1. Topology

**Problem 1.** Let $X$ be a nonempty compact Hausdorff space.

(a) Prove that $X$ is normal.

(b) State the Tietze extension theorem.

(c) Prove that if $X$ is also connected, then either $X$ consists of a single point or $X$ is uncountable.

**Problem 2.** Give $[0, 1]$ the usual topology. Let $X$ be a product of uncountably many copies of $[0, 1]$; view $X$ as the set of tuples $(x_\alpha)$, where $\alpha$ ranges over the nonnegative reals $\mathbb{R}^+$ and $x_\alpha \in [0, 1]$ for all $\alpha \in \mathbb{R}^+$. Give $X$ the product topology. Prove that $X$ is not first countable as follows.

(a) Let $A \subset X$ be the set of tuples $(x_\alpha)$ such that $x_\alpha = 1/2$ for all but finitely many values of $\alpha$. Let $0$ denote the tuple in $X$ with all entries equal to 0. Prove that $0 \in A$.

(b) Prove that no sequence in $A$ converges to $0$.

**Problem 3.** The Klein bottle $K$ is the quotient space obtained by starting with the unit square $\{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1\}$ and then making the identifications $(0, y) \sim (1, 1 - y)$ for all $y \in [0, 1]$ and $(x, 0) \sim (x, 1)$ for all $x \in [0, 1]$. Use the Seifert/van Kampen theorem to compute the fundamental group of $K$.

**Problem 4.** Let $X_1 \supset X_2 \supset X_3 \supset \cdots$ be a nested sequence of nonempty compact connected subsets of $\mathbb{R}^n$. Prove that the intersection $X = \bigcap_{i=1}^{\infty} X_i$ is nonempty, compact, and connected.

**Problem 5.** Let $p$ be an odd prime integer. Define $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ as follows. If $m = n$, set $d(m, n) = 0$. If $m \neq n$, set $d(m, n) = 1/(r+1)$, where $r$ is the largest nonnegative integer such that $p^r$ divides $m - n$.

(a) Prove that $d$ is a metric on $\mathbb{Z}$.

(b) With respect to the topology on $\mathbb{Z}$ induced by the metric $d$, is the set of even integers closed?

**Problem 6.** Let $D^2$ denote the closed unit disk in $\mathbb{R}^2$. Let $v : D^2 \to \mathbb{R}^2 \setminus \{0\}$ be a continuous, nonvanishing vector field on $D^2$. Prove that there exists a point $z \in S^1$ at which $v(z)$ points directly inward. Hint: argue by contradiction.

2. Manifold theory

**Problem 7.** Let $v \in \mathbb{R}^n$ be a nonzero vector. For $c \in \mathbb{R}$, define

$$L_c = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m : \langle x, v \rangle^2 = \|y\|^2 + c \right\}.$$ 

For $c \neq 0$, show that $L_c$ is an embedded submanifold of $\mathbb{R}^n \times \mathbb{R}^m$ of codimension 1. Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^m$ and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^n$.

**Problem 8.**

(a) State the Sard theorem.

(b) Let $f : S^1 \to S^2$ be a smooth map. Prove that $f$ cannot be surjective.
(c) For a plane $P$ in $\mathbb{R}^3$, let $\pi_P : \mathbb{R}^3 \to P$ denote orthogonal projection onto $P$. Suppose that $g : \mathbb{S}^1 \to \mathbb{R}^3$ is a smooth embedding. Prove that there exists a plane $P$ for which $\pi_P \circ g$ is an immersion.

**Problem 9.** Let $(s, t)$ be coordinates on $\mathbb{R}^2$ and let $(x, y, z)$ be coordinates on $\mathbb{R}^3$. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$f(s, t) = (\sin(t), st^2, s^3 - 1).$$

(a) Let $X_p$ be the tangent vector in $T_p \mathbb{R}^2$ defined by $X_p = \frac{\partial}{\partial s}|_p - \frac{\partial}{\partial t}|_p$. Compute the push-forward $f_\ast X_p$.

(b) Let $\omega$ be the smooth 1-form on $\mathbb{R}^3$ defined by $\omega = dx + xdy + y^2dz$. Compute the pullback $f^\ast \omega$.

**Problem 10.** Let $\theta$ and $\gamma$ be smooth 3-forms on $\mathbb{S}^7$. Prove that

$$\int_{\mathbb{S}^7} \theta \wedge d\gamma = \int_{\mathbb{S}^7} d\theta \wedge \gamma.$$

Hint: recall that if $\omega$ is a smooth $k$-form and $\eta$ is a smooth $l$-form, we have

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$