Geometry/Topology PhD Qualifying Examination
January 2012

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the natural numbers, the integers, the rational numbers, and the real numbers, respectively.

1. Topology

Problem 1. Let $(X, \mathcal{T})$ be a topological space.

(a) Let D be a directed set and let $p \in X$. What does it mean for a net $(x_\beta)_{\beta \in D}$ in X to converge to p?

(b) Let $A \subset X$. Prove that $q \in \bar{A}$ if and only if there exists a net in A that converges to q.

(c) Prove that X is Hausdorff if and only if every convergent net in X has a unique limit.

Problem 2. (quotient spaces)

(a) Let $X = \frac{\mathbb{R}}{\sim}$, where the equivalence relation $\sim$ is defined by $x \sim y$ if $x - y \in \mathbb{Z}$. Prove that X (with the quotient topology) is homeomorphic to the unit circle $\mathbb{S}^1$.

(b) Let $Y = \frac{\mathbb{R}}{\sim}$, where the equivalence relation $\sim$ is defined by $x \sim y$ if $x - y \in \mathbb{Q}$. Prove that the quotient topology on Y is the indiscrete (trivial) topology.

Problem 3. (extension theorems)

(a) State the Urysohn lemma and the Tietze extension theorem.

(b) Let $(X, \mathcal{T})$ be a connected normal topological space that contains at least two points. Prove that X contains uncountably many points.

Problem 4. (continuous bijections)

(a) Let X and Y be topological spaces with X compact and Y Hausdorff. Prove that if $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.

(b) Give an example of two topological spaces X and Y and a continuous one-to-one map $f : X \to Y$ such that X and $f(X)$ are not homeomorphic. (Here $f(X)$ is given the relative topology).

Problem 5. Let $X = [0, 1]^{\mathbb{N}}$. View X as the space of sequences $(x_n)_{n=1}^{\infty}$ with $x_n \in [0, 1]$ for all $n \in \mathbb{N}$. We define two topologies on X. First, let $\mathcal{T}_1$ be the smallest topology for which the projection map $\pi_i : X \to [0, 1]$ defined by $\pi_i((x_n)) = x_i$ is continuous for every $i \in \mathbb{N}$. Second, let $\mathcal{T}_2$ be the metric topology on X defined by the metric

$$d((x_n), (y_n)) = \sqrt{\sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|^2}.$$

Let $\mathrm{Id} : X \to X$ be the identity map. Prove or disprove the following statements.

(a) The map $\mathrm{Id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous.

(b) The map $\mathrm{Id} : (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ is continuous.

(c) The topological spaces $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are homeomorphic.

Problem 6. Let (X, d) be a compact metric space and let $g : X \to X$ be an isometry ($d(g(x), g(y)) = d(x, y)$ for all $x, y \in X$). Prove that g is onto. (Hint: Suppose that g is not onto and let $z \in X \setminus g(X)$. Show that there exists $\varepsilon > 0$ such that $d(y, z) > \varepsilon$ for every $y \in g(X)$. Since X is compact, it can be covered by finitely many balls of radius $\varepsilon/2$. Let N denote the minimal number of balls of radius $\varepsilon/2$ needed to cover X. Now derive a contradiction to the minimality of N.)

2. Smooth manifolds

Problem 7. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) = (x - 1)^2 - yz$. For what values $t \in \mathbb{R}$ is $f^{-1}(t)$ an embedded submanifold of $\mathbb{R}^3$ of dimension 2? Justify your answer.

Problem 8. Let (s, t) be coordinates on $\mathbb{R}^2$ and let (x, y, z) be coordinates on $\mathbb{R}^3$. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$f(s, t) = (\sin(t), st^2, s^3 - 1).$$

(a) Let X_p be the tangent vector in $T_p\mathbb{R}^2$ defined by $X_p = \frac{\partial}{\partial s}\big|_p - \frac{\partial}{\partial t}\big|_p$. Compute the push-forward $f_* X_p$.

(b) Let ω be the smooth 1-form on $\mathbb{R}^3$ defined by $\omega = \mathrm{d}x + x\mathrm{d}y + y^2\mathrm{d}z$. Compute the pullback $f^*\omega$.

Problem 9. (injective immersions and embeddings)

(a) Let M and N be smooth manifolds. Give the definition of an injective immersion $f : M \to N$.

(b) Let $\mathbb{T}^2 = [0,1) \times [0,1)$ be the 2-torus. Let λ be an irrational real number. Prove that the map $\gamma : \mathbb{R} \to \mathbb{T}^2$ defined by $\gamma(t) = (t, \lambda t) \pmod 1$ is an injective immersion but not an embedding.

Problem 10. Let X be a topological space and let $A \subset X$. A retraction $r : X \to A$ is a map such that $r(x) = x$ for all $x \in A$.

(a) State the Stokes theorem for smooth orientable manifolds with boundary.

(b) Let M be a smooth n-dimensional compact connected orientable manifold with boundary. Prove that there exists no smooth retraction $r : M \to \partial M$. Hint: proceed by contradiction and consider a nonvanishing smooth $(n-1)$-form on ∂M.