

# PERIODIC ATTRACTORS VERSUS NONUNIFORM EXPANSION IN SINGULAR LIMITS OF FAMILIES OF RANK ONE MAPS

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ABSTRACT. We analyze parametrized families of multimodal  $1D$  maps that arise as singular limits of parametrized families of rank one maps. For a generic 1-parameter family of such maps that contains a Misiurewicz-like map, it has been shown that in a neighborhood of the Misiurewicz-like parameter, a subset of parameters of positive Lebesgue measure exhibits nonuniformly expanding dynamics characterized by the existence of a positive Lyapunov exponent and an absolutely continuous invariant measure. Under a mild combinatoric assumption, we prove that each such parameter is an accumulation point of the set of parameters admitting superstable periodic sinks.

## 1. INTRODUCTION

This paper is about how the dynamics depend on the parameter in certain parametrized families of dynamical systems. We study the 1-parameter families of multimodal  $1D$  maps that are obtained by passing to the singular limits of parametrized families of rank one maps. The theory of rank one maps developed by Wang and Young [35, 39] provides checkable conditions that imply the existence of strange attractors with strong stochastic properties in parametrized families of dissipative diffeomorphisms. This theory is based on the parameter exclusion techniques of Jakobson [17], the analysis of the Hénon family by Benedicks and Carleson [6, 7], the construction of SRB measures for the Hénon family by Benedicks and Young [9], and the theory of Young towers [40, 41]. We briefly summarize the theory of rank one maps in order to motivate the setting and results of this paper.

Let  $N \geq 2$  and let  $M = I \times D_{N-1}$ , where  $I$  is an interval or a circle and  $D_{N-1}$  is the closed unit disk in  $\mathbb{R}^{N-1}$ . Consider a 1-parameter family  $\{T_a : a \in [a_0, a_1]\}$  of diffeomorphisms  $T_a : M \rightarrow M$  that map  $M$  diffeomorphically into its interior. Suppose that the family  $T_a$  is sufficiently close in the  $C^3$  topology to a singular family of maps  $S_a : M \rightarrow I \times \{\mathbf{0}\}$ . By restricting each  $S_a$  to  $I \times \{\mathbf{0}\}$ , we obtain the singular limit, a multimodal family of  $1D$  maps  $f_a : I \rightarrow I$ . Roughly speaking, the theory of rank one maps asserts that if the singular limit contains a strongly expanding Misiurewicz-type map and if certain transversality and nondegeneracy conditions are met, then there exists a set  $\Lambda \subset [a_0, a_1]$  of positive Lebesgue measure such that for  $a \in \Lambda$ ,  $T_a$  admits a strange attractor that supports a unique SRB measure  $\mu$ . The SRB measure  $\mu$  satisfies the central limit theorem and exhibits exponential decay of correlations for Hölder continuous observables. See [35, 39] for a precise description of the theory of rank one maps. The theory has been applied in a variety of settings, including simple mechanical systems [36], Hopf bifurcations and limit cycles driven by periodic pulsatile drives [37, 26], periodically-forced systems with dissipative homoclinic loops [34], systems with multiple time scales [16], certain parabolic partial differential equations [19], and electronic circuitry [33]. This paper studies how the dynamics depend on the parameter in singular limit families  $f_a$  associated with families of diffeomorphisms  $T_a$  to which the theory of rank one maps applies. We call such families admissible families.

A 1-parameter  $C^2$  family  $\{f_a : a \in [a_0, a_1]\}$  of multimodal maps  $f_a : I \rightarrow I$  is an admissible family if the family contains a strongly expanding Misiurewicz-type map  $f_{a^*}$  (see Definition 2.1) and if a certain parameter transversality condition is satisfied (see Definition 2.2). Wang and Young [38] prove that in a neighborhood of  $a^*$ , there exists a set  $\Delta$  of positive Lebesgue measure such that for  $a \in \Delta$ ,  $f_a$  admits an absolutely continuous invariant probability measure and satisfies **(G1)** and **(G2)** (see Section 2.3). Moreover,  $a^*$  is a Lebesgue

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density point of  $\Delta$ . Conditions **(G1)** and **(G2)** are of Collet-Eckmann type. Roughly speaking, **(G1)** controls the rate at which critical orbits may approach the critical set and **(G2)** says that derivatives along the critical orbits grow at an exponential rate. The techniques used in [38] are not new and many similar results have been obtained in various contexts. See *e.g.* [6, 7, 8, 13, 14, 17, 20, 24, 27, 28, 31]. Tsujii [30] proves perhaps the strongest version of the result in [38] by obtaining a set analogous to  $\Delta$  under weaker assumptions on the starting parameter  $a^*$ . Note also that the existence of an absolutely continuous invariant measure follows from conditions substantially weaker than **(G1)** and **(G2)** (see *e.g.* [10, 11, 14, 23, 25]). We work in the setting of [38] because we are interested in the admissible families that are obtained by passing to the singular limits of families of rank one maps. We hope that both this paper and [38] will serve as points of entry into the theory of rank one maps.

We prove (Theorem 1) that if an admissible family satisfies a mild combinatoric assumption, then every parameter  $a \in \Delta$  is an accumulation point of parameters corresponding to maps with superstable periodic sinks (a superstable periodic sink of a map  $g : I \rightarrow I$  is a critical point  $c \in I$  such that  $g^n(c) = c$  for some  $n \in \mathbb{N}$ ). Theorem 1 is the main result of this paper. We show that the combinatoric assumption is satisfied by admissible families of long-branched maps (Theorem 2) and that the combinatoric assumption is always satisfied by admissible families of unimodal maps (Theorem 3). These results constitute a first step toward the goal of understanding how the dynamics depend on the parameter in admissible families of  $1D$  maps.

This goal is relevant to any parametrized family of dynamical systems and significant progress has been made in various contexts. Much is known about the quadratic family  $\{q_a : a \in [1, 2]\}$  of  $1D$  maps defined by  $q_a(x) = 1 - ax^2$  on  $J = [-1, 1]$ . The dynamical properties of  $q_a$  depend sensitively on the value of the parameter  $a$ . For this family, there are 2 primary dynamical scenarios competing in the space of parameters: the existence of periodic sinks and the existence of an absolutely continuous invariant measure. Let  $\Delta_s$  be the set of all  $a \in [1, 2]$  such that  $q_a$  admits a periodic sink and let  $\Delta_e$  be the set of all  $a \in [1, 2]$  such that  $q_a$  admits an absolutely continuous invariant measure. It is known that

- (1)  $\Delta_s$  is open and dense in  $[1, 2]$  ([15, 21]),
- (2)  $a = 2$  is a Lebesgue density point of  $\Delta_e$  ([17, 6, 7]), and
- (3)  $\Delta_s \cup \Delta_e$  has full Lebesgue measure in  $[1, 2]$  ([22]).

Statement (1) implies that maps with periodic sinks dominate parameter space in the topological sense. On the other hand, (2) implies that, at least in the vicinity of  $a = 2$ , maps with absolutely continuous invariant measures dominate in the measure-theoretic sense. See [1, 2, 5, 3, 4, 12] for extensions and refinements of statement (3) in the context of more general unimodal families. Kozlovski, Shen, and van Strien [18] have shown that in the one-dimensional setting, real polynomials with periodic attractors are dense in the space of real polynomials. It follows from this that maps with periodic attractors are dense in  $C^k(I, I)$  for all  $k \in \mathbb{N}$ , for  $k = \infty$ , and for  $k = \omega$ . We remark that Theorem 1 does not follow from the fact that maps with periodic attractors are dense in  $C^2(I, I)$  because the intersection of a given admissible family with the set of maps with periodic attractors may not be dense in the topology on the parameter interval. In fact, the following is a major challenge.

**Question 1.1.** Let  $\{f_a : a \in [a_0, a_1]\}$  be a  $C^2$  admissible family. Does  $f_a$  have a periodic attractor for a dense set of parameter values  $a$ ?

We conclude the introductory material by noting that a version of Theorem 3 is proved by Thunberg in [29].

This paper is organized as follows. We present the setting of [38] and we state our results in Section 2. In Section 3, we prove Theorems 2 and 3 by applying Theorem 1. Theorem 1 is proved in Section 4. The proof of Theorem 1 is based on techniques presented in [38]. We gather what we need from [38] into one proposition to provide a coherent presentation. We then prove this proposition in Section 5. This paper is self-contained modulo one isolated technical result from [38].

## 2. STATEMENT OF RESULTS

**2.1. The class  $\mathcal{E}$ .** First we introduce Misiurewicz maps. Let  $I$  denote the unit interval or the circle. For  $f \in C^2(I, I)$ , let  $C = C(f) = \{x \in I : f'(x) = 0\}$  denote the critical set of  $f$ . For  $\delta > 0$ , let  $C_\delta$  denote the  $\delta$ -neighborhood of  $C$  in  $I$ . Let  $C_\delta(c)$  denote the  $\delta$ -neighborhood of  $c \in C$ .

**Definition 2.1 (Misiurewicz map).** We say  $f \in C^2(I, I)$  is a Misiurewicz map and we write  $f \in \mathcal{E}$  if the following hold.

- (A) There exists  $\delta_0 > 0$  such that
  - (1) for all  $x \in C_{\delta_0}$ ,  $f''(x) \neq 0$ , and
  - (2) for all  $c \in C(f)$  and  $n > 0$ ,  $d(f^n(c), C(f)) \geq \delta_0$ .
- (B) There exist positive constants  $b_0$  and  $\lambda_0$  such that the following hold for all  $\delta < \delta_0$  and  $n > 0$ .
  - (1) If  $f^k(x) \notin C_\delta$  for  $0 \leq k \leq n-1$ , then  $|(f^n)'(x)| \geq b_0 \delta e^{\lambda_0 n}$ .
  - (2) If  $f^k(x) \notin C_\delta$  for  $0 \leq k \leq n-1$  and  $f^n(x) \in C_{\delta_0}$ , then  $|(f^n)'(x)| \geq b_0 e^{\lambda_0 n}$ .

