RECURRENCE IN PAIRS

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Abstract. We introduce and study the notion of weak product recurrence. Two sufficient conditions for this type of recurrence are established. We deduce that any point with a dense orbit in either the full one-sided shift on a finite number of symbols or a mixing subshift of finite type is weakly product recurrent. This observation implies that distality does not follow from weak product recurrence. We have therefore answered a question posed by Auslander and Furstenberg in the negative.

1. Introduction

The notion of recurrence is central in the study of dynamical systems and especially in topological dynamics. Classically, a topological dynamical system consists of a topological space X with a one-parameter group $\{T^n : n \in \mathbb{Z}\}$ of homeomorphisms acting on X. Poincaré's original nineteenth-century notion of recurrence, interpreted in the setting of topological dynamics, is that of a point returning to every neighborhood of itself. A compact dynamical system always contains recurrent points. This notion has since been strengthened to various degrees. A point x is said to be uniformly recurrent (or almost periodic) if for every neighborhood U of x, the return time set $R(x, U) = \{n \in \mathbb{Z} : T^n x \in U\}$ of x to U has bounded gaps. In other words, x returns to U with a bound on the 'waiting time'. Further strengthening the notion of recurrence, one may require some form of synchronized recurrence with other recurrent points. A point x is said to be product recurrent if given any recurrent point y in any dynamical system and any neighborhoods U of x and V of y, the return time sets $R(x, U)$ and $R(y, V)$ intersect nontrivially.

By associating product recurrence with a combinatorial property of sets of return times (x is product recurrent if and only if $R(x, U)$ is IP* for each neighborhood U of x), Furstenberg [3] proves that product recurrence implies uniform recurrence. Furstenberg [3] also shows that for $\mathbb{Z}$-actions, product recurrence and distality are equivalent (a point x is said to be distal if whenever y is in the orbit closure of x and a sequence (n_i) in $\mathbb{Z}$ exists with $\lim_i T^{n_i} x = \lim_i T^{n_i} y$, then $x = y$). Using algebraic properties of idempotents in the enveloping semigroup, Auslander and Furstenberg [1] extend the equivalence of product recurrence and distality to more general semigroup actions.

We study a form of synchronized recurrence that generalizes the notion of product recurrence. A point x is said to be weakly product recurrent if given any uniformly recurrent point y in any dynamical system and any neighborhoods U of x and V of y, the return time sets $R(x, U)$ and $R(y, V)$ intersect nontrivially. We

Date: June 2007.
2000 *Mathematics Subject Classification.* Primary: 37B05, 37B20.
Key words and phrases. Auslander-Furstenberg question, Distality, Mixing subshift of finite type, Product recurrence, Recurrence, Topological dynamics, Weak product recurrence.

establish two sufficient conditions for weak product recurrence. The first sufficient condition is presented in the noninvertible case. Let Γ be a compact metric space and let $\tau : \Gamma \to \Gamma$ be a continuous map. The point $\gamma \in \Gamma$ is weakly product recurrent if γ satisfies the following recurrence property. For every neighborhood V of γ there exists $n = n(V) \in \mathbb{N}$ such that if $S \subset \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ is any finite set satisfying $|s - t| \geqslant n$ for all distinct s, $t \in S$, then there exists $\ell \in \mathbb{Z}^+$ such that $\tau^{\ell+s}(\gamma) \in V$ for all $s \in S$. The proof of this result uses the van der Waerden theorem [4]. The theorem of van der Waerden states that every subset of $\mathbb{N}$ with bounded gaps contains arithmetic progressions of arbitrary length.

Auslander and Furstenberg [1] ask if weak product recurrence implies distality. The first sufficient condition for weak product recurrence answers this question in the negative for actions of $\mathbb{N}$. This first result implies that any point with a dense orbit in a 'sufficiently mixing' system is weakly product recurrent. In particular, any point with a dense orbit in either the full one-sided shift on a finite number of symbols or a mixing subshift of finite type is weakly product recurrent. Any point with a dense orbit in the full shift on a finite number of symbols is proximal to a fixed point in its orbit closure. Such a point is therefore weakly product recurrent but not distal.

