}_k)_{k=1}^T$ in $\mathcal{N}(\hat{g}, \delta_0; s, K_2)$ we have

$$\zeta_1 < \frac{\lambda\big(J_1 \cap (F_T)^{-1}(J_2)\big)}{\lambda(J_1)\lambda(J_2)} < \zeta_2 \tag{13}$$

for all $J_1, J_2 \in \mathcal{Q}$. This can be done because for every $J_1, J_2 \in \mathcal{Q}$ and for every $\eta > 0$, there exists $\varepsilon > 0$ such that for every sequence of holes $(H_j)_{j=1}^T$ with complexity bound L and with $\lambda(H_j) \leqslant \varepsilon$ for all $1 \leqslant j \leqslant T$, we have

$$\left| \frac{\lambda\big(J_1 \cap (\widehat{F}_T)^{-1}(J_2)\big)}{\lambda(J_1)\lambda(J_2)} - \frac{\lambda\big(J_1 \cap (F_T)^{-1}(J_2)\big)}{\lambda(J_1)\lambda(J_2)} \right| < \eta$$

for every $(\hat{f}_k)_{k=1}^T$ in $\mathcal{N}(\hat{g}, \delta_0; s, K_2)$.

3.3. **Invariance of a suitable convex cone.** Define

$$\mathcal{C}_a = \left\{\varphi \in L^1(\lambda) : \varphi \geqslant 0,\ \ \varphi \not\equiv 0,\ \ \mathrm{Var}(\varphi) \leqslant a\mathbb{E}[\varphi|\mathcal{Q}]\right\}. \tag{14}$$

We study the action of $\mathcal{L}_{F_m}$ on $\mathcal{C}_a$. For positive integers $m > i$, define

$$\widehat{F}_{m,i} = \hat{f}_m \circ \hat{f}_{m-1} \circ \cdots \circ \hat{f}_i, \quad F_{m,i} = f_m \circ f_{m-1} \circ \cdots \circ f_i,$$

where f_k is the open system corresponding to $\hat{f}_k$ ($i \leqslant k \leqslant m$).

Lemma 3.2. *Let δ_0 and ε_0 be as in Section 3.2. For every $\varphi \in \mathcal{C}_a$ and $i \in \mathbb{N}$ we have*

$$(\zeta_1 - \zeta_2 a \cdot \mathrm{diam}(\mathcal{Q})) \int_{[0,1]} \varphi \,\mathrm{d}\lambda \leqslant \mathbb{E}[\mathcal{L}_{F_{i+T-1,i}}(\varphi)|\mathcal{Q}] \leqslant \zeta_2(1 + a \cdot \mathrm{diam}(\mathcal{Q})) \int_{[0,1]} \varphi \,\mathrm{d}\lambda. \tag{15}$$

Proof of Lemma 3.2. Write $F = F_{i+T-1,i}$. For $x \in [0,1]$, let $Q(x)$ denote the element of $\mathcal{Q}$ that contains x. We have

$$\begin{aligned}
\mathbb{E}[\mathcal{L}_F(\varphi)|\mathcal{Q}](x) &= \frac{1}{\lambda(Q(x))} \int_{Q(x)} \mathcal{L}_F(\varphi) \,\mathrm{d}\lambda \\
&= \frac{1}{\lambda(Q(x))} \int_{F^{-1}(Q(x))} \varphi \,\mathrm{d}\lambda \\
&= \frac{1}{\lambda(Q(x))} \sum_{Q' \in \mathcal{Q}} \int_{Q' \cap F^{-1}(Q(x))} \varphi(z) \,\mathrm{d}\lambda(z).
\end{aligned} \tag{16}$$

Bounding φ from below, for every $z \in Q' \cap F^{-1}(Q(x))$ we have

$$\begin{aligned}
\varphi(z) &\geqslant \inf_{y \in Q'} \varphi(y) \\
&\geqslant \sup_{y \in Q'} \varphi(y) - \mathrm{Var}(\varphi, Q') \\
&\geqslant \frac{1}{\lambda(Q')} \int_{Q'} \varphi \,\mathrm{d}\lambda - \mathrm{Var}(\varphi, Q') \\
&= \frac{1}{\lambda(Q')} \left(\int_{Q'} \varphi \,\mathrm{d}\lambda - \lambda(Q') \,\mathrm{Var}(\varphi, Q') \right).
\end{aligned} \tag{17}$$

