Geometry/Topology PhD Qualifying Examination
August 2013

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the natural numbers, the integers, the rational numbers, and the real numbers, respectively. You are free to use well-known results in your arguments.

1. Topology

Problem 1. Let X and Y be topological spaces and let $f : X \to Y$ be a map. Prove that f is continuous if and only if for every $x \in X$ and every net (z_α) such that (z_α) converges to x, we have that $(f(z_\alpha))$ converges to $f(x)$.

Problem 2. For $n \in \mathbb{N}$, let $\mathbb{S}^n$ denote the unit sphere in $\mathbb{R}^{n+1}$.

- **(a)** Prove that $\mathbb{S}^n$ is connected and compact for every $n \in \mathbb{N}$.
- **(b)** Let $\mathbb{R}^\infty$ be the space of sequences $(x_i)_{i=1}^\infty$ of real numbers such that at most finitely many of the x_i are nonzero. Embedding $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ via $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$, we may view $\mathbb{R}^\infty$ as the union of the $\mathbb{R}^n$ as n ranges over $\mathbb{N}$. Define a topology on $\mathbb{R}^\infty$ by declaring that a set $C \subset \mathbb{R}^\infty$ is closed if and only if $C \cap \mathbb{R}^n$ is closed in $\mathbb{R}^n$ for every $n \in \mathbb{N}$. Now let $\mathbb{S}^\infty$ be the subset of $\mathbb{R}^\infty$ consisting of the union of the $\mathbb{S}^n$ as n ranges over $\mathbb{N}$. Prove that $\mathbb{S}^\infty$ is connected but not compact in $\mathbb{R}^\infty$.

Problem 3.

- **(a)** State the Urysohn lemma.
- **(b)** Let X be a normal topological space. Suppose that $X = V \cup W$, where V and W are open in X. Prove that there exist open sets V_1 and W_1 such that $\overline{V_1} \subset V$, $\overline{W_1} \subset W$, and $X = V_1 \cup W_1$.

Problem 4. Let A be an annulus bounded by inner circle C_1 and outer circle C_2. Define a quotient space Q by starting with A, identifying antipodal points on C_2, and then identifying points on C_1 that differ by $2\pi/3$ radians. Use the Seifert/van Kampen theorem to compute the fundamental group $\pi_1(Q)$.

Problem 5.

Recall that a topological space Y is said to be locally compact if for every $y \in Y$, there exists an open neighborhood U_y of y such that $\overline{U_y}$ is compact.

- **(a)** Give the definition of a *second countable* topological space.
- **(b)** Let X be a second countable, locally compact, Hausdorff space. Let $X^+ = X \cup \{\infty\}$ be the one-point compactification of X. Recall that a set V is open in X^+ if and only if V is open in X or $V = X^+ \setminus C$ for some compact set $C \subset X$. Prove that X^+ is second countable.

Problem 6.

- **(a)** Let X be a path connected topological space and let A be a path connected subset of X. Suppose there exists a continuous map $r : X \to A$ such that $r(a) = a$ for every $a \in A$. Prove that $r_* : \pi_1(X) \to \pi_1(A)$ is surjective.
- **(b)** Let D^2 denote the closed unit disk in $\mathbb{R}^2$ and notice that the unit circle $\mathbb{S}^1$ forms the boundary of D^2. Prove that there does not exist a continuous map $r : D^2 \to \mathbb{S}^1$ such that $r(z) = z$ for every $z \in \mathbb{S}^1$.

2. Manifold theory

Problem 7. Let $\boldsymbol{v} \in \mathbb{R}^n$ be a nonzero vector. For $c \in \mathbb{R}$, define

$$L_c = \left\{ (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \langle \boldsymbol{x}, \boldsymbol{v} \rangle^2 = \|\boldsymbol{y}\|^2 + c \right\}.$$

For $c \neq 0$, show that L_c is an embedded submanifold of $\mathbb{R}^n \times \mathbb{R}^m$ of codimension 1. Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^m$ and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^n$.

Problem 8.

(a) State the Sard theorem.

(b) Let $f : \mathbb{S}^1 \to \mathbb{S}^2$ be a smooth map. Prove that f cannot be surjective.

Problem 9. Let G be a Lie group with multiplication $m : G \times G \to G$ defined by $m(g, h) = gh$ and inversion $\mathrm{inv} : G \to G$ defined by $\mathrm{inv}(g) = g^{-1}$. Let e denote the identity element of G.

(a) Show that the push-forward map $m_* : T_eG \oplus T_eG \to T_eG$ is given by $m_*(X, Y) = X + Y$.

(b) Show that the push-forward map $\mathrm{inv}_* : T_eG \to T_eG$ is given by $\mathrm{inv}_*(X) = -X$.

(c) Show that $m : G \times G \to G$ is a submersion.

Problem 10. Let X be a topological space and let $A \subset X$. A retraction $r : X \to A$ is a map such that $r(x) = x$ for all $x \in A$.

(a) State the Stokes theorem for smooth orientable manifolds with boundary.

(b) Let M be a smooth n-dimensional compact connected orientable manifold with boundary. Prove that there exists no smooth retraction $r : M \to \partial M$. Hint: proceed by contradiction and consider a nonvanishing smooth $(n - 1)$-form on ∂M.