The second sufficient condition for weak product recurrence is based on the observation that the return time sets of points in arbitrary dynamical systems are related to the return time sets of special points in the symbolic dynamical system (Ω, σ), where $\Omega = \{0, 1\}^{\mathbb{Z}}$ and σ denotes the shift map. We prove that for actions of $\mathbb{Z}$, the point x is weakly product recurrent if (x, ω) is recurrent for all uniformly recurrent $\omega \in \Omega$. The shift space is therefore universal in this context. We believe that this universality may constructively inform the study of recurrence.

The material is organized as follows. In Section 2, we present dynamical preliminaries and we discuss the work of Auslander and Furstenberg. Section 3 contains the statement and proof of the first sufficient condition for weak product recurrence. The universality of the shift space in the context of weak product recurrence is established in Section 4. We conclude the paper in Section 5 by stating several open questions and discussing the relationship of weak product recurrence to other notions of recurrence.

2. Dynamical preliminaries

We present the relevant aspects of the theory of dynamical systems, introduce the notion of weak product recurrence, and state the Auslander-Furstenberg question.

Definition 2.1. A ***dynamical system*** (X, G) consists of a compact metric space X together with a group or semigroup G acting on X by continuous transformations.

Recurrence is a form of asymptotic behavior related to the action of sequences of group elements 'tending to infinity'. Various notions of recurrence may be defined for a point $x \in X$ either in terms of the elements of the Stone-Čech compactification βG that fix x or in terms of the 'size' of the sets of return times to neighborhoods of x. We adopt the latter approach, restricting our study to the case $G = \mathbb{Z}$.

Definition 2.2. Let $x \in X$. The ***orbit*** of x, denoted $\mathcal{O}(x)$, is given by $\mathcal{O}(x) = \{T^n(x) : n \in \mathbb{Z}\}$.

Definition 2.3. Let U denote a neighborhood of $x \in X$. The ***return time set*** $R(x,U)$ is given by $R(x,U) = \{n \in \mathbb{Z} : T^n(x) \in U\}$.

Definition 2.4. A point $x \in X$ is said to be ***recurrent*** if for every neighborhood U of x, we have $(R(x,U) \cap \mathbb{Z}) \setminus \{0\} \neq \emptyset$.

It follows from the definition of recurrence that if x is recurrent, then $R(x,U)$ is infinite for every neighborhood U of x. One obtains the notion of uniform recurrence by demanding that the sets $R(x,U)$ be syndetic.

Definition 2.5. A subset S of $\mathbb{Z}$ is ***syndetic*** if there exists a positive integer M such that for all $z \in \mathbb{Z}$,

$$\{y \in \mathbb{Z} : z \leqslant y \leqslant z + M\} \cap S \neq \emptyset.$$

Such a set is said to have bounded gaps.

Definition 2.6. A point $x \in X$ is said to be ***uniformly recurrent*** (or almost periodic) if for every neighborhood U of x, the return time set $R(x,U)$ is syndetic.

The notion of recurrence may also be strengthened by requiring that the recurrent point x recur in a synchronized way with any other recurrent point.

Definition 2.7. A point $x \in X$ is said to be ***product recurrent*** if given any dynamical system (Y,S) and any recurrent point $y \in Y$, the point (x,y) is a recurrent point of the product dynamical system $(X \times Y, T \times S)$.

Recurrence shares deep connections with the notions of proximality and distality.

Definition 2.8. Two points x and y in X are said to be ***proximal*** if there exists a sequence $(n_k) \subset \mathbb{Z}$ and a point $z \in X$ such that $n_k \to \pm\infty$ and

$$\lim_{k\to\infty} T^{n_k}(x) = \lim_{k\to\infty} T^{n_k}(y) = z.$$

Definition 2.9. A point $x \in X$ is said to be ***distal*** if x is not proximal to any point in its orbit closure $\overline{\mathcal{O}(x)}$ other than itself.