This definition is equivalent to the definition of class  $\mathcal{M}$  in Section 1.1 of [38] in the sense that  $f \in \mathcal{E}$  if and only if  $f \in \mathcal{M}$ . Definition 2.1(B) asserts that derivatives grow at a uniform exponential rate (modulo a prefactor) along orbits that remain outside  $C_\delta$ . For every  $c \in C(f)$ , the derivative  $(f^n)'(f(c))$  grows exponentially by Definition 2.1(A2) and (B1).

**2.2. Admissible families.** Let  $F : I \times [a_1, a_2] \rightarrow I$  be a  $C^2$  map. The map  $F$  defines a one-parameter family  $\{f_a \in C^2(I, I) : a \in [a_1, a_2]\}$  via  $f_a(x) = F(x, a)$ . We assume that there exists  $a^* \in (a_1, a_2)$  such that  $f_{a^*} \in \mathcal{E}$ . For each  $c \in C(f_{a^*})$ , there exists a continuation  $c(a) \in C(f(a))$  provided  $a$  is sufficiently close to  $a^*$  by Definition 2.1(A1).

Let  $C(f_{a^*}) = \{c^{(1)}(a^*), \dots, c^{(q)}(a^*)\}$ , where  $c^{(i)}(a^*) < c^{(i+1)}(a^*)$  for  $1 \leq i \leq q-1$ . For  $c(a^*) \in C(f_{a^*})$  we denote  $\beta_c(a^*) = f_{a^*}(c(a^*))$ . For all parameters  $a$  sufficiently close to  $a^*$ , there exists a unique continuation  $\beta_c(a)$  of  $\beta_c(a^*)$  such that the orbits

$$\{f_{a^*}^n(\beta_c(a^*)) : n \geq 0\} \text{ and } \{f_a^n(\beta_c(a)) : n \geq 0\}$$

have the same itineraries with respect to the partitions of  $I$  induced by  $C(f_{a^*})$  and  $C(f_a)$ . This means that for all  $n \geq 0$ ,  $f_{a^*}^n(\beta_c(a^*)) \in (c^{(j)}(a^*), c^{(j+1)}(a^*))$  if and only if  $f_a^n(\beta_c(a)) \in (c^{(j)}(a), c^{(j+1)}(a))$ .<sup>1</sup> Moreover, the map  $a \mapsto \beta_c(a)$  is differentiable (see Proposition 4.1 in [38]).

**Definition 2.2.** Let  $F : I \times [a_1, a_2] \rightarrow I$  be a  $C^2$  map. The associated one-parameter family  $\{f_a : a \in [a_1, a_2]\}$  is *admissible* if

- (1) there exists  $a^* \in (a_1, a_2)$  such that  $f_{a^*} \in \mathcal{E}$ ;
- (2) for all  $c \in C(f_{a^*})$ ,

$$(2.1) \quad \xi(c) = \left. \frac{d}{da}(f_a(c(a)) - \beta_c(a)) \right|_{a=a^*} \neq 0.$$

**2.3. Main results.** We begin by describing the main result of [38]. Let  $\{f_a : a \in [a_1, a_2]\}$  be an admissible family and let  $a^* \in (a_1, a_2)$  be such that  $f_{a^*} \in \mathcal{E}$ . Set  $b_1 = b_0 \delta_0$  and fix  $\lambda \leq \lambda_0/5$ . For  $\alpha > 0$ , let  $\Delta(\lambda, \alpha)$  denote the set of parameters  $a \in [a_1, a_2]$  such that  $f_a$  satisfies the following: for all  $c \in C(f_a)$  and for all  $n > 0$ ,

- (G1)  $d(f_a^n(c), C(f_a)) \geq \min\{\delta_0/2, 2e^{-\alpha n}\}$ , and
- (G2)  $|(f_a^n)'(f_a(c))| \geq 2b_1 e^{\lambda n}$ .

It is proved in [38] that maps satisfying (G1) and (G2) admit invariant probability measures absolutely continuous with respect to Lebesgue measure provided  $\alpha$  is sufficiently small and  $a$  is sufficiently close to  $a^*$ . It follows directly from the proofs in [38] that the relative measure  $|\Delta(\lambda, \alpha) \cap (a^* - \epsilon, a^* + \epsilon)|/2\epsilon \rightarrow 1$  as  $\epsilon \rightarrow 0$ . We now introduce the combinatorics we need to state the main theorem of this paper.

For  $f \in \mathcal{E}$ , let  $C = C(f) = \{c^{(1)}, \dots, c^{(q)}\}$  be the set of critical points of  $f$ . For  $\delta > 0$  and  $1 \leq i \leq q$ , define  $C_\delta^{(i)} = C_\delta(c^{(i)}) = \{x \in I : |x - c^{(i)}| < \delta\}$ . Let  $\delta < \delta_0$  be fixed.

For  $1 \leq i \leq q$ , let  $J^{(i)}$  be a subinterval of  $C_\delta^{(i)}$  and assume that there exist  $n = n(i)$  and  $j = j(i)$  associated with  $J^{(i)}$  such that

- (1)  $f^k(J^{(i)}) \cap C_\delta = \emptyset$  for all  $0 < k < n$ , and
- (2)  $f^n(J^{(i)}) = C_\delta^{(j)}$ .

<sup>1</sup>If  $I$  is an interval, let  $c^{(0)}$  and  $c^{(q+1)}$  denote the endpoints of  $I$ . If  $I$  is a circle, we use the cyclic convention  $c^{(1)}(a) < \dots < c^{(q)}(a) < c^{(q+1)}(a) = c^{(1)}(a)$ .

Define the collection

$$\mathcal{J}_\delta = \{(J^{(i)}, n(i), j(i)) : 1 \leq i \leq q\}.$$

We associate a directed graph  $\mathcal{P}(\mathcal{J}_\delta)$  with  $\mathcal{J}_\delta$  as follows. The graph  $\mathcal{P}(\mathcal{J}_\delta)$  contains  $q$  vertices  $v_1, \dots, v_q$  representing  $c^{(1)}, \dots, c^{(q)}$ . There exists a directed edge from  $v_i$  to  $v_\ell$  in  $\mathcal{P}(\mathcal{J}_\delta)$  if and only if  $j(i) = \ell$ .

**Definition 2.3.** We say that a vertex  $v_{i_0}$  in  $\mathcal{P}(\mathcal{J}_\delta)$  is **completely accessible** if for every  $1 \leq i \leq q$ , there exists a directed path from  $v_i$  to  $v_{i_0}$  in  $\mathcal{P}(\mathcal{J}_\delta)$ .

We now state the main theorem of this paper.

**Theorem 1.** *Let  $\{f_a : a \in [a_1, a_2]\}$  be an admissible family and let  $a^* \in (a_1, a_2)$  be such that  $f_{a^*} \in \mathcal{E}$ . Fix  $\lambda \leq \frac{\lambda_0}{5}$ . Then for  $\alpha < \lambda$  sufficiently small, there exists  $\delta_1 > 0$  sufficiently small such that the following holds. If  $f_{a^*}$  admits a collection  $\mathcal{J}_\delta$  such that the directed graph  $\mathcal{P}(\mathcal{J}_\delta)$  has a completely accessible vertex for some  $\delta \leq \delta_1$ , then for every  $\hat{a} \in \Delta(\lambda, \alpha)$  sufficiently close to  $a^*$ , there exists a sequence  $a_n \rightarrow \hat{a}$  such that for every  $n \in \mathbb{N}$ , the map  $f_{a_n}$  admits a superstable periodic sink.*

**Remark 2.4.** Once a directed graph  $\mathcal{P} = \mathcal{P}(\mathcal{J}_\delta)$  is obtained, checking for the existence of a completely accessible vertex is straightforward. We associate a  $q \times q$  matrix  $D = (d_{ij})$  with  $\mathcal{P}$  as follows. Set  $d_{ij} = 1$  if there exists a directed edge from  $v_i$  to  $v_j$  in  $\mathcal{P}$  and set  $d_{ij} = 0$  otherwise. For  $n > 0$ , write  $D^n = (d_{ij}^{(n)})$ . The vertex  $v_{j_0}$  is completely accessible if and only if for every  $1 \leq i \leq q$ , there exists  $n > 0$  such that  $d_{ij_0}^{(n)} \neq 0$ .

Theorem 2 states that Theorem 1 applies to admissible families of maps with long branches. Admissible families satisfying the hypotheses of Theorem 2 frequently arise in applications of the theory of rank one maps.

**Theorem 2.** *Assume that  $I = S^1$ . Let  $\{f_a : a \in [a_1, a_2]\}$  be an admissible family and let  $a^* \in (a_1, a_2)$  be such that  $f_{a^*} \in \mathcal{E}$ . Suppose that  $f_{a^*}$  satisfies*

- (1)  $e^{\lambda_0} > 2$ , and
- (2)  $f_{a^*}((c^{(i)}, c^{(i+1)})) \supset I$  for all  $1 \leq i \leq q$ .

*Then for every  $\alpha$  sufficiently small and for every  $\hat{a} \in \Delta(\lambda, \alpha)$  sufficiently close to  $a^*$ , there exists a sequence  $a_n \rightarrow \hat{a}$  such that for every  $n \in \mathbb{N}$ , the map  $f_{a_n}$  admits a superstable periodic sink.*

The combinatorial hypothesis in Theorem 1 is always satisfied by admissible families of unimodal maps:

**Theorem 3.** *Let  $\{f_a : a \in [a_1, a_2]\}$  be an admissible family of unimodal maps and let  $a^* \in (a_1, a_2)$  be such that  $f_{a^*} \in \mathcal{E}$ . Then for every  $\alpha$  sufficiently small and for every  $\hat{a} \in \Delta(\lambda, \alpha)$  sufficiently close to  $a^*$ , there exists a sequence  $a_n \rightarrow \hat{a}$  such that for every  $n \in \mathbb{N}$ , the map  $f_{a_n}$  admits a superstable periodic sink.*

**Remark 2.5.** Thunberg [29] independently proves Theorem 3 for admissible families of unimodal maps. Ures [32] proves an analogous theorem for Hénon-like maps.