Using (16), (17), and (13), we have

$$\begin{aligned}
\mathbb{E}[\mathcal{L}_F(\varphi)|\mathcal{Q}](x) &\geqslant \frac{1}{\lambda(Q(x))} \sum_{Q' \in \mathcal{Q}} \int_{Q' \cap F^{-1}(Q(x))} \frac{1}{\lambda(Q')} \left(\int_{Q'} \varphi \,\mathrm{d}\lambda - \lambda(Q') \,\mathrm{Var}(\varphi, Q') \right) \mathrm{d}\lambda(z) \\
&= \sum_{Q' \in \mathcal{Q}} \frac{\lambda(Q' \cap F^{-1}(Q(x)))}{\lambda(Q(x))\lambda(Q')} \left(\int_{Q'} \varphi \,\mathrm{d}\lambda - \lambda(Q') \,\mathrm{Var}(\varphi, Q') \right) \\
&\geqslant \zeta_1 \int_{[0,1]} \varphi \,\mathrm{d}\lambda - \zeta_2 \cdot \mathrm{diam}(\mathcal{Q}) \cdot \mathrm{Var}(\varphi, [0,1]) \\
&\geqslant \zeta_1 \int_{[0,1]} \varphi \,\mathrm{d}\lambda - \zeta_2 a \cdot \mathrm{diam}(\mathcal{Q}) \cdot \int_{[0,1]} \varphi \,\mathrm{d}\lambda \\
&= (\zeta_1 - \zeta_2 a \cdot \mathrm{diam}(\mathcal{Q})) \int_{[0,1]} \varphi \,\mathrm{d}\lambda.
\end{aligned} \tag{18}$$

The upper bound

$$\mathbb{E}[\mathcal{L}_F(\varphi)|\mathcal{Q}](x) \leqslant \zeta_2(1 + a \cdot \mathrm{diam}(\mathcal{Q})) \int_{[0,1]} \varphi \,\mathrm{d}\lambda$$

follows from an analogous line of reasoning. ∎

Proposition 3.3. *In the setting of Lemma 3.2, for every $i \in \mathbb{N}$ we have*

$$\mathcal{L}_{F_{i+T-1,i}}(\mathcal{C}_a) \subset \mathcal{C}_{\sigma a}.$$

Proof of Proposition 3.3. Write $F = F_{i+T-1,i}$ and let $\varphi \in \mathcal{C}_a$. Using (5) and (15), we have

$$\begin{aligned}\operatorname{Var}(\mathcal{L}_F(\varphi), [0,1]) &\leqslant \theta^T \operatorname{Var}(\varphi, [0,1]) + C_{\mathrm{LY}} \|\varphi\|_{L^1(\lambda)} \\ &\leqslant (a\theta^T + C_{\mathrm{LY}}) \|\varphi\|_{L^1(\lambda)} \\ &\leqslant \frac{a\theta^T + C_{\mathrm{LY}}}{\zeta_1 - \zeta_2 a \cdot \operatorname{diam}(\Omega)} \mathbb{E}[\mathcal{L}_F(\varphi)|\Omega] \\ &\leqslant \sigma a \mathbb{E}[\mathcal{L}_F(\varphi)|\Omega].\end{aligned}$$

■

3.4. **Cones, Hilbert metrics, and positive operators.** Following [13], we review a theory of cones developed by Birkhoff [3]. We will use this theory to show that $\mathcal{L}_{F_{i+T-1,i}}$ is a contraction with respect to a projective metric known as the Hilbert metric.

Definition 3.4. Let $\mathcal{V}$ be a vector space. A *convex cone* is a subset $\mathcal{C} \subset \mathcal{V}$ with the following properties.

(a) $\mathcal{C} \cap -\mathcal{C} = \emptyset$
(b) $\gamma\mathcal{C} = \mathcal{C}$ for all $\gamma > 0$
(c) $\mathcal{C}$ is a convex set
(d) For all $\varphi, \psi \in \mathcal{C}$, every $c \in \mathbb{R}$, and every sequence (c_n) in $\mathbb{R}$ such that $c_n \to c$, if $\varphi - c_n\psi \in \mathcal{C}$ for all n, then $\varphi - c\psi \in \mathcal{C} \cup \{0\}$.