In the special case $G = \mathbb{Z}$, product recurrence and distality are equivalent [3, Theorem 9.11]. This equivalence does not hold in the context of $\mathcal{E}$-semigroup actions. Distality implies product recurrence, but the converse does not hold in general. Auslander and Furstenberg [1] formulate conditions under which product recurrence implies distality. Their algebraic approach is based on the study of idempotents in the $\mathcal{E}$-semigroup. Innovations in [1] include the illumination of the key role played by maximal idempotents and the introduction of the cancellation semigroup. For a systematic and thorough study of the fine structure of recurrence for semigroup actions, see the work of Ellis, Ellis, and Nerurkar [2].

Inspired by [1], we introduce the notion of weak product recurrence.

Definition 2.10. Let (X,T) be a dynamical system. A point $x \in X$ is said to be ***weakly product recurrent*** if given any dynamical system (Y,S) and any uniformly recurrent point $y \in Y$, the point (x,y) is a recurrent point of the product dynamical system $(X \times Y, T \times S)$.

With this definition in place, we now state the Auslander-Furstenberg question.

Question 2.11 (Auslander-Furstenberg [1])**.** Let (X,T) be a dynamical system. If $x \in X$ is weakly product recurrent, is x necessarily a distal point?

Since distality and product recurrence are equivalent for $\mathbb{Z}$-actions, the Auslander-Furstenberg question may be restated as follows. Do the notions of product recurrence and weak product recurrence coincide for $\mathbb{Z}$-actions?

3. A sufficient condition for weak product recurrence

We present a sufficient condition for weak product recurrence. We then describe several situations in which this condition is satisfied. The recurrence result presented in this section implies the existence of weakly product recurrent points that are not distal. We have therefore answered the Auslander-Furstenberg question in the negative. The material in this section is presented in the context of $\mathbb{N}$-actions.

Theorem 3.1. *Let Γ be a compact metric space and let $\tau : \Gamma \to \Gamma$ be a continuous map. The point $\gamma \in \Gamma$ is weakly product recurrent if γ has the following property: For every neighborhood V of γ there exists $n = n(V) \in \mathbb{N}$ such that if $S \subset \mathbb{Z}^+$ is any finite set satisfying $|s-t| \geqslant n$ for all distinct s, $t \in S$, then there exists $\ell \in \mathbb{Z}^+$ such that $\tau^{\ell+s}(\gamma) \in V$ for all $s \in S$.*

The proof of Theorem 3.1 is based on the deep fact that syndetic subsets of $\mathbb{Z}$ contain arithmetic progressions of arbitrary length. This fact is known as the van der Waerden theorem [4].

Proof of Theorem 3.1. Suppose that γ has the property described in Theorem 3.1. Let Λ be a compact metric space and let $\rho : \Lambda \to \Lambda$ be a continuous map. Suppose $\lambda \in \Lambda$ is uniformly recurrent. We must show that (λ, γ) is recurrent. Let U and V be neighborhoods of λ and γ, respectively. We prove that

$$(R(\lambda, U) \cap R(\gamma, V)) \setminus \{0\} \neq \emptyset.$$

Choose $n = n(V)$ such that for any finite set $S \subset \mathbb{Z}^+$ satisfying $|s - t| \geqslant n$ for all distinct s, $t \in S$, there exists $\ell \in \mathbb{Z}^+$ such that $\tau^{\ell+s}(\gamma) \in V$ for every $s \in S$.