**Remark 2.6.** Theorems 2 and 3 are simply two specific propositions derived from Theorem 1. One could formulate and prove other variations, including a version of Theorem 2 for interval maps.

### 3. PROOFS OF THEOREMS 2 AND 3 ASSUMING THEOREM 1

We begin with two lemmas.

**Lemma 3.1.** *Let  $f \in \mathcal{E}$  and let  $C = C(f) = \{c^{(1)}, \dots, c^{(q)}\}$  be the set of critical points of  $f$ . The following holds provided  $\delta$  is sufficiently small. For every  $i \in \{1, \dots, q\}$ , there exists a subinterval  $J^{(i)} \subset C_\delta^{(i)}$ , an integer  $n(i) \leq 6\lambda_0^{-1} \log(\delta^{-1})$ , and  $j(i) \in \{1, \dots, q\}$  such that*

- (1)  $f^k(J^{(i)}) \cap C_\delta = \emptyset$  for all  $0 < k < n(i)$ , and
- (2)  $f^{n(i)}(J^{(i)}) = C_\delta^{(j(i))}$ .

**Proof of Lemma 3.1.** For  $i \in \{1, \dots, q\}$ , let  $S_i$  be one of the components of  $C_\delta^{(i)} \setminus \{c^{(i)}\}$ . We prove the existence of  $J^{(i)}$ ,  $n(i)$ , and  $j(i)$  in two steps. First, we iterate  $S_i$  under  $f$  until the image intersects  $C_\delta$  for

the first time. Let  $m \geq 2$  be such that  $f^k(S_i) \cap C_\delta = \emptyset$  for all  $1 \leq k < m$ . We have

$$(3.1) \quad \begin{aligned} |f^m(S_i)| &\geq |f(S_i)| \cdot \inf_{y \in f(S_i)} |(f^{m-1})'(y)| \\ &\geq \kappa \delta^2 \cdot b_0 \delta e^{\lambda_0(m-1)} \end{aligned}$$

by Definition 2.1(A1) and (B1), where  $\kappa = \frac{1}{2} \min_{x \in C_\delta} |f''(x)|$ . This exponential growth estimate implies that the images of  $S_i$  must intersect  $C_\delta$ . Let  $m_1$  be the largest  $m$  as in the above. We have  $f^{m_1}(S_i) \cap C_\delta \neq \emptyset$ . Setting  $m = m_1$  in (3.1), the inequality

$$\kappa \delta^3 b_0 e^{\lambda_0(m_1-1)} \leq 1$$

yields

$$(3.2) \quad m_1 \leq \frac{4}{\lambda_0} \log(\delta^{-1}).$$

If there exists  $\ell \in \{1, \dots, q\}$  such that  $f^{m_1}(S_i) \supset C_\delta^{(\ell)}$ , then we let  $J^{(i)}$  be one of the connected components of  $S_i \cap f^{-m_1}(C_\delta^{(\ell)})$  and we set  $j(i) = \ell$  and  $n(i) = m_1$ . Otherwise, let  $L_0 = f^{m_1}(S_i) \setminus C_\delta$ . We have  $|L_0| > \frac{1}{2} \delta_0$  if  $\delta < \frac{\delta_0}{2}$  because  $f \in \mathcal{E}$ .

We now iterate  $L_0$  forward in time and let  $L_{k+1} = L_k \setminus f^{-(k+1)}(C_\delta)$  for  $k \geq 0$ . Let  $k_0$  be the smallest  $k$  such that  $f^k(L_{k-1}) \supset C_\delta^{(\ell)}$  for some  $\ell$ . We argue that  $k_0 \leq \frac{2}{\lambda_0} \log \delta^{-1}$ . First, observe that for all  $k < k_0$  we must have  $|f^k(L_{k-1}) \cap C_\delta| \leq 4\delta$ . Moreover, for  $x \in L_{k-1}$  such that  $f^k(x) \in C_\delta$ , we have  $|(f^k)'(x)| \geq b_0 e^{\lambda_0 k}$  by Definition 2.1(B2). Therefore,

$$|L_k| \geq |L_0| - 4\delta b_0^{-1} \sum_{\ell=1}^k e^{-\lambda_0 \ell} \geq |L_0| - 4\delta b_0^{-1} e^{-\lambda_0} (1 - e^{-\lambda_0})^{-1} \geq \frac{\delta_0}{4}$$

provided

$$\delta \leq \frac{\delta_0 b_0 e^{\lambda_0} (1 - e^{-\lambda_0})}{16}.$$

This implies

$$(3.3) \quad |f^k(L_k)| \geq \frac{\delta_0}{4} b_0 \delta e^{\lambda_0 k}$$

for  $k < k_0$  by Definition 2.1(B1). On the other hand, we must have

$$|f^k(L_k)| < 1,$$

inducing a contradiction to (3.3) if  $k_0 > \frac{2}{\lambda_0} \log \delta^{-1}$ . ■

**Lemma 3.2.** *Assume that  $f \in \mathcal{E}$  satisfies assumptions (1) and (2) of Theorem 2. For all  $\delta$  sufficiently small,  $f$  admits a collection  $\mathcal{J}_\delta$  such that the directed graph  $\mathcal{P}(\mathcal{J}_\delta)$  has a completely accessible vertex. Furthermore, for  $(J^{(i)}, n(i), j(i)) \in \mathcal{J}_\delta$  we have  $n(i) \leq K_0 \log(\delta^{-1})$ , where  $K_0 = \frac{6}{\lambda_0 - \ln 2}$ .*

**Proof of Lemma 3.2.** Let  $j_0 \in \{1, \dots, q\}$  be fixed. We construct a directed graph  $\mathcal{P}(\mathcal{J}_\delta)$  with completely accessible vertex  $v_{j_0}$ . Fix  $i \in \{1, \dots, q\}$  and let  $S_i$  be one of the components of  $C_\delta^{(i)} \setminus \{c^{(i)}\}$ . Arguing as in the proof of Lemma 3.1, there exists  $m_1 \in \mathbb{Z}^+$  satisfying

$$(3.4) \quad m_1 \leq \frac{4}{\lambda_0} \log(\delta^{-1})$$

such that  $f^k(S_i) \cap C_\delta = \emptyset$  for all  $1 \leq k < m_1$  and  $f^{m_1}(S_i) \cap C_\delta \neq \emptyset$ . If  $f^{m_1}(S_i) \supset C_\delta^{(j_0)}$ , then set  $n(i) = m_1$  and let  $J^{(i)}$  be one of the components of  $S_i \cap f^{-m_1}(C_\delta^{(j_0)})$ . If  $f^{m_1}(S_i) \supset (c^{(\ell)}, c^{(\ell+1)})$  for some  $1 \leq \ell < q$ , then there exists a subinterval  $\widehat{S}_i \subset S_i$  such that  $f^{m_1}(\widehat{S}_i) \subset (c^{(\ell)}, c^{(\ell+1)})$  and  $f^{m_1+1}(\widehat{S}_i) = C_\delta^{(j_0)}$  by assumption (2) of Theorem 2. In this case, define  $n(i) = m_1 + 1$  and set  $J^{(i)} = \widehat{S}_i$ . Otherwise, let  $L_0$  be the component of  $f^{m_1}(S_i) \setminus C_\delta$  with one end at  $f^{m_1}(c^{(i)})$ . We have  $|L_0| \leq \frac{1}{2} \delta_0$  if  $\delta \leq \frac{\delta_0}{2}$  because  $f \in \mathcal{E}$ .

We claim that there exists  $m_2 \in \mathbb{Z}^+$  and a subinterval  $L_1 \subset L_0$  such that  $f^k(L_1) \cap C_\delta = \emptyset$  for  $k < m_2$  and  $f^{m_2}(L_1) = (c^{(\ell)}, c^{(\ell+1)})$  for some  $\ell \in \{1, \dots, q\}$ . This claim is proved as follows. We iterate  $L_0$  under  $f$ , deleting all parts that fall into  $C_\delta$ . After  $k$  steps, the undeleted part of  $f^k(L_0)$  is made up of finitely many

segments. Suppose that for  $k \leq n$ , none of these segments contain some  $(c^{(\ell)}, c^{(\ell+1)})$ . This implies that the undeleted part of  $f^n(L_0)$  consists of at most  $2^n$  segments. We estimate the average length of these segments at time  $n$ . The pull-back to  $L_0$  of all of the deleted parts has measure bounded above by

$$4\delta \sum_{k=1}^n 2^{k-1} b_0^{-1} e^{-\lambda_0 k} \leq \frac{\delta_0}{4}$$

by Definition 2.1(B2) provided

$$\delta \leq \frac{\delta_0 b_0 e^{\lambda_0} (1 - 2e^{-\lambda_0})}{16}.$$

The undeleted segments of  $f^n(L_0)$  therefore have total length at least

$$\frac{\delta_0}{4} b_0 \delta e^{\lambda_0 n}$$

by Definition 2.1(B1) and because  $|L_0| \geq \frac{\delta_0}{2}$ . Since  $f^n(L_0)$  consists of at most  $2^n$  segments, their average length is bounded below by

$$\frac{\delta_0}{4} b_0 \delta 2^{-n} e^{\lambda_0 n}.$$

This estimate implies the claim since  $e^{\lambda_0} > 2$ .