Definition 3.5. Let $\mathcal{C}$ be a convex cone. The Hilbert metric $d_{\mathcal{C}}$ is defined on $\mathcal{C}$ by

$$d_{\mathcal{C}}(\varphi, \psi) = \log\left(\frac{\inf\{c > 0 : c\varphi - \psi \in \mathcal{C}\}}{\sup\{r > 0 : \psi - r\varphi \in \mathcal{C}\}}\right).$$

The following result asserts that in the current context, a positive linear operator is a contraction in the Hilbert metric provided the diameter of the image is finite.

Theorem 3.6 ([3])**.** *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be vector spaces containing convex cones $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Let $\mathcal{L} : \mathcal{V}_1 \to \mathcal{V}_2$ be a positive linear operator, meaning $\mathcal{L}(\mathcal{C}_1) \subset \mathcal{C}_2$. Define*

$$\Delta = \sup_{\varphi^*, \psi^* \in \mathcal{L}(\mathcal{C}_1)} d_{\mathcal{C}_2}(\varphi^*, \psi^*).$$

Then for all $\varphi, \psi \in \mathcal{C}_1$, we have

$$d_{\mathcal{C}_2}(\mathcal{L}\varphi, \mathcal{L}\psi) \leqslant \tanh\left(\frac{\Delta}{4}\right) d_{\mathcal{C}_1}(\varphi, \psi).$$

We conclude this review by relating the Hilbert metric to *adapted* norms on $\mathcal{V}$.

Proposition 3.7. *Let $\mathcal{C} \subset \mathcal{V}$ be a convex cone and let $\|\cdot\|$ be an adapted norm on $\mathcal{V}$; that is, a norm such that for all $\varphi, \psi \in \mathcal{V}$, if $\psi - \varphi \in \mathcal{C}$ and $\psi + \varphi \in \mathcal{C}$, then $\|\psi\| \leqslant \|\varphi\|$. Then for all $\varphi, \psi \in \mathcal{C}$, we have*

$$\|\varphi\| = \|\psi\| \Longrightarrow \|\varphi - \psi\| \leqslant \left(e^{d_{\mathcal{C}}(\varphi, \psi)} - 1\right) \|\varphi\|.$$

3.5. **Completion of the proof of Theorem 2.5.**

Proposition 3.8. *Assume the setting of Proposition 3.3. For every $i \in \mathbb{N}$ and for all $\varphi, \psi \in \mathcal{C}_a$, we have*

$$d_{\mathcal{C}_a}(\mathcal{L}_{F_{i+T-1,i}}(\varphi), \mathcal{L}_{F_{i+T-1,i}}(\psi)) \leqslant \Delta_0 := 2\log\left(\frac{1+\sigma}{1-\sigma}\right) + 2\log\left(\frac{\zeta_2(1 + a \cdot \operatorname{diam}(\Omega))}{\zeta_1 - \zeta_2 a \cdot \operatorname{diam}(\Omega)}\right). \tag{19}$$

Proof of Proposition 3.8. Let $\varphi^*, \psi^* \in \mathcal{C}_{\sigma a}$. Suppose $c > 0$. We have

$$\begin{aligned}\operatorname{Var}(c\varphi^* - \psi^*, [0,1]) &\leqslant c \operatorname{Var}(\varphi^*, [0,1]) + \operatorname{Var}(\psi^*, [0,1]) \\ &\leqslant c\sigma a \mathbb{E}[\varphi^*|\Omega] + \sigma a \mathbb{E}[\psi^*|\Omega].\end{aligned}$$

Therefore $c\varphi^* - \psi^* \in \mathcal{C}_a$ if

$$c\sigma a \mathbb{E}[\varphi^*|\Omega] + \sigma a \mathbb{E}[\psi^*|\Omega] \leqslant a \mathbb{E}[c\varphi^* - \psi^*|\Omega].$$

This is equivalent to

$$\left(\frac{1+\sigma}{1-\sigma}\right)\left(\frac{\mathbb{E}[\psi^*|\mathcal{Q}]}{\mathbb{E}[\varphi^*|\mathcal{Q}]}\right) \leqslant c. \tag{20}$$