Let $A = \{a + jd : 0 \leqslant j \leqslant n\} \subset \mathbb{N}$ be an arithmetic progression of length $n + 1$ such that the common difference d is a multiple of n and $\rho^{a+jd}(\lambda) \in U$ for $0 \leqslant j \leqslant n$. We justify the existence of A as follows. The set $R(\lambda, U)$ is syndetic because λ is uniformly recurrent. Therefore, the van der Waerden theorem implies that $R(\lambda, U)$ contains arithmetic progressions of arbitrary length. Choose an arithmetic progression $B = \{b + kq : 0 \leqslant k \leqslant n(n + 1)\} \subset R(\lambda, U)$. Now define $A = \{(b + nq) + j(nq) : 0 \leqslant j \leqslant n\}$. Here $a = b + nq$ and $d = nq$. Since ρ is continuous, there exists a neighborhood $U_1 \subset U$ such that if $\xi \in U_1$ then $\rho^{a+jd}(\xi) \in U$ for $0 \leqslant j \leqslant n$. The set $R(\lambda, U_1)$ is syndetic and therefore the set

$$\{k : \rho^{k+jd}(\lambda) \in U \text{ for } 0 \leqslant j \leqslant n\}$$

is syndetic. Choose $M \in \mathbb{N}$ such that for any $\ell \in \mathbb{Z}^+$ there exists $k \in [\ell, \ell + M - nd]$ satisfying $\rho^{k+jd}(\lambda) \in U$ for $0 \leqslant j \leqslant n$.

We now define a configuration S to which the hypothesis on γ applies. Choose $r \in \mathbb{N}$ such that $r(d + 1) \geqslant M$. Define

$$S = \{x(d + 1) + yn : 0 \leqslant x \leqslant r,\ 0 \leqslant y \leqslant q - 1\}.$$

We show that $|s - t| \geqslant n$ for all distinct s, $t \in S$. Suppose $s = x_1(d + 1) + y_1 n$ and $t = x_2(d+1)+y_2 n$ are distinct elements of S with $x_2 \geqslant x_1$. We consider two cases.

(1) If $x_2 = x_1$, then $y_1 \neq y_2$ and therefore $|s - t| \geqslant n$.

(2) If $x_2 > x_1$, then $x_2(d+1) + y_2 n \geqslant (x_1+1)(d+1)$ and

$$\begin{aligned} x_1(d+1) + y_1 n &\leqslant x_1(d+1) + (q-1)n \\ &= x_1(d+1) + qn - n \\ &= x_1(d+1) + d - n \\ &= (x_1+1)(d+1) - n - 1 \\ &< (x_1+1)(d+1) - n. \end{aligned}$$

Therefore, we have $|s-t| > n$ in this case.

Since S is an admissible configuration for γ, there exist $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}^+$ such that

(1) $\tau^{\ell+s}(\gamma) \in V$ for all $s \in S$,
(2) $\rho^{k+jd}(\lambda) \in U$ for $0 \leqslant j \leqslant n$,
(3) $k \in [\ell, \ell + M - nd]$.

We find $j \in \{0, \ldots, n\}$ such that $k + jd \in \ell + S$.

Let h be the largest integer such that $h(d+1) \leqslant k - \ell$. Since $qn = d$, there exist unique integers $i \in \{0, \ldots, q-1\}$ and $j \in \{0, \ldots, n\}$ such that

$$k - \ell = h(d+1) + in + j.$$

Note that $k - (\ell + h(d+1)) \leqslant d$. If $k - (\ell + h(d+1)) = d$, then $i = q-1$ and $j = n$. If $k - (\ell + h(d+1)) < d$, then $j < n$. Transforming $k + jd$, we have

$$\begin{aligned} k + jd &= \ell + (k - \ell) + jd \\ &= \ell + h(d+1) + in + j + jd \\ &= \ell + (h+j)(d+1) + in. \end{aligned}$$

Since $k + jd \leqslant \ell + M$, we must have $h + j \leqslant r$. We also have $i \in \{0, \ldots, q-1\}$. Therefore, $(h+j)(d+1) + in \in S$ and $k + jd \in \ell + S$. We conclude that

$$k + jd \in (R(\lambda, U) \cap R(\gamma, V)) \setminus \{0\}.$$

Since U and V were chosen arbitrarily, we conclude that (λ, γ) is recurrent. ∎

We now state conditions under which the hypothesis of Theorem 3.1 is satisfied. This hypothesis will be satisfied if γ is a point with a dense orbit in a 'sufficiently mixing' system.

