Analysis PhD Qualifying Examination: August 2015

Instructions. Each exercise is worth 10 total points. Solve 5 of the first 7 exercises and 1 of the final 3. In the *Graded exercises* area, please clearly list the 6 exercises you wish to have graded. Whenever you provide a counterexample, you must prove that your counterexample works.

Notation and conventions. $\mathcal{F}[\cdot]$ denotes the Fourier transform. Let m denote one-dimensional Lebesgue measure. All functions are real-valued unless explicitly stated otherwise.

Graded exercises:

Exercise 1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Throughout this problem, all functions are real-valued on X and measurable. For each part, either prove the statement or provide a counterexample.

(a) If $f_n \in L^1(\mu)$ for every $n \in \mathbb{N}$, $f \in L^1(\mu)$, and $f_n \to f$ in the $L^1(\mu)$ sense, then $f_n \to f$ in measure.

(b) If $f_n \in L^1(\mu)$ for every $n \in \mathbb{N}$, $f \in L^1(\mu)$, $f_n \to f$ in measure, and $\mu(X) < \infty$, then $f_n \to f$ in the $L^1(\mu)$ sense.

(c) If $f_n \to f$ almost uniformly, then $f_n(x) \to f(x)$ for μ-a.e. $x \in X$. (Recall that $f_n \to f$ almost uniformly if for every $\varepsilon > 0$, there exists $E \subset X$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on $X \setminus E$.

Exercise 2. Let μ and ν be finite (positive) measures on a measurable space $(X, \mathcal{M})$ and suppose that

$$\nu(E) = \int_E f \,\mathrm{d}\mu$$

for every $E \in \mathcal{M}$, where f is some nonnegative function in $L^1(\mu)$. Prove that

$$\int_X g \,\mathrm{d}\nu = \int_X gf \,\mathrm{d}\mu$$

for all $g \in L^1(\nu)$.

Exercise 3. (Analysis of a singularity)

(a) Prove that if $f \in L^p([0,1])$ and if $2 < p < \infty$, then the integral

$$\int_0^1 \frac{|f(x)|}{\sqrt{x}} \,\mathrm{d}m(x) \tag{1}$$

is finite.

(b) Prove or provide a counterexample: If $f \in L^2([0,1])$, then integral (1) is finite.

Exercise 4. (On translation) For a real-valued function f on $\mathbb{R}$, define the translate f^t by $f^t(x) = f(x-t)$.

(a) Suppose f is continuous on $\mathbb{R}$ and has compact support. Prove that $\|f^t - f\|_{L^\infty(\mathbb{R})} \to 0$ as $t \to 0$.

(b) Show that if $f \in L^1(\mathbb{R})$, then $\|f^t - f\|_{L^1(\mathbb{R})} \to 0$ as $t \to 0$.

Exercise 5. (An absolute continuity result for the integral) Let $f \in L^1(\mathbb{R})$.

(a) Prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f| \cdot \mathbf{1}_{\{x \in \mathbb{R} : |f(x)| > n\}} \,\mathrm{d}m = 0.$$

Here $\mathbf{1}$ denotes the indicator function.

(b) Prove that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every measurable set $E \subset \mathbb{R}$ satisfying $m(E) < \delta$, we have

$$\int_E |f(x)| \,\mathrm{d}m(x) < \varepsilon.$$

Exercise 6. (Absolute continuity)

- **(a)** A function $f : [0,1] \to \mathbb{R}$ is said to be *Lipschitz* if there exists $L > 0$ such that $|f(x) - f(y)| \leqslant L|x - y|$ for all $x, y \in [0,1]$. Prove that if $f : [0,1] \to \mathbb{R}$ is Lipschitz, then f is absolutely continuous.
- **(b)** Suppose $f : [0,1] \to \mathbb{R}$ is continuous on $[0,1]$ and absolutely continuous on $[\varepsilon, 1]$ for every $\varepsilon \in (0,1)$. Show that f need not be absolutely continuous on $[0,1]$ by giving a counterexample. Hint: Consider functions of the form $x^a \cos(1/x^b)$.

Exercise 7. Let $F : \mathbb{R} \to \mathbb{R}$ be a measurable function for which there exists $C > 0$ such that $|F(x)| \leqslant C|x|$ for every $x \in \mathbb{R}$. Suppose further that F is differentiable at 0 with $F'(0) = a$. Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{nF(x)}{x(1+n^2x^2)} \,\mathrm{d}m(x) = \pi a.$$

Hints: Consider the change of variable $u = nx$. You may use the fact that

$$\int_{-\infty}^{\infty} \frac{1}{1+u^2} \,\mathrm{d}m(u) = \pi.$$

Exercise 8. (On the closed graph theorem)

- **(a)** State the closed graph theorem.
- **(b)** Let $\mathcal{X}$ and $\mathcal{Y}$ be (complex) Banach spaces. Prove that if $T : \mathcal{X} \to \mathcal{Y}$ is a linear map such that $\varphi \circ T \in \mathcal{X}^*$ for every $\varphi \in \mathcal{Y}^*$, then T is a bounded linear map. Here $\mathcal{X}^*$ and $\mathcal{Y}^*$ denote the dual spaces of $\mathcal{X}$ and $\mathcal{Y}$, respectively.

Exercise 9. (Convolution) Functions in this exercise take values in $\mathbb{C}$.

- **(a)** Define the *convolution* $f * g$ of two measurable functions $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$.
- **(b)** Suppose $f, g \in L^1(\mathbb{R})$. Prove that

  $$\|f * g\|_1 \leqslant \|f\|_1 \|g\|_1 .$$

  You must prove this special case of the Young inequality yourself; do not simply quote the Young inequality.
- **(c)** Suppose that $f, g \in L^1(\mathbb{R})$. Prove that $\mathcal{F}[f * g](\gamma) = \mathcal{F}[f](\gamma)\mathcal{F}[g](\gamma)$ for all $\gamma \in \mathbb{R}$.
- **(d)** Prove that there does not exist $u \in L^1(\mathbb{R})$ such that $f = f * u$ a.e. for every $f \in L^1(\mathbb{R})$. Hint: Proceed by contradiction. Assume u exists and let $