Applying assumption (2) of Theorem 2, there exists a subinterval  $L_2 \subset L_1$  such that  $f^{m_2+1}(L_2) = C_\delta^{(j_0)}$ . Set  $n(i) = m_1 + m_2 + 1$  and define  $J^{(i)} = S_i \cap f^{-m_1}(L_2)$ . The estimate

$$(3.5) \quad \frac{\delta_0}{4} b_0 \delta 2^{-(m_2-1)} e^{\lambda_0(m_2-1)} \leq 1$$

and (3.4) imply  $n(i) \leq K_0 \log(\delta^{-1})$ .

The vertex  $v_{j_0}$  of the directed graph  $\mathcal{P}(\mathcal{J}_\delta)$  associated with the collection  $\mathcal{J}_\delta = \{(J^{(i)}, n(i), j_0), i \in \{1, \dots, q\}\}$  is completely accessible. ■

**Proof of Theorem 2.** (Assume Theorem 1.) Let  $\delta \leq \delta_1$  be small enough that Lemma 3.2 applies. By Lemma 3.2, the map  $f_{a^*}$  admits a collection  $\mathcal{J}_\delta$  such that the directed graph  $\mathcal{P}(\mathcal{J}_\delta)$  has a completely accessible vertex. Theorem 2 is now a direct corollary of Theorem 1. ■

**Proof of Theorem 3.** (Assume Theorem 1.) Let  $\delta \leq \delta_1$  be small enough that Lemma 3.1 applies. By Lemma 3.1, the map  $f_{a^*}$  admits a directed graph  $\mathcal{P}(\mathcal{J}_\delta)$  consisting of one vertex and one directed edge. This vertex is completely accessible, so Theorem 3 follows from Theorem 1. ■

#### 4. PROOF OF THEOREM 1

**4.1. Preliminaries.** Let  $\mathcal{F} = \{f_a : a \in [a_1, a_2]\}$  be an admissible family and let  $a^* \in (a_1, a_2)$  be such that  $f_{a^*} \in \mathcal{E}$ . Let  $\lambda \leq \frac{\lambda_0}{5}$  be fixed throughout. In what follows, three positive constants are critically important. They are  $\alpha$ ,  $\delta$ , and  $\epsilon$ . The constant  $\alpha$  is as in (G1),  $\delta$  represents the size of the critical interval around each critical point, and  $\epsilon$  is used to define a small parameter interval  $\Delta_0 = [a^* - \epsilon, a^* + \epsilon]$  around  $a^*$ . In the rest of this paper, we will only consider parameters  $a \in \Delta_0$ . The constants  $\alpha$ ,  $\delta$ , and  $\epsilon$  are small numbers chosen in the order that they are listed here. That is,  $\delta$  depends on  $\alpha$  and  $\epsilon$  depends on both  $\alpha$  and  $\delta$ . All three constants are assumed to be sufficiently small and are reduced in size if necessary as we go along. The letter  $K$  represents generic constants *independent* of  $\alpha$ ,  $\delta$  and  $\epsilon$ . We allow the value of  $K$  to vary from line to line. Specific values of  $K$  we wish to track will be given subscripts.

For notational simplicity, let us assume from this point on that  $C(f_a) = C(f_{a^*}) = C$  where  $C = \{c^{(1)}, \dots, c^{(q)}\}$  for  $a \in \Delta_0$ . Let  $\mathcal{Q}_0 = \{I_\mu\}$  be the following partition on  $(-\delta, \delta)$ . Assume  $\delta = e^{-\mu_*}$  for some  $\mu_* \in \mathbb{Z}^+$ . For  $\mu \geq \mu_*$ , let  $I_\mu = (e^{-(\mu+1)}, e^{-\mu})$ . For  $\mu \leq -\mu_*$ , let  $I_\mu$  be the reflection of  $I_{-\mu}$  about 0. We define a partition  $\mathcal{Q}(c)$  on  $C_\delta(c)$  for a critical point  $c \in C$  by shifting the center of  $\mathcal{Q}_0$  from 0 to  $c$ . We refer to the members of  $\mathcal{Q}(c)$  simply as ' $I_\mu$ '. For  $I_\mu \in \mathcal{Q}(c)$ , let  $I_\mu^+$  denote the union of  $I_\mu$  and the two elements of  $\mathcal{Q}(c)$  adjacent to it (for  $I_{\mu_*}$  we use  $I_{\mu_*-1}$  as one of the adjacent intervals). For an interval  $\omega \subset C_\delta$ , we write  $\omega \approx I_\mu$  if  $I_\mu \subset \omega \subset I_\mu^+$ .

Recall that  $\{f_a\}$  is defined by  $f_a(x) = F(x, a)$  for some  $C^2$  map  $F : I \times [a_1, a_2] \rightarrow I$ . Also recall that  $\Delta(\lambda, \alpha)$  is defined through (G1) and (G2) in Section 2.3. First we have

**Lemma 4.1.** *Let  $\alpha$  be sufficiently small and assume that  $\hat{a} \in \Delta(\lambda, \alpha)$ . There exists a constant  $L > 1$  independent of  $\alpha$ ,  $\delta$ , and  $\epsilon$  such that for all  $N$  sufficiently large,  $a \in [\hat{a} - L^{-N}, \hat{a} + L^{-N}]$ ,  $c \in C$ , and  $n \leq N$ , we have*

- (1)  $d(f_a^n(c), C) \geq \min\{\delta_0/2, e^{-\alpha n}\}$ ;
- (2)  $|(f_a^n)'(f_a(c))| \geq b_1 e^{\lambda n}$ .

**Proof of Lemma 4.1.** In this proof we denote  $\widehat{K} = \|F\|_{C^2}$ . For  $n \leq N$ , let  $x_n = f_a^n(c)$  and  $\hat{x}_n = f_{\hat{a}}^n(c)$ . For  $a \in [\hat{a} - L^{-N}, \hat{a} + L^{-N}]$ , we have

$$\begin{aligned} |x_n - \hat{x}_n| &\leq \widehat{K}(|a - \hat{a}| + |x_{n-1} - \hat{x}_{n-1}|) \\ &\leq (\widehat{K} + \widehat{K}^2 + \dots + \widehat{K}^n)|a - \hat{a}| \\ &< K_1^n |a - \hat{a}| \end{aligned}$$

for some  $K_1 > 0$  depending on  $\widehat{K}$ . Let  $L > 2K_1$ . We have  $|x_n - \hat{x}_n| \ll 1$ . This implies  $|d(x_n, C) - d(\hat{x}_n, C)| = |x_n - \hat{x}_n|$ , and

$$\frac{d(x_n, C)}{d(\hat{x}_n, C)} \geq 1 - \frac{|x_n - \hat{x}_n|}{d(\hat{x}_n, C)} > 1 - \frac{1}{2} e^{\alpha n} K_1^n |a - \hat{a}| > \frac{1}{2}.$$

This combined with **(G1)** for  $d(\hat{x}_n, C)$  implies (1) of this lemma.

To prove (2) we let  $w_n = |(f_a^n)'(f_a(c))|$ ,  $\widehat{w}_n = |(f_{\hat{a}}^n)'(f_{\hat{a}}(c))|$ . We have

$$\begin{aligned} \log \frac{\widehat{w}_n}{w_n} &= \sum_{i=1}^{n-1} \log \left( 1 + \frac{f'_{\hat{a}}(\hat{x}_i) - f'_a(x_i)}{f'_a(x_i)} \right) \\ &\leq \sum_{i=1}^{n-1} \frac{\widehat{K}|x_i - \hat{x}_i|}{K^{-1}d(x_i, C)} \\ &< \left( \sum_{i=1}^{n-1} K K_1^i e^{\alpha i} \right) |a - \hat{a}| \\ &< \log 2. \end{aligned}$$

for  $L$  sufficiently large (independent of  $\alpha$ ,  $\delta$  and  $\epsilon$ ). Property (2) now follows from **(G2)** for  $\widehat{w}_n$ . ■

**4.2. Evolution of critical curves.** For  $\hat{a} \in \Delta(\lambda, \alpha)$ , denote  $\Delta_N(\hat{a}) = [\hat{a} - L^{-N}, \hat{a} + L^{-N}]$  where  $L$  is as in Lemma 4.1. It suffices to prove that there exists  $N_0$  sufficiently large such that for every  $N > N_0$ , there exists  $a_N \in \Delta_N(\hat{a})$  such that  $f_{a_N}$  admits periodic sinks. In this subsection we fix  $i_0 \in \{1, \dots, q\}$ . We define  $c = c^{(i_0)}$  and  $\gamma_n(a) = f_a^n(c)$ . First we need to study the evolutions of curves  $\gamma_n : \Delta_N(\hat{a}) \rightarrow I$ . We denote

$$\psi = \frac{3\alpha}{\lambda}.$$

For the evolutions of  $\gamma_n$ , there are four time indices worth noting. The first is the time the derivatives of  $\gamma_n$  start to grow exponentially. This time will be denoted as  $m_0$ . The second is a time index  $N \gg m_0$ , sufficiently large such that  $\Delta_N(\hat{a}) \subset \Delta_0$ . Note that  $N$  is chosen after  $\epsilon$ . The third time index is  $\frac{1}{\psi}N$ . This is a time before which the size of  $\gamma_n$  must be relatively long. Observe that  $\frac{1}{\psi}N \sim \frac{1}{\alpha}N \gg N$ . Finally, let  $N_1$  be the time we are about to find the parameter  $a_N$ . The time  $N_1$  is in general larger than  $N$  but could be either smaller or larger than  $\frac{1}{\psi}N$ . The existence of all these time indices will soon become clear.