Corollary 3.2. *Let Γ be a compact metric space and let $\tau : \Gamma \to \Gamma$ be a continuous map. The point $\gamma \in \Gamma$ is weakly product recurrent if the following hold.*

(1) The orbit of γ is dense in Γ.
(2) For any neighborhood V of γ, there exists $N = N(V)$ such that for any $k \in \mathbb{N}$, if $n_i \geqslant N$ for $1 \leqslant i \leqslant k$, then the intersection

$$V \cap \tau^{-n_1}(V) \cap \cdots \cap \tau^{-(n_1+\cdots+n_k)}(V)$$

is nonempty.

Proof of Corollary 3.2. We show that γ satisfies the hypothesis of Theorem 3.1. Let V be a neighborhood of γ. Using (2), there exists $N = N(V)$ such that for any $k \in \mathbb{N}$, if $n_i \geqslant N$ for $1 \leqslant i \leqslant k$, then the intersection

$$V \cap \tau^{-n_1}(V) \cap \cdots \cap \tau^{-(n_1+\cdots+n_k)}(V)$$

is nonempty. Let $S = \{s_i : 1 \leqslant i \leqslant k\} \subset \mathbb{Z}^+$ be a finite set such that $s_{i+1} - s_i \geqslant N$ for $1 \leqslant i \leqslant k-1$. The intersection

$$W = V \cap \tau^{-s_1}(V) \cap \cdots \cap \tau^{-s_k}(V)$$

is nonempty and open. Since the orbit of γ is dense in Γ, there exists $\ell \in \mathbb{Z}^+$ such that $\tau^{\ell}(\gamma) \in W$. We have $\tau^{\ell+s}(\gamma) \in V$ for all $s \in S$. ■

Mixing property (3.2)(2) is satisfied by the full one-sided shift on a finite number of symbols and by any mixing subshift of finite type. Therefore, Corollary 3.2 implies the following.

Corollary 3.3. *Let (Γ, τ) be either the full one-sided shift on a finite number of symbols or a mixing subshift of finite type. If $\gamma \in \Gamma$ has a dense orbit, then γ is weakly product recurrent.*

We now address the question posed by Auslander and Furstenberg. Corollary 3.3 implies the existence of weakly product recurrent points that are not distal, thereby answering the Auslander-Furstenberg question in the negative. In the full shift on a finite number of symbols, any point with a dense orbit has a fixed point in its orbit closure and is therefore weakly product recurrent but not distal. For example, let $\Sigma^+ = \{0,1\}^{\mathbb{Z}^+}$ and let σ denote the shift map. For $m \in \mathbb{N}$, let α^m denote the binary representation of m. We define $\gamma \in \Sigma^+$ by

$$\gamma = \lim_{m \to \infty} \alpha^1 * \alpha^2 * \cdots * \alpha^m,$$

where $*$ denotes concatenation. Therefore, γ takes the form

$$\gamma = 1\,10\,11\,100\,\cdots.$$

The point γ is the binary Champernowne sequence. Since γ contains every binary word, the orbit of γ is dense in Σ^+ and therefore γ is weakly product recurrent. However, γ is proximal to the fixed point $\rho \in \Sigma^+$ defined by $\rho_i = 1$ for all $i \in \mathbb{Z}^+$ and $\rho \in \overline{\mathcal{O}(\gamma)}$, so γ is not distal.

4. Universality of the shift space

Let (X, T) denote a dynamical system and let $x \in X$. A priori, it may be difficult to determine if x is weakly product recurrent because we must examine the orbit of the pair (x, y) for every uniformly recurrent point y in every dynamical system (Y, S). One encounters this difficulty whenever one wishes to verify any notion of recurrence that requires synchronized recurrence with points of a certain type in every dynamical system. We show that when establishing weak product recurrence or product recurrence, it suffices to consider uniformly recurrent points or recurrent points, respectively, in the shift space $\Omega = \{0,1\}^{\mathbb{Z}}$. This universal feature of the shift space follows from a general proposition relating the return time sets of points in arbitrary dynamical systems to the return time sets of special points in the shift space.

