Geometry/Topology PhD Qualifying Examination: January 2014

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{S}^n$ denote the natural numbers, the integers, the rational numbers, the real numbers, and the unit sphere in $\mathbb{R}^{n+1}$, respectively. You are free to use well-known results in your arguments.

1. Topology

Problem 1. Let X be a nonempty compact Hausdorff space.

(a) Prove that X is normal.

(b) State the Tietze extension theorem.

(c) Prove that if X is also connected, then either X consists of a single point or X is uncountable.

Problem 2. Give $[0,1]$ the usual topology. Let X be a product of uncountably many copies of $[0,1]$; view X as the set of tuples (x_α), where α ranges over the nonnegative reals $\mathbb{R}^+$ and $x_\alpha \in [0,1]$ for all $\alpha \in \mathbb{R}^+$. Give X the product topology. Prove that X is not first countable as follows.

(a) Let $A \subset X$ be the set of tuples (x_α) such that $x_\alpha = 1/2$ for all but finitely many values of α. Let $\mathbf{0}$ denote the tuple in X with all entries equal to 0. Prove that $\mathbf{0} \in \overline{A}$.

(b) Prove that no sequence in A converges to $\mathbf{0}$.

Problem 3. The Klein bottle $\mathbb{K}$ is the quotient space obtained by starting with the unit square

$$\{(x,y) \in \mathbb{R}^2 : 0 \leqslant x, y \leqslant 1\}$$

and then making the identifications $(0,y) \sim (1, 1-y)$ for all $y \in [0,1]$ and $(x,0) \sim (x,1)$ for all $x \in [0,1]$. Use the Seifert/van Kampen theorem to compute the fundamental group of $\mathbb{K}$.

Problem 4. Let $X_1 \supset X_2 \supset X_3 \supset \cdots$ be a nested sequence of nonempty compact connected subsets of $\mathbb{R}^n$. Prove that the intersection

$$X = \bigcap_{i=1}^{\infty} X_i$$

is nonempty, compact, and connected.

Problem 5. Let p be an odd prime integer. Define $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ as follows. If $m = n$, set $d(m,n) = 0$. If $m \neq n$, set $d(m,n) = 1/(r+1)$, where r is the largest nonnegative integer such that p^r divides $m-n$.

(a) Prove that d is a metric on $\mathbb{Z}$.

(b) With respect to the topology on $\mathbb{Z}$ induced by the metric d, is the set of even integers closed?

Problem 6. Let D^2 denote the closed unit disk in $\mathbb{R}^2$. Let $\boldsymbol{v} : D^2 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ be a continuous, nonvanishing vector field on D^2. Prove that there exists a point $z \in \mathbb{S}^1$ at which $\boldsymbol{v}(z)$ points directly inward. Hint: argue by contradiction.

2. Manifold theory

Problem 7. Let $\boldsymbol{v} \in \mathbb{R}^n$ be a nonzero vector. For $c \in \mathbb{R}$, define

$$L_c = \left\{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \langle \boldsymbol{x}, \boldsymbol{v}\rangle^2 = \|\boldsymbol{y}\|^2 + c\right\}.$$

For $c \neq 0$, show that L_c is an embedded submanifold of $\mathbb{R}^n \times \mathbb{R}^m$ of codimension 1. Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^m$ and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^n$.

Problem 8.

(a) State the Sard theorem.

(b) Let $f : \mathbb{S}^1 \to \mathbb{S}^2$ be a smooth map. Prove that f cannot be surjective.

(c) For a plane P in $\mathbb{R}^3$, let $\pi_P : \mathbb{R}^3 \to P$ denote orthogonal projection onto P. Suppose that $g : \mathbb{S}^1 \to \mathbb{R}^3$ is a smooth embedding. Prove that there exists a plane P for which $\pi_P \circ g$ is an immersion.

Problem 9. Let (s,t) be coordinates on $\mathbb{R}^2$ and let (x,y,z) be coordinates on $\mathbb{R}^3$. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$f(s,t) = (\sin(t), st^2, s^3 - 1).$$

(a) Let X_p be the tangent vector in $T_p\mathbb{R}^2$ defined by $X_p = \frac{\partial}{\partial s}\big|_p - \frac{\partial}{\partial t}\big|_p$. Compute the push-forward $f_* X_p$.

(b) Let ω be the smooth 1-form on $\mathbb{R}^3$ defined by $\omega = \mathrm{d}x + x\mathrm{d}y + y^2\mathrm{d}z$. Compute the pullback $f^*\omega$.

Problem 10. Let θ and γ be smooth 3-forms on $\mathbb{S}^7$. Prove that

$$\int_{\mathbb{S}^7} \theta \wedge \mathrm{d}\gamma = \int_{\mathbb{S}^7} \mathrm{d}\theta \wedge \gamma.$$

Hint: recall that if ω is a smooth k-form and η is a smooth l-form, we have

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta.$$