Let us now define the set of *good parameters*  $\Pi(N_1)$  in  $\Delta_N(\hat{a})$  for the proofs of this paper. For  $N < N_1 \leq \frac{1}{\psi}N$ , we define  $\Pi(N_1)$  as the set of all  $a \in \Delta_N(\hat{a})$  such that the rule of distance exclusion (1) in Lemma 4.1, that is

$$(4.1) \quad d(f_a^n(c), C) \geq \min\{\delta_0/2, e^{-\alpha n}\},$$

holds for all  $n \leq N_1 \leq \frac{1}{\psi}N$ . For  $N_1 > \frac{1}{\psi}N$ , we define  $\Pi(N_1)$  as the set of all  $a \in \Delta_N(\hat{a})$  such that the rule of distance exclusion (4.1) holds up to time  $\frac{1}{\psi}N$  and for  $\frac{1}{\psi}N < n \leq N_1$ ,

$$(4.2) \quad d(f_a^n(c), C) \geq e^{-\alpha \frac{1}{\psi}N}.$$

We now state the properties of the evolutions  $\gamma_n$  we need in proving Theorem 1. In what follows,

$$\tau_n(a) = \frac{d}{da}\gamma_n(a).$$

**Proposition 4.2.** *Assume that  $\alpha$ ,  $\delta$ , and  $\epsilon$  are sufficiently small and suitably related. Then*

**(D1)** *there exists  $m_0 > 0$  such that  $|\tau_{m_0}(a)| > e^{\frac{1}{3}\lambda m_0}$  for all  $a \in \Delta_0$ .*

*In addition, assume that  $N \gg m_0$  is sufficiently large and let  $a \in \Delta_N(\hat{a})$  be such that  $a \in \Pi(n)$  for some  $n \geq m_0$ . Then there exist constants  $K_1, K_2 > 1$  independent of  $\alpha$ ,  $\delta$ , and  $\epsilon$  such that*

**(D2) (Outside  $C_\delta$ )** *for  $m > 0$ ,*

**(a)** *if  $\gamma_{n+k}(a) \notin C_\delta$  for all  $0 \leq k < m$ , then  $|\tau_{n+m}(a)| \geq K_1^{-1}\delta e^{\frac{1}{2}\lambda_0 m}|\tau_n(a)|$ ;*

**(b)** *if, in addition,  $\gamma_{n+m}(a) \in C_{\delta_0}$ , then  $|\tau_{n+m}(a)| \geq K_1^{-1}e^{\frac{1}{2}\lambda_0 m}|\tau_n(a)|$ ;*

**(D3) (Recovery)** *if  $\gamma_n(a) \in I_\mu$ , then there exists  $K_2^{-1}|\mu| \leq p(a) \leq \frac{3}{\lambda}|\mu|$  such that  $a \in \Pi(n + p(a) - 1)$ . We also have*

**(a)**  $|\tau_{n+p(a)}(a)| \geq e^{\frac{1}{3}\lambda p(a)}|\tau_n(a)|$ ;

**(b)** *if there is an interval  $\omega \subset \Pi(n)$  such that  $\gamma_n(\omega) \approx I_\mu$ , then  $|\gamma_{n+p}(\omega)| \geq e^{-\frac{8\alpha}{\lambda}|\mu|}$  for some  $K_2^{-1}|\mu| \leq p \leq \frac{3}{\lambda}|\mu|$ .*

Property **(D1)** states that there is a time  $m_0$  when we see exponential growth of derivatives of the critical curves with respect to  $a$  for all  $a \in \Delta_0$ . Property **(D2)** states that along the critical orbit of a good parameter (a parameter satisfying (4.1) and (4.2)), the derivatives of the critical curves grow exponentially as long as the orbit stays out of  $C_\delta$ . Property **(D3)** states that the potential drop in the derivative caused by a return to  $C_\delta$  will be compensated for by growth in future iterates. In a relatively short period of time, exponential growth of derivatives will again be observed. Note that there is no need for us to put an upper bound on  $n$  in this proposition.  $\Pi(n)$  is well-defined for all  $n > m_0$ .

Let  $a \in \Pi(N_1)$ . This proposition implies the following for the evolutions of the derivatives  $\tau_n(a)$  for  $n \leq N_1$ . The derivative for the first  $m_0$  iterates is not relevant. Let  $t_1 \geq m_0$  be the first time  $\gamma_{t_1}(a) \in C_\delta$ . We call  $t_1$  the first free return time. For  $m_0 \leq n \leq t_1$ ,  $\tau_n$  grows exponentially according to **(D2)**. The derivative will drop at  $\gamma_{t_1}(a)$  because  $\gamma_{t_1}(a)$  is close to  $C$ . However, **(D3)** claims that there exists  $p_1$  relatively small such that the derivative at  $t_1 + p_1$  has regained a definite amount of exponential growth. We call the time period from  $t_1$  to  $t_1 + p_1$  a bound period. We then have the next free return time  $t_2$  and the next bound period  $p_2$ , and so on. In this way, the time interval from  $m_0$  to  $N_1$  is divided into an alternating sequence of **free intervals**  $(t_k + p_k, t_{k+1}]$  and **bound periods**  $(t_{k+1}, t_{k+1} + p_{k+1}]$ . We have the following corollary of **(D1)-(D3)**.

**Corollary 4.3.** *For  $a \in \Pi(n)$ , let  $i > t_1 + p_1$  be a free return time such that  $i < n$ . Then*

$$|\tau_i(a)| > e^{\frac{\lambda}{4}i}.$$

**Proof of Corollary 4.3.** Let  $i$  be such that

$$m_0 \leq t_1 < t_1 + p_1 \leq \dots < t_k + p_k \leq i,$$

where  $t_j$  are the times of free returns and  $p_j$  are the corresponding bound periods before time  $i$ . Combining **(D2)(b)** and **(D3)(a)**, we have

$$\begin{aligned} |\tau_i| &> |\tau_{m_0}| K^{-1} e^{\frac{\lambda_0}{2}(t_1 - m_0)} \cdot e^{\frac{\lambda}{3}p_1} \cdot K^{-1} e^{\frac{\lambda_0}{2}(t_2 - t_1 - p_1)} \cdot e^{\frac{\lambda}{3}p_2} \dots K^{-1} e^{\frac{\lambda_0}{2}(i - t_k - p_k)} \\ &> e^{\frac{\lambda}{4}i}. \end{aligned}$$

Note that each copy of  $K^{-1}$  is absorbed by reducing the exponent of growth from  $\frac{\lambda}{3}$  in **(D3)(a)** to  $\frac{\lambda}{4}$ . It suffices to take  $\delta = e^{-\mu^*}$  small enough so that  $K^{-1}e^{\frac{1}{12}K_2^{-1}\lambda\mu^*} > 1$ . ■

4.3. **Proof of Theorem 1 assuming Proposition 4.2.** Let  $\alpha$  be small enough so that

$$(4.3) \quad e^{\frac{1}{12\alpha}} > L,$$

where  $L$  is as in Lemma 4.1. Assume that for  $\delta > 0$  sufficiently small, there exists a collection  $\mathcal{J}_\delta$  such that the directed graph  $\mathcal{P}(\mathcal{J}_\delta)$  admits a completely accessible vertex  $v_{i_0}$ . Denote  $c = c^{(i_0)}$  and let  $\gamma_n : \Delta_N(\hat{a}) \rightarrow I$  be such that  $\gamma_n(a) = f_a^n(c)$ . We assume that  $\delta > 0$  is small enough so that Proposition 4.2 holds. Choose  $N$  such that  $N > m_0$  and  $\Delta_N(\hat{a}) \subset \Delta_0$ .

**Step 1.** We prove that there exists a time  $n_0$ ,  $m_0 \ll n_0 < \frac{N}{\psi}$ , such that

- (1) for each  $n < n_0$ , either  $\gamma_n(\Delta_N(\hat{a}))$  is completely out of  $C_\delta$  or there exists  $\mu \in \mathbb{Z}$  satisfying  $|\mu| \geq \mu^*$  ( $\mu^* = \log \delta^{-1}$ ) such that  $\gamma_n(\Delta_N(\hat{a})) \subset I_\mu^+$ ;
- (2) there exists  $\mu \in \mathbb{Z}$  satisfying  $|\mu| \geq \mu^*$  such that  $\gamma_{n_0}(\Delta_N(\hat{a})) \supset I_\mu$ .

If these are false, then we have

- (a)  $|\gamma_{\frac{N}{\psi}}(\Delta_N(\hat{a}))| < 1$ , and
- (b)  $\Delta_N(\hat{a}) = \Pi(\frac{1}{\psi}N)$ .

Statement (b) holds because for  $n < \frac{N}{\psi}$ , when  $\gamma_n(\hat{a})$  returns to  $C_\delta$ , all points of  $\gamma_n(\Delta_N(\hat{a}))$  fall into the same  $I_\mu^+$  interval where  $\gamma_n(\hat{a})$  is located. It follows that we can define identical bound periods and free time intervals for all  $a \in \Delta_N(\hat{a})$ . We now argue that (b) contradicts (a) because from (b), (D2)(a), and Corollary 4.3, we have

$$|\gamma_{\frac{N}{\psi}}(\Delta_N(\hat{a}))| > K^{-1} \delta e^{\frac{\lambda}{4} \frac{N}{\psi}} L^{-N} \gg 1$$

provided that  $\frac{1}{\psi}N$  is a free time. Note that the last estimate uses (4.3). If  $\frac{1}{\psi}N$  is not free, then it is inside of a bound period, say  $[t_k, t_k + p_k)$ . We argue that  $t_k \geq \frac{1}{\psi}N - N$ . This is because  $p_k \leq N$  by (D3) and (4.1). Using Corollary 4.3 for  $t_k$  instead of  $\frac{1}{\psi}N$  induces a similar contradiction.