Arguing analogously, for $r > 0$ we have $\psi^* - r\varphi^* \in \mathcal{C}_a$ if

$$r \leqslant \left(\frac{1-\sigma}{1+\sigma}\right)\left(\frac{\mathbb{E}[\psi^*|\mathcal{Q}]}{\mathbb{E}[\varphi^*|\mathcal{Q}]}\right). \tag{21}$$

Bounds (20) and (21) imply

$$\begin{aligned} d_{\mathcal{C}_a}(\varphi^*,\psi^*) &\leqslant \log\left(\left(\frac{1+\sigma}{1-\sigma}\right)\sup_{x\in[0,1]}\frac{\mathbb{E}[\psi^*|\mathcal{Q}]}{\mathbb{E}[\varphi^*|\mathcal{Q}]}\right) - \log\left(\left(\frac{1-\sigma}{1+\sigma}\right)\inf_{x\in[0,1]}\frac{\mathbb{E}[\psi^*|\mathcal{Q}]}{\mathbb{E}[\varphi^*|\mathcal{Q}]}\right) \\ &\leqslant 2\log\left(\frac{1+\sigma}{1-\sigma}\right) + \log\left(\sup_{x\in[0,1]}\frac{\mathbb{E}[\psi^*|\mathcal{Q}]}{\mathbb{E}[\varphi^*|\mathcal{Q}]}\right) - \log\left(\inf_{x\in[0,1]}\frac{\mathbb{E}[\psi^*|\mathcal{Q}]}{\mathbb{E}[\varphi^*|\mathcal{Q}]}\right). \end{aligned} \tag{22}$$

Proposition 3.3 and estimates (15) and (22) imply (19) with

$$\Delta_0 = 2\log\left(\frac{1+\sigma}{1-\sigma}\right) + 2\log\left(\frac{\zeta_2(1+a\cdot\mathrm{diam}(\mathcal{Q}))}{\zeta_1-\zeta_2 a\cdot\mathrm{diam}(\mathcal{Q})}\right).$$

■

Corollary 3.9 (corollary of Proposition 3.8)**.** *Assume the setting of Proposition 3.8. For every $i \in \mathbb{N}$ and for all $\varphi,\psi \in \mathcal{C}_a$, we have*

$$d_{\mathcal{C}_a}(\mathcal{L}_{F_{i+T-1,i}}(\varphi),\mathcal{L}_{F_{i+T-1,i}}(\psi)) \leqslant \tanh\left(\frac{\Delta_0}{4}\right) d_{\mathcal{C}_a}(\varphi,\psi). \tag{23}$$

Proof of Corollary 3.9. The result follows directly from Theorem 3.6 and Proposition 3.8. ■

We are nearly in position to derive (2). One additional ingredient is needed: a Lipschitz estimate involving $\mathcal{R}$.

Lemma 3.10. *Assume the setting of Corollary 3.9. There exists $C_{\mathrm{Lip}} > 0$ such that for all integers n satisfying $1 \leqslant n < T$, for every $i \in \mathbb{N}$, and for all $\varphi,\psi \in \mathcal{D}\cap\mathcal{C}_a$, we have*

$$\left\|\mathcal{R}_{F_{i+n-1,i}}(\varphi) - \mathcal{R}_{F_{i+n-1,i}}(\psi)\right\|_{L^1(\lambda)} \leqslant C_{\mathrm{Lip}}\left\|\varphi-\psi\right\|_{L^1(\lambda)}. \tag{24}$$

Proof of Lemma 3.10. Write $F = F_{i+n-1,i}$ and $\|\cdot\| = \|\cdot\|_{L^1(\lambda)}$. Let $\varphi,\psi \in \mathcal{D}\cap\mathcal{C}_a$. We have