We begin with notation and definitions. We represent a point $\omega \in \Omega$ as a bi-infinite sequence $(\omega_n)_{n \in \mathbb{Z}}$. The shift map $\sigma : \Omega \to \Omega$ is given by $\omega \mapsto \sigma(\omega)$, where $(\sigma(\omega))_n = \omega_{n+1}$. For $k \in \mathbb{Z}^+$ and $\omega \in \Omega$, define

$$V_k(\omega) = \{\beta \in \Omega : \beta_j = \omega_j \text{ for } |j| \leqslant k\}.$$

The family of neighborhoods $\{V_k(\omega) : k \in \mathbb{Z}^+\}$ forms a neighborhood basis at ω.

Let (Y, S) be a dynamical system and let $y \in Y$. Let (γ_m) be a descending sequence in $(0, 1)$ such that $\gamma_m \to 0$ and $d(S^n(y), y) \neq \gamma_m$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. Such a sequence exists because the set $\{S^n y : n \in \mathbb{Z}\}$ is countable. Let $B_m = B(y, \gamma_m)$, where $B(y, \gamma_m)$ is the ball of radius γ_m centered at y. Define $\omega^{(y,m)} \in \Omega$ by

$$\omega_i^{(y,m)} = \begin{cases} 1, & \text{if } i \in R(y, B_m); \\ 0, & \text{if } i \notin R(y, B_m). \end{cases}$$

Think of $\omega^{(y,m)}$ as the characteristic function of the set $R(y, B_m)$.

Proposition 4.1. *Let (Y, S) be a dynamical system and let $y \in Y$ be arbitrary. Fix $m \in \mathbb{N}$ and let γ_m, B_m, and $\omega^{(y,m)}$ be as above. For each $k \in \mathbb{N}$, there exists m_k such that*

$$R(\omega^{(y,m)}, V_k(\omega^{(y,m)})) \supset R(y, B_{m_k}).$$

Proof of Proposition 4.1. For each $j \in \{-k, \ldots, k\}$, let $\epsilon_j > 0$ be such that either $B(S^j y, \epsilon_j) \subset B_m$ if $S^j y \in B_m$ or $B(S^j y, \epsilon_j) \cap B_m = \emptyset$ if $S^j y \notin B_m$. This may be done because $S^j y \notin \partial B_m$ for each j. Since each mapping S^j is uniformly continuous, there exists $\delta > 0$ such that if $d(y, z) < \delta$, then $d(S^j y, S^j z) < \epsilon_j$ for all $|j| \leqslant k$. Choose m_k such that $\gamma_{m_k} < \delta$. Let $n \in R(y, B_{m_k})$. Then $d(y, S^n y) < \gamma_{m_k} < \delta$, so $d(S^j y, S^{n+j} y) < \epsilon_j$ for all $|j| \leqslant k$. This implies that $\omega_j^{(y,m)} = \omega_{n+j}^{(y,m)}$ for all $|j| \leqslant k$. We conclude that $\sigma^n(\omega^{(y,m)}) \in V_k(\omega^{(y,m)})$, so $n \in R(\omega^{(y,m)}, V_k(\omega^{(y,m)}))$. ∎

Corollary 4.2 (Universality of the shift space)**.** *Let (X, T) be a dynamical system and let $x \in X$. The following statements hold.*

(1) The point x is recurrent if and only if $\omega^{(x,m)}$ is recurrent for all $m \in \mathbb{N}$.
(2) The point x is uniformly recurrent if and only if $\omega^{(x,m)}$ is uniformly recurrent for all $m \in \mathbb{N}$.
(3) If (x, ω) is recurrent in $(X \times \Omega, T \times \sigma)$ for all uniformly recurrent $\omega \in \Omega$, then x is weakly product recurrent.
(4) If (x, ω) is recurrent in $(X \times \Omega, T \times \sigma)$ for all recurrent $\omega \in \Omega$, then x is product recurrent.