**Step 2.** We prove that there exists a subinterval  $\omega \subset \Delta_N(\hat{a})$  and  $N_1 > n_0$  such that  $\gamma_{N_1}(\omega) \supset C_\delta(c^{(j)})$  for some  $j \in \{1, \dots, q\}$ . This is proved as follows. Let  $|\mu_0|$  be the smallest  $|\mu| \in \mathbb{Z}^+$  such that  $\gamma_{n_0}(\Delta_N(\hat{a})) \supset I_{\mu_0}$ , and let  $\rho_0 \subset \Delta_N(\hat{a})$  be an interval such that  $\gamma_{n_0}(\rho_0) = I_{\mu_0}$ . Since  $\hat{a} \in \Delta(\lambda, \alpha)$ , the choice of  $\mu_0$  implies that  $\rho_0 \subset \Pi(n_0)$ .

Applying (D3)(b), there exists a recovery time  $p_0 \leq \frac{3}{\lambda} |\mu_0|$  such that

$$(4.4) \quad |\gamma_{n_0+p_0}(\rho_0)| \geq e^{-\frac{8\alpha}{\lambda} |\mu_0|}.$$

Suppose that for some  $\ell \geq 0$ ,  $\gamma_{n_0+p_0+k}(\rho_0)$  contains no  $I_\mu$  for all  $0 \leq k \leq \ell$ , and  $n_0 + p_0 + \ell$  is a free return time. From  $n_0 + p_0$  to  $n_0 + p_0 + \ell$ , the image of  $\rho_0$  is either free and stays outside of  $C_\delta$ , or it returns to  $C_\delta$  freely with the image completely contained inside of some  $I_\mu^+$ . Properties (D2) and (D3) together imply that  $\rho_0 \subset \Pi(n_0 + p_0 + \ell)$ , and from Corollary 4.3 we have

$$(4.5) \quad |\gamma_{n_0+p_0+\ell}(\rho_0)| \geq e^{\frac{1}{4}\lambda\ell - \frac{8\alpha}{\lambda} |\mu_0|}.$$

It then follows that there exists a free return time  $n_1 > n_0 + p_0$  such that  $\gamma_{n_1}(\rho_0) \supset I_\mu$  for some  $I_\mu$  for the first time. Let  $I_{\mu_1}$  be the longest  $I_\mu$  inside of  $\gamma_{n_1}(\rho_0)$  and let  $\rho_1 \subset \rho_0$  be a subinterval such that  $\gamma_{n_1}(\rho_1) = I_{\mu_1}$ . We have

$$|\mu_1| \leq \frac{8\alpha}{\lambda} |\mu_0| - \frac{\lambda}{4} (n_1 - (n_0 + p_0))$$

by (4.5). The choice of  $\mu_1$  implies that  $\rho_1 \subset \Pi(n_1)$ .

Inductively, suppose we have constructed  $\rho_k \subset \rho_{k-1}$ ,  $\mu_k \in \mathbb{Z}$ , and  $n_k \in \mathbb{Z}^+$  such that  $\rho_k \subset \Pi(n_k)$  and  $\gamma_{n_k}(\rho_k) = I_{\mu_k}$ . Applying (D3)(b), there exists  $p_k \leq \frac{3}{\lambda} |\mu_k|$  such that

$$(4.6) \quad |\gamma_{n_k+p_k}(\rho_k)| \geq e^{-\frac{8\alpha}{\lambda} |\mu_k|}.$$

Suppose that for some  $\ell \geq 0$ ,  $\gamma_{n_k+p_k+j}(\rho_k)$  contains no  $I_\mu$  for all  $0 \leq j \leq \ell$ , and  $n_k + p_k + \ell$  is a free return time. Then  $\rho_k \subset \Pi(n_k + p_k + \ell)$  with

$$(4.7) \quad |\gamma_{n_k+p_k+\ell}(\rho_k)| \geq e^{-\frac{8\alpha}{\lambda} |\mu_k| + \frac{1}{4}\lambda\ell}.$$

Again, let  $n_{k+1} \geq n_k + p_k$  be the smallest free return time such that  $\gamma_{n_{k+1}}(\rho_k) \supset I_\mu$  for some  $I_\mu$ . Let  $I_{\mu_{k+1}}$  be the longest  $I_\mu$  such that  $\gamma_{n_{k+1}}(\rho_k) \supset I_\mu$  and let  $\rho_{k+1} \subset \rho_k$  be a subinterval such that  $\gamma_{n_{k+1}}(\rho_{k+1}) = I_{\mu_{k+1}}$ . We have

$$(4.8) \quad |\mu_{k+1}| \leq \frac{8\alpha}{\lambda} |\mu_k| - \frac{\lambda}{4} (n_{k+1} - (n_k + p_k))$$

by (4.7). The choice of  $\mu_{k+1}$  implies that  $\rho_{k+1} \subset \Pi(n_{k+1})$ .

This inductive procedure must terminate after finitely many steps because of (4.8). So there exists a free return time  $n_k$ , a subinterval  $\rho_k \subset \Delta_N(\hat{a})$  and  $|\mu_k| = \mu^*$  such that  $\gamma_{n_k}(\rho_k) = I_{\mu_k}$ . Letting the bound period for this free return be  $p_k$ , we also have

$$|\gamma_{n_k+p_k}(\rho_k)| \geq e^{-\frac{8\alpha}{\lambda} \mu^*} \gg \delta.$$

At this point we repeat the proof of Lemma 3.1 (regarding  $\delta_0$  in that proof as  $|\gamma_{n_k+p_k}(\rho_k)|$  and putting (D2) in the position of Definition 2.1(B)). We conclude that there exists  $\omega \subset \rho_k$ ,  $N_1 > n_k + p_k$ , and a critical point  $c^{(j)} \in C$  such that

$$(4.9) \quad \gamma_{N_1}(\omega) = C_\delta(c^{(j)}).$$

**Step 3.** Recall that  $v_{i_0}$  is a completely accessible vertex of the directed graph  $\mathcal{P}(\mathcal{J}_\delta)$  associated with a collection

$$\mathcal{J} = \{(J^{(i)}, n(i), j(i)) : i \in \{1, \dots, q\}\}$$

satisfying (1) and (2) at the beginning of Section 3 for  $f = f_{a^*}$ . We start with the index  $j$  in (4.9). Since  $i_0$  is completely accessible, there is a path

$$v_j = v_{j_0} \rightarrow v_{j_1} \rightarrow \dots \rightarrow v_{j_m} = v_{i_0}$$

in  $\mathcal{P}(\mathcal{J}_\delta)$ . Let

$$T = n(j_0) + n(j_1) + \dots + n(j_{m-1}).$$

We conclude that there exists an interval  $J \subset J^{(j)}$  such that

$$(4.10) \quad f_{a^*}^T(J) = C_\delta^{(i_0)}.$$

We are finally ready to finish our construction of a parameter admitting a superstable periodic sink. Let  $T \in \mathbb{Z}^+$  and let the subinterval  $J$  be as in (4.10). Set  $\hat{K} = \|F\|_{C^1}$ . Let  $\epsilon$  be sufficiently small so that

$$\hat{K}^T |a - a^*| \leq \frac{\delta}{10}$$

for all  $a \in \Delta_0$ . Let  $\zeta \subset \omega$  be such that  $\gamma_{N_1}(\zeta) = J$ . For  $a \in \zeta$ ,

$$|\gamma_{N_1+T}(a) - f_{a^*}^T(\gamma_{N_1}(a))| = |F^T(\gamma_{N_1}(a), a) - F^T(\gamma_{N_1}(a), a^*)| \leq \hat{K}^T |a - a^*| \leq \frac{\delta}{10}.$$

Since  $f_{a^*}^T(J) = C_\delta^{(i_0)}$ , we conclude that there exists  $a \in \zeta$  such that  $f_a^{N_1+T}(c^{(i_0)}) = \gamma_{N_1+T}(a) = c^{(i_0)}$ . This finishes the proof of Theorem 1.

## 5. PROOF OF PROPOSITION 4.2

In this section we prove Proposition 4.2. The conclusions we gather in this proposition are proved in [38]. These conclusions and their proofs, however, are mixed with other more complicated considerations in [38], such as estimates on global distortions, a large deviation argument and interactions of different critical curves. To make a coherent presentation, we provide a self-contained proof in this section. We hope this will save the reader the trouble of going through the entire length of [38] to achieve a complete proof of Proposition 4.2.

**5.1. Phase space dynamics.** In this subsection we fix  $a \in \Delta_N(\hat{a})$  and assume that  $N$  is large enough so that  $\Delta_N(\hat{a}) \subset \Delta_0$  where  $\Delta_0 = [a^* - \epsilon, a^* + \epsilon]$ . Let  $\alpha$ ,  $\delta$ , and  $\epsilon$  be the same as before. See the first paragraph of Section 4 for a discussion about these constants.

5.1.1. **Outside of  $C_\delta$ .** We start with exponential growth of derivatives for orbit segments staying out of  $C_\delta$ . Let  $f = f_a$  where  $a \in \Delta_N(\hat{a}) \subset \Delta_0$ .

**Lemma 5.1 (Outside of  $C_\delta$ ).** *Let  $\epsilon$  be sufficiently small depending on  $\delta$ . We have*

- (a) *for any  $n \geq 1$ , if  $f^k(x) \notin C_\delta$  for  $0 \leq k \leq n-1$ , then  $|(f^n)'(x)| \geq b_0 \delta e^{\frac{1}{2}\lambda_0 n}$ ;*
- (b) *if, in addition,  $f^n(x) \in C_{\delta_0}$ , then  $|(f^n)'(x)| \geq b_0 e^{\frac{1}{2}\lambda_0 n}$ .*

**Proof of Lemma 5.1.** By Definition 2.1(B1), there exists  $M = M(\delta)$  such that for all  $y \in I$ , if  $f_{a^*}^k(y) \notin C_{\delta/2}(f_{a^*})$  for  $0 \leq k < M$ , then

$$|(f_{a^*}^M)'(y)| > e^{\frac{3}{4}\lambda_0 M}.$$

We choose  $\epsilon$  small enough so that  $f$  is sufficiently close to  $f_{a^*}$  for  $M$  iterates in the following sense.