$$\begin{aligned} \|\mathcal{R}_F(\varphi) - \mathcal{R}_F(\psi)\| &= \left\|\frac{\mathcal{L}_F(\varphi)}{\|\mathcal{L}_F(\varphi)\|} - \frac{\mathcal{L}_F(\psi)}{\|\mathcal{L}_F(\psi)\|}\right\| \\ &= \left\|\frac{\mathcal{L}_F(\varphi)}{\|\mathcal{L}_F(\varphi)\|} - \frac{\mathcal{L}_F(\varphi)}{\|\mathcal{L}_F(\psi)\|} + \frac{\mathcal{L}_F(\varphi)}{\|\mathcal{L}_F(\psi)\|} - \frac{\mathcal{L}_F(\psi)}{\|\mathcal{L}_F(\psi)\|}\right\| \\ &\leqslant \frac{\left|\,\|\mathcal{L}_F(\psi)\| - \|\mathcal{L}_F(\varphi)\|\,\right|}{\|\mathcal{L}_F(\varphi)\|\cdot\|\mathcal{L}_F(\psi)\|}\|\mathcal{L}_F(\varphi)\| + \frac{1}{\|\mathcal{L}_F(\psi)\|}\|\mathcal{L}_F(\varphi)-\mathcal{L}_F(\psi)\| \\ &\leqslant 2(\zeta_1-\zeta_2 a\cdot\mathrm{diam}(\mathcal{Q}))^{-1}\|\mathcal{L}_F(\varphi)-\mathcal{L}_F(\psi)\| \\ &\leqslant 2(\zeta_1-\zeta_2 a\cdot\mathrm{diam}(\mathcal{Q}))^{-1}\|\varphi-\psi\| \end{aligned}$$

using (15) and the fact that $\|\mathcal{L}_F(\gamma)\| \leqslant \|\gamma\|$ for every $\gamma \in \mathrm{BV}([0,1],\mathbb{R})$. Set

$$C_{\mathrm{Lip}} = 2(\zeta_1-\zeta_2 a\cdot\mathrm{diam}(\mathcal{Q}))^{-1}.$$

■

We now derive (2). Write $\|\cdot\|_1$ for the L^1 norm. Let $\varphi, \psi \in \mathcal{D} \cap \mathcal{C}_a$. Let $m \in \mathbb{Z}^+$ and write $m = kT + n$ where $k \in \mathbb{Z}^+$ and $0 \leqslant n < T$. If $k \geqslant 1$, we have

$$\begin{aligned}
\|\mathcal{R}_{F_m}(\varphi) - \mathcal{R}_{F_m}(\psi)\|_1 &\leqslant C_{\text{Lip}} \|\mathcal{R}_{F_{kT}}(\varphi) - \mathcal{R}_{F_{kT}}(\psi)\|_1 && (24)\\
&\leqslant C_{\text{Lip}} \left(\exp\left(d_{\mathcal{C}_a}(\mathcal{R}_{F_{kT}}(\varphi), \mathcal{R}_{F_{kT}}(\psi))\right) - 1\right) && (\text{P}3.7)\\
&= C_{\text{Lip}} \left(\exp\left(d_{\mathcal{C}_a}(\mathcal{L}_{F_{kT}}(\varphi), \mathcal{L}_{F_{kT}}(\psi))\right) - 1\right) && (\text{projectivity})\\
&\leqslant C_{\text{Lip}} \left(\exp\left(\left(\tanh\left(\frac{\Delta_0}{4}\right)\right)^{k-1} d_{\mathcal{C}_a}(\mathcal{L}_{F_T}(\varphi), \mathcal{L}_{F_T}(\psi))\right) - 1\right) && (23)\\
&\leqslant C_{\text{Lip}} \Delta_0 e^{\Delta_0} \left(\tanh\left(\frac{\Delta_0}{4}\right)\right)^{k-1} && (19)\\
&\leqslant C_{\text{Lip}} \Delta_0 e^{\Delta_0} \tanh^{-2}\left(\frac{\Delta_0}{4}\right) \left(\left(\tanh\left(\frac{\Delta_0}{4}\right)\right)^{1/T}\right)^m
\end{aligned}$$

Consequently, for any $m \in \mathbb{Z}^+$ we have

$$\|\mathcal{R}_{F_m}(\varphi) - \mathcal{R}_{F_m}(\psi)\|_1 \leqslant C_{\text{Lip}} \max\{\Delta_0, 1\} e^{\Delta_0} \tanh^{-2}\left(\frac{\Delta_0}{4}\right) \left(\left(\tanh\left(\frac{\Delta_0}{4}\right)\right)^{1/T}\right)^m.$$

This establishes (2) with

$$\begin{aligned}
C_1 &= C_{\text{Lip}} \max\{\Delta_0, 1\} e^{\Delta_0} \tanh^{-2}\left(\frac{\Delta_0}{4}\right),\\
\Lambda &= \left(\tanh\left(\frac{\Delta_0}{4}\right)\right)^{1/T}.
\end{aligned}$$

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