Houston analysis PhD exam: syllabus

1. Overview

Philosophy. The PhD qualifying exam in analysis will test your ability to complete exercises using the material from 6320 and 6321. While the exam will primarily focus on exercises, you may be asked to state definitions and state/prove standard results. You will not be asked to prove standard results that require long, technical proofs. For example, you may be asked to prove Chebyshev or Egorov, but you will not be asked to prove Lebesgue-Radon-Nikodym.

Preparation. Assuming thorough knowledge of the theoretical material, the best way to prepare for the qualifying exam is to solve problems. Books by Folland [1], Royden and Fitzpatrick [3], and Rudin [4] contain numerous useful exercises. Many universities upload their past qualifying exams to the internet; these past exams provide excellent practice material. I recommend starting with exams from Texas A&M and the University of Maryland.

2. Outline

(1) **Measures**
- (a) Algebras and σ-algebras of sets, product σ-algebra, Borel σ-algebra
- (b) Measures and their general properties
- (c) Types of measures: finite, σ-finite, semifinite
- (d) Complete measure, completion of a measure space
- (e) Outer measure (μ^*), μ^*-measurability
- (f) Carathéodory construction of measures: Start with a premeasure μ_0 on an algebra $\mathcal{A}$, use μ_0 to define an outer measure μ^*, and then extend μ_0 to a complete measure μ defined on the σ-algebra of μ^*-measurable sets.
- (g) Lebesgue-Stieltjes measures and their regularity properties, Lebesgue measure

(2) **Lebesgue integration**
- (a) Extended real number system
- (b) Measurable functions: definitions and basic results
- (c) Existence of sequences of simple functions converging pointwise to measurable functions (Folland, Theorem 2.10)
- (d) Integration of nonnegative functions, integrability of complex-valued functions, integral of integrable complex-valued functions
- (e) Approximation of integrable functions by 'simpler' or more regular functions (e.g. Folland, Theorem 2.26)
- (f) Monotone convergence theorem, Fatou lemma, dominated convergence theorem
- (g) Uniform integrability and the Vitali convergence theorem (see Folland, Problem 6.1.15)
- (h) Differentiation under the integral sign (e.g. Folland, Theorem 2.27)
- (i) Product measures, theorems of Fubini and Tonelli
- (j) Lebesgue integral on $\mathbb{R}^n$ (see the results of Folland, Chapter 2.6)

(3) **Modes of convergence**
- (a) Convergence in measure, pointwise a.e. convergence, L^p convergence; relationships between the various modes of convergence
- (b) Egorov theorem, Lusin theorem as a consequence of Egorov (see Folland, Problem 2.4.44)

(4) **L^p spaces**
- (a) Hölder and Minkowski inequalities
- (b) L^p is complete (and therefore a Banach space) for $1 \leqslant p \leqslant \infty$.
- (c) Interpolation (Folland, Proposition 6.10), relationships between L^p spaces (e.g. Folland, Proposition 6.12)
- (d) Representation of the dual space $(L^p)^*$ (Folland, Theorem 6.15)
- (e) Chebyshev inequality, Jensen inequality, Minkowski inequality for integrals
- (f) Characterization of sequentially relatively compact subsets of L^p in the weak topology for $1 < p < \infty$ (using the Alaoglu theorem)
- (g) Kolmogorov-Riesz compactness theorem (See [2] for an intuitive presentation of Kolmogorov-Riesz and related results.)

(5) **Elements of functional analysis**
- (a) Seminorm, norm, normed vector space, Banach space

   (b) Bounded linear map, operator norm
   (c) Linear functional, dual space of a vector space
   (d) Hahn-Banach theorems and consequences (Folland, Theorem 5.8)
   (e) Reflexive spaces
   (f) Baire category theorem, open mapping theorem, closed graph theorem, uniform boundedness principle
   (g) Alaoglu theorem

(6) **Hilbert spaces**
   (a) Cauchy-Schwarz inequality
   (b) Parallelogram law
   (c) Orthogonality, Pythagorean theorem, orthogonal complement of a closed subspace
   (d) Representation of the dual space of a Hilbert space
   (e) Orthonormal sets, Bessel inequality
   (f) Completeness of an orthonormal set, Parseval, orthonormal basis (see Folland, Theorem 5.27)
   (g) Separable Hilbert spaces
   (h) Basic notions for bounded linear operators on Hilbert spaces (unitary operators, orthogonal projections, adjoints, et cetera)

(7) **Differentiation of measures**
   (a) Signed measures, Hahn decomposition theorem, mutually singular measures, Jordan decomposition theorem
   (b) Lebesgue-Radon-Nikodym theorem
   (c) Lebesgue differentiation theorems, Lebesgue set

(8) **Differentiation of functions of one variable** (See [3] for an alternative to the presentation in Folland.)
   (a) Functions of bounded variation and their properties
   (b) Absolutely continuous functions
   (c) Fundamental theorem of calculus for Lebesgue integrals

(9) **The Fourier transform and convolution**
   (a) Basic properties of convolution, Young inequality
   (b) Approximate identities (See Pages 242–243 of Folland.)
   (c) Fourier transform on $\mathbb{R}^n$
   (d) Riemann-Lebesgue lemma
   (e) Fourier inversion theorem (Folland, Theorem 8.26), uniqueness theorem (Folland, Corollary 8.27)
   (f) Plancherel theorem

References

[1] G. B. Folland, *Real analysis*, Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, second ed., 1999. Modern techniques and their applications, A Wiley-Interscience Publication.
[2] H. Hanche-Olsen and H. Holden, *The Kolmogorov-Riesz compactness theorem*, Expo. Math., 28 (2010), pp. 385–394.
[3] H. Royden and P. Fitzpatrick, *Real Analysis (Fourth Edition)*, Prentice Hall, 2010.
[4] W. Rudin, *Real and complex analysis*, McGraw-Hill Book Co., New York, third ed., 1987.