Proof of Corollary 4.2. Statements (1) and (2) follow immediately from Proposition 4.1. The proofs of statements (3) and (4) are structurally similar. We prove statement (3). Assume that (x, ω) is recurrent in $(X \times \Omega, T \times \sigma)$ for all uniformly recurrent $\omega \in \Omega$. Let (Y, S) be a dynamical system and suppose that $y \in Y$ is uniformly recurrent. Let U and V be neighborhoods of x and y, respectively. Choose m such that $B_m \subset V$. By statement (2) and the hypothesis on x, $(x, \omega^{(y,m)})$ is recurrent in $(X \times \Omega, T \times \sigma)$. Therefore,

$$(R(x, U) \cap R(\omega^{(y,m)}, V_0(\omega^{(y,m)}))) \setminus \{0\} \neq \emptyset.$$

Since $R(y, V) \supset R(y, B_m) = R(\omega^{(y,m)}, V_0(\omega^{(y,m)}))$, we conclude that

$$(R(x, U) \cap R(y, V)) \setminus \{0\} \neq \emptyset.$$

Consequently, x is weakly product recurrent. ∎

5. Discussion

We have shown that any point with a dense orbit in either the full one-sided shift on a finite number of symbols or a mixing subshift of finite type is weakly product recurrent. This result suggests the following questions.

Question 5.1. How 'large' is the class of weakly product recurrent points?

Question 5.2. Algebraically, a product recurrent point is characterized by the fact that it is fixed by all maximal idempotents. Can one establish an algebraic characterization of weak product recurrence?

We now discuss the relationship of weak product recurrence to other notions of recurrence. Uniform recurrence does not imply weak product recurrence. We prove this by constructing uniformly recurrent points α and β in the symbolic dynamical system $\Omega^+ = \{0,1\}^{\mathbb{Z}^+}$ such that $\alpha_0 = 0$, $\beta_0 = 1$, and $\alpha_i = \beta_i$ for $i \geqslant 1$. The points α and β are constructed as follows. For a word ω, let $\ell(\omega)$ denote the length of ω. Writing $\omega = \omega_0 \cdots \omega_{\ell(\omega)-1}$, let $\overline{\omega} = \omega_{\ell(\omega)-1} \cdots \omega_0$ denote the reflection of ω. Define $a^{(0)} = 0$ and $b^{(0)} = 10$. Inductively, define

$$a^{(n+1)} = a^{(n)} * \overline{b^{(n)}} \text{ and } b^{(n+1)} = b^{(n)} * \overline{a^{(n+1)}},$$

where $*$ denotes concatenation. Define

$$\alpha = \lim_{n\to\infty} a^{(n)} \text{ and } \beta = \lim_{n\to\infty} b^{(n)}.$$

We show inductively that $\alpha_i = \beta_i$ for $i \geqslant 1$. The inductive assumptions are as follows.

(1) We have $a_i^{(k)} = b_i^{(k)}$ for $1 \leqslant i \leqslant \ell(a^{(k)}) - 1$.
(2) For $0 \leqslant j \leqslant k$, the subwords $b_0^{(j)} \cdots b_{\ell(a^{(j)})-1}^{(j)}$ and $b_{\ell(a^{(j)})}^{(j)} \cdots b_{\ell(b^{(j)})-1}^{(j)}$ are symmetric.