- (1) If  $x$  and  $n$  are as in Lemma 5.1 and  $n \leq M$ , then  $|(f^n)'(x) - (f_{a^*}^n)'(x)|$  is small enough that the conclusions of Lemma 5.1 follow from Definition 2.1(B).
- (2) If  $f^i(y) \notin C_\delta$  for  $0 \leq i < M$ , then  $|(f^M)'(y)| > e^{\frac{1}{2}\lambda_0 M}$ .

For  $n > M$ , we let  $k$  be such that  $kM \leq n < (k+1)M$ . We estimate  $|(f^n)'(x)|$  using the chain rule, comparing  $(f^M)'(f^{iM}(x))$  with  $(f_{a^*}^M)'(f^{iM}(x))$  for  $i \leq k$  using (2) above, and comparing  $(f^{n-kM})'(f^{kM}(x))$  with  $(f_{a^*}^{n-kM})'(f^{kM}(x))$  using (1). ■

5.1.2. **Bound periods and recovery.** Let  $N$  be fixed and let  $f = f_a$ , where  $a \in \Delta_N(\hat{a}) \subset \Delta_0$ . Let  $C(f) = \{c^{(1)}, \dots, c^{(q)}\}$  denote the set of critical points of  $f$ . By Lemma 4.1 we have, for all  $c \in C(f)$  and  $n \leq N$ ,

- (G1)'  $d(f^n(c), C) \geq \min\{\delta_0/2, e^{-\alpha n}\}$ ;
- (G2)'  $|(f^n)'(f(c))| \geq b_1 e^{\lambda n}$ .

Let  $c = c^{(i)} \in C(f)$  be a critical point and let  $x \in \overline{C_\delta(c^{(i)})}$ . Intuitively, the derivative growth of the critical orbit given by (G2)' is copied to a certain extent by the orbit of  $x$ . We make this intuition precise.

**Definition 5.2.** Let  $c \in C(f)$  and suppose  $x \in C_\delta(c)$ . We define  $p(x)$ , the **bound period** of  $x$ , to be the largest positive integer  $j$  such that  $|f^i(x) - f^i(c)| \leq e^{-2\alpha i}$  for all  $i < j$ .

In what follows, let  $x_k = f^k(x)$ . We have

**Lemma 5.3 (local distortion estimate).** *Let  $c \in C(f)$  and  $x \in C_\delta(c)$ . Then for all  $y \in [c, x]$  and  $k \leq \min\{p(x), N\}$ , we have*

$$\frac{1}{2} \leq \left| \frac{(f^k)'(f(y))}{(f^k)'(f(c))} \right| \leq 2.$$

**Proof of Lemma 5.3.** First, we have

$$\log \left| \frac{(f^k)'(y_1)}{(f^k)'(c_1)} \right| \leq \sum_{j=1}^k \frac{|f'(y_j) - f'(c_j)|}{|f'(c_j)|} \leq K \sum_{j=1}^k \frac{|y_j - c_j|}{d(c_j, C)}.$$

We choose  $h_0$  large enough that  $e^{-\alpha h_0} < \frac{\delta_0}{2}$  and

$$\sum_{j=h_0+1}^{\infty} e^{-\alpha j} \ll 1.$$

Next, we choose  $\delta$  small enough that

$$\delta \sum_{j=1}^{h_0} \frac{2}{\delta_0} \left( \max_{z \in I} |f'(z)| \right)^j \ll 1.$$

Finally, we let  $\epsilon$  be small enough that  $d(c_j, C) > \frac{\delta_0}{2}$  for all  $j \leq h_0$ . Then

$$\sum_{j=1}^k \frac{|y_j - c_j|}{d(c_j, C)} \leq \delta \sum_{j=1}^{h_0} \frac{2}{\delta_0} \left( \max_{z \in I} |f'(z)| \right)^j + \sum_{j=h_0+1}^k \frac{e^{-2\alpha j}}{e^{-\alpha j}} \ll 1.$$

■

Our next lemma is a version of **(D3)** in phase space.

**Lemma 5.4.** *Let  $c \in C(f)$  and let  $x \in I_\mu \subset C_\delta(c)$  with  $|\mu| < \frac{\alpha N}{\psi}$ . Let  $p(x)$  be the bound period. Then*

$$(5.1) \quad (3 \log \|f\|_{C^1})^{-1} |\mu| \leq p(x) \leq \frac{3}{\lambda} |\mu|.$$

We also have the following.

- (a)  $|(f^{p(x)})'(x)| > e^{\frac{\lambda}{3} p(x)}$ .  
 (b) Let  $\omega \approx I_\mu \subset C_\delta(c)$  be an interval such that  $|\mu| < \frac{\alpha N}{\psi}$ . Let

$$p(\omega) = \min_{x \in I_\mu^+} p(x).$$

Then we have

$$|f^{p(\omega)}(\omega)| \geq e^{-\frac{7\alpha}{\lambda} |\mu|}.$$

**Proof of Lemma 5.4.** First we prove (5.1). Suppose  $|x - c| = e^{-h}$ . We first establish the upper bound on  $p(x)$ . For  $n < \min\{p(x), N\}$ , Lemma 5.3 and **(G2)'** imply

$$|x_n - c_n| \geq \frac{1}{2} |(f^{n-1})'(c_1)| \cdot |x_1 - c_1| \geq \frac{1}{2} b_1 e^{\lambda(n-1)} |x_1 - c_1| \geq K^{-1} e^{\lambda(n-1)} (x - c)^2.$$

This inequality implies that  $p(x) \leq \frac{3h}{\lambda} \leq N$  provided  $h$  is sufficiently large (or  $\delta$  is sufficiently small). For the lower bound, we observe that for  $n \in \mathbb{N}$ ,

$$K \|f\|_{C^1}^{n-1} e^{-2h} \geq |x_n - c_n|.$$

This inequality gives

$$p(x) \geq \frac{h}{3 \log (\|f\|_{C^1})}$$

provided  $h$  is sufficiently large (or  $\delta$  is sufficiently small).

We now prove (a). The inequality

$$K |(f^{p-1})'(c_1)| (x - c)^2 \geq |x_p - c_p| \geq e^{-2\alpha p}$$

yields

$$(5.2) \quad K |(f^{p-1})'(c_1)|^{\frac{1}{2}} |x - c| \geq e^{-\alpha p}.$$

Bounding  $|(f^p)'(x)|$  using (5.2), we have

$$\begin{aligned} |(f^p)'(x)| &= |(f^{p-1})'(x_1)| \cdot |f'(x)| \\ &\geq K^{-1} |(f^{p-1})'(c_1)|^{\frac{1}{2}} |x - c| \cdot |(f^{p-1})'(c_1)|^{\frac{1}{2}} \\ &\geq K^{-1} e^{-\alpha p} e^{\frac{1}{2} \lambda (p-1)}. \end{aligned}$$

The final quantity is greater than  $e^{\lambda p/3}$  provided that  $p$  is sufficiently large, or, equivalently, that  $\delta$  is sufficiently small.

We finish with the proof of (b). Applying Lemma 5.3, we have

$$\begin{aligned} |f^{p(\omega)}(\omega)| &= \frac{|f^{p(\omega)}(\omega)|}{|f^{p(\omega)}([c - e^{-|\mu|+1}, c + e^{-|\mu|+1}])|} \left| f^{p(\omega)}([c - e^{-|\mu|+1}, c + e^{-|\mu|+1}]) \right| \\ &\geq \frac{|f(\omega)|}{2 |f([c - e^{-|\mu|+1}, c + e^{-|\mu|+1}])|} \left| f^{p(\omega)}([c - e^{-|\mu|+1}, c + e^{-|\mu|+1}]) \right| \\ &\geq K^{-1} e^{-2\alpha p(\omega)}. \end{aligned}$$

By (5.1), we have  $p(\omega) \leq \frac{3}{\lambda} |\mu|$ . Therefore,

$$|f^{p(\omega)}(\omega)| \geq K^{-1} e^{-\frac{6\alpha}{\lambda} |\mu|} \geq e^{-\frac{7\alpha}{\lambda} |\mu|}$$

provided  $|\mu|$  is sufficiently large (or  $\delta$  is sufficiently small). ■

As a direct corollary we have

**Corollary 5.5.** *Let  $x \in I$  be such that*

$$d(f^i(x), C) > \max\{\min\{\frac{\delta_0}{2}, e^{-\alpha i}\}, e^{-\alpha \frac{N}{\psi}}\}$$

for all  $i \leq n$ . Then

$$|(f^n)'(x)| > K^{-1} \delta e^{\frac{1}{4} \lambda n}.$$

**Proof of Corollary 5.5.** We define finite sequences  $(t_k)$  and  $(p_k)$  satisfying

$$t_1 < t_1 + p_1 \leq t_2 < t_2 + p_2 \leq \dots < n$$

as follows. Let  $t_1$  be the smallest value of  $j \geq 0$  such that  $f^j(x) \in C_\delta$ . For  $k \geq 1$ , let  $p_k$  be the bound period of  $f^{t_k}(x)$  and let  $t_{k+1}$  be the smallest  $j \geq t_k + p_k$  such that  $f^j(x) \in C_\delta$ . This decomposition partitions the orbit of  $x$  into segments corresponding to time intervals  $(t_k, t_k + p_k)$  and  $[t_k + p_k, t_{k+1}]$ , during which we think of the orbit as **bound** and **free**, respectively.