Notice that $a^{(1)} = 001$ and $b^{(1)} = 10100$. Inductive assumption (1) holds for $k = 1$ because $a_1^{(1)} = b_1^{(1)} = 0$ and $a_2^{(1)} = b_2^{(1)} = 1$. Inductive assumption (2) holds for $k = 1$ because the words $b_0^{(1)} \cdots b_2^{(1)} = 101$ and $b_3^{(1)} b_4^{(1)} = 00$ are symmetric. We now show that inductive assumptions (1) and (2) hold at level $k+1$ if they hold at level k. Since $b_{\ell(a^{(k)})}^{(k)} \cdots b_{\ell(b^{(k)})-1}^{(k)}$ is symmetric, we have $a_i^{(k+1)} = b_i^{(k+1)}$ for $\ell(a^{(k)}) \leqslant i \leqslant \ell(b^{(k)}) - 1$. Since $b_0^{(k)} \cdots b_{\ell(a^{(k)})-1}^{(k)}$ is symmetric, we have $a_i^{(k+1)} = b_i^{(k+1)}$ for $\ell(b^{(k)}) \leqslant i \leqslant \ell(a^{(k+1)}) - 1$. Therefore, inductive assumption (1) holds at level $k+1$. We have

$$b_0^{(k+1)} \cdots b_{\ell(a^{(k+1)})-1}^{(k+1)} = b_0^{(k)} \cdots b_{\ell(a^{(k)})-1}^{(k)} * b_{\ell(a^{(k)})}^{(k)} \cdots b_{\ell(b^{(k)})-1}^{(k)} * b_0^{(k)} \cdots b_{\ell(a^{(k)})-1}^{(k)},$$

$$b_{\ell(a^{(k+1)})}^{(k+1)} \cdots b_{\ell(b^{(k+1)})-1}^{(k+1)} = \overline{a_0^{(k+1)} \cdots a_{\ell(b^{(k)})-1}^{(k+1)}},$$

$$a_0^{(k+1)} \cdots a_{\ell(b^{(k)})-1}^{(k+1)} = b_{\ell(a^{(k)})}^{(k)} \cdots b_{\ell(b^{(k)})-1}^{(k)} * b_0^{(k-1)} \cdots b_{\ell(a^{(k-1)})-1}^{(k-1)} * b_{\ell(a^{(k)})}^{(k)} \cdots b_{\ell(b^{(k)})-1}^{(k)}.$$

These equalities imply that inductive assumption (2) holds at level $k+1$.

We show that α and β are uniformly recurrent. The length of the gap between any two occurrences of $a^{(n)}$ in α is bounded above by $2\ell(b^{(n)})$. This is so because if we choose $k \geqslant 2$ and assume that this bound holds for all $a^{(n+j)}$ with $1 \leqslant j \leqslant k$, then the equality

$$a^{(n+k+1)} = a^{(n+k)} * \left(a^{(n+k)} * a^{(n+k-1)} * \cdots * a^{(n+2)}\right) * a^{(n)} * \overline{b^{(n)}} * \overline{b^{(n)}}$$

implies that the bound holds for $a^{(n+k+1)}$ as well. The length of the gap between any two occurrences of $b^{(n)}$ in β is bounded above by $\ell(a^{(n)})$. This is so because if we choose $k \geqslant 1$ and assume that this bound holds for all $b^{(n+j)}$ with $1 \leqslant j \leqslant k$, then the equality

$$b^{(n+k+1)} = b^{(n+k)} * \left(b^{(n+k)} * b^{(n+k-1)} * \cdots * b^{(n)}\right) * \overline{a^{(n)}}$$

implies that the bound holds for $b^{(n+k+1)}$ as well. Finally, α is not weakly product recurrent because (α, β) is clearly not recurrent.

Any point with a dense orbit in the full one-sided shift on a finite number of symbols is weakly product recurrent but not distal. Such a point is not uniformly recurrent. This observation suggests the following question.

Question 5.3. Do weak product recurrence and uniform recurrence together imply product recurrence?

Acknowledgments. We would like to thank Ahmad El-Guindy and Laird Taylor for useful discussions leading to the results in Section 4 of this paper. We would also like to thank the referee for substantially simplifying the statement and proof of Theorem 3.1 and for noting that Corollary 3.2 applies to mixing subshifts of finite type.

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