Let us first observe that we have  $p_k \leq N$  for all  $k$ . This follows from the estimate of Lemma 5.4 on  $p$  and the restriction on  $d(f^i(x), C)$  assumed by this lemma. The derivatives on time intervals  $[t_k, t_k + p_k)$  and  $[t_k + p_k, t_{k+1})$  are now estimated using Lemmas 5.1(b) and 5.4(a) respectively, provided these orbit segments are completed before time  $n$ . We assume that  $p_k$  is sufficiently large (or  $\delta$  is sufficiently small) so that

$$|(f^{t_{k+1}-t_k})'(f^{t_k}(x))| \geq e^{\frac{2\lambda}{T}(t_{k+1}-t_k)}.$$

If  $f^n(x)$  is in a bound period initiated at time  $j$ , then Lemma 5.3 and (G2)' imply

$$\begin{aligned} |(f^{n-j})'(f^j(x))| &= |(f^{n-j-1})'(f^{j+1}(x))| \cdot |f'(f^j(x))| \\ &\geq \frac{1}{2} b_1 e^{\lambda(n-j-1)} |f'(f^j(x))| \\ &\geq K^{-1} d(f^j(x), C(f)) e^{\lambda(n-j-1)} \\ &\geq K^{-1} e^{-\alpha n} e^{\lambda(n-j-1)}. \end{aligned}$$

If  $t_k + p_k < n < t_{k+1}$  for some  $k \geq 1$  or  $n < t_1$ , then we have

$$|(f^{n-(t_k+p_k)})'(f^{t_k+p_k}(x))| \geq K^{-1} \delta e^{\frac{1}{2} \lambda_0 (n-(t_k+p_k))}$$

by Lemma 5.1(a). The chain rule therefore yields

$$|(f^n)'(x)| \geq K^{-1} \delta e^{\frac{1}{4} \lambda n}.$$

■

**5.2. Duality between phase space and parameter space dynamics.** Let  $C = \{c^{(1)}, \dots, c^{(q)}\}$  be the set of critical points for  $\{f_a\}$  and let  $c = c^{(i_0)} \in C$ . We now study the evolution of critical curves  $\gamma_n : \Delta_N(\hat{a}) \rightarrow I$  where  $\gamma_n(a) = f_a^n(c)$ . Remember that  $F(x, a) = f_a(x)$  is  $C^2$  and  $\hat{a} \in \Delta_0 = [a^* - \epsilon, a^* + \epsilon]$  is such that  $\hat{a} \in \Delta(\lambda, \alpha)$ . For  $n > 0$  and  $a \in \Delta_N$ , we say that  $a \in \Pi(n)$  if  $\gamma_i(a)$  satisfies (4.1) and (4.2) for  $i \leq n$ . That is, we have

$$d(\gamma_i(a), C) > \max\{\min\{\frac{1}{2} \delta_0, e^{-\alpha i}\}, e^{-\alpha \frac{N}{\psi}}\}$$

for  $i \leq n$ , where  $\psi = \frac{3\alpha}{\lambda}$ .

Recall that there are two conditions for  $\{f_a\}$  to be admissible. The first is the existence of a Misiurewicz map  $f_{a^*}$  and the second is a parameter transversality condition  $\xi(c) \neq 0$ . For  $c \in C$ , let  $\beta_c(a)$  and  $\xi(c)$  be as in Section 2.2. We have

**Lemma 5.6** (Corollary 4.2, [38]). *Let  $\epsilon$  be sufficiently small and let  $c \in C$ . Then  $\beta_c(a)$  is well-defined on  $\Delta_0$  and  $a \mapsto \beta_c(a)$  is differentiable. In addition, we have*

$$\xi(c) = \frac{d}{da} (f_a(c) - \beta_c(a)) \Big|_{a=a^*} = \frac{dc_1}{da} (a^*) + \sum_{i=1}^{\infty} \frac{\partial_a F(c_i(a^*), a^*)}{(f_{a^*}^{i_0})'(c_1(a^*))}.$$

For a proof of this lemma we refer the reader to Section 4.2 of [38]. This is a proof the reader can pick up directly from [38] without interference from other parts of that paper. We skip it here because the arguments used in proving this lemma are not related to the techniques developed so far and they are not used anywhere else in this paper.

The assumption  $\xi(c) \neq 0$  implies the equivalence of spatial and parametric derivatives.

**Proposition 5.7 (Derivative Equivalence).** *Let  $\epsilon$  be sufficiently small. Then there exists  $m_0 \in \mathbb{Z}^+$  such that the following holds. For all  $n > m_0$ ,  $a \in \Delta_N$ , and under the assumption that  $a \in \Pi(n)$ , we have*

$$(5.3) \quad \frac{1}{2} |\xi(c)| \leq \frac{\left| \frac{d}{da} \gamma_i(a) \right|}{|(f_a^{i-1})'(c_1(a))|} \leq 2 |\xi(c)|$$

for  $m_0 < i \leq n$ , where  $c_1(a) = f_a(c)$ .

**Proof of Proposition 5.7.** Computing the parametric derivative of  $\gamma_i(a)$ , we have

$$\frac{d}{da} \gamma_i(a) = (f_a)'(\gamma_{i-1}(a)) \frac{d}{da} \gamma_{i-1}(a) + \partial_a F(\gamma_{i-1}(a), a).$$

Inductively, we obtain

$$(5.4) \quad \frac{\frac{d}{da} \gamma_i(a)}{(f_a^{i-1})'(c_1(a))} = \frac{d}{da} c_1(a) + \sum_{j=1}^{i-1} \frac{\partial_a F(\gamma_j(a), a)}{(f_a^j)'(c_1(a))}.$$

Let  $W(a, i)$  denote the expression on the right side of (5.4). We choose  $m_0$  large enough that the following hold.

- (1)  $e^{-\alpha m_0 / \psi} < \delta$ ;
- (2)  $|W(a^*, m_0) - \xi(c)| \ll |\xi(c)|$ ;
- (3) for  $m_0 < i \leq n$ , we have

$$\left| \sum_{j=m_0}^{i-1} \frac{\partial_a F(\gamma_j(a), a)}{(f_a^j)'(c_1(a))} \right| \ll |\xi(c)|.$$

Condition (2) follows from Lemma 5.6 and  $\xi(c) \neq 0$ . Condition (3) is achievable because for  $m_0 \leq j \leq i-1$ , we have  $|\partial_a F(\gamma_j(a), a)| \leq K$  and  $|(f_a^j)'(c_1(a))| \geq K^{-1} \delta e^{\frac{\lambda}{2} j}$  by Corollary 5.5. Finally, let  $\epsilon > 0$  be sufficiently small so that  $|W(a, m_0) - W(a^*, m_0)| \ll |\xi(c)|$  for all  $a \in \Delta_0$ . We conclude that for  $m_0 < i \leq n$ ,

$$\begin{aligned} |W(a, i) - \xi(c)| &\leq |W(a, i) - W(a, m_0)| + |W(a, m_0) - W(a^*, m_0)| \\ &\quad + |W(a^*, m_0) - \xi(c)| \ll |\xi(c)|. \end{aligned}$$

■

We now prove Proposition 4.2.

**Proof of (D1).** Observe that the condition  $a \in \Delta_N(\hat{a})$  can be replaced by  $a \in \Delta_0$  if  $n$  in Proposition 5.7 is  $m_0 + 1$ . Property (D1) follows from Proposition 5.7 and Lemma 5.1(a) because by making  $\epsilon$  sufficiently small, we keep  $\gamma_k(\Delta_0)$  out of  $C_\delta$  for  $1 \leq k \leq m_0$ . ■

**Proof of (D2).** Property (D2) follows from Proposition 5.7 and Lemma 5.1. ■

**Proof of (D3).** Property (D3)(a) follows directly from Proposition 5.7 and Lemma 5.4(a). For (D3)(b) we assume  $\alpha$  satisfies

$$(5.5) \quad (\widehat{K})^{\frac{3\alpha}{\lambda}} < e^{\frac{1}{8}\lambda}$$

where  $\widehat{K} = \|F\|_{C^1}$ . For  $a \in \omega$ , let  $p_a$  denote the bound period of  $I_\mu$  with respect to the fixed map  $f_a$ . Define

$$p = \min_{a \in \omega} p_a.$$

For all  $a, \bar{a} \in \omega$  and  $j \leq p$ , we have

$$\begin{aligned} |\gamma_{n+j}(\bar{a}) - f_a^j(\gamma_n(\bar{a}))| &= |F^j(\gamma_n(\bar{a}), \bar{a}) - F^j(\gamma_n(\bar{a}), a)| \\ &\leq \widehat{K}^j |\bar{a} - a| \leq \widehat{K}^j |\omega|. \end{aligned}$$

Observe that  $|\omega| \leq Ke^{-\lambda n/4}$  by Corollary 5.5 and  $j \leq p \leq \frac{3}{\lambda}|\mu|$  by (5.1). Since  $\alpha$  satisfies (5.5),

$$\widehat{K}^j|\omega| \leq Ke^{-\frac{1}{4}\lambda n} \widehat{K}^{\frac{3}{\lambda}|\mu|} \leq Ke^{-\frac{1}{4}\lambda n} \widehat{K}^{\frac{3}{\lambda}\alpha n} < Ke^{-\frac{1}{4}\lambda n} e^{\frac{1}{8}\lambda n} < e^{-\frac{1}{9}\lambda n}.$$

Therefore, for  $j \leq p$  we have

$$(5.6) \quad |\gamma_{n+j}(\bar{a}) - f_a^j(\gamma_n(\bar{a}))| < e^{-\frac{1}{9}\lambda n}.$$

The estimate  $|\gamma_{n+p}(\omega)| \geq e^{-\frac{8\alpha}{\lambda}|\mu|}$  now follows from Lemma 5.4(b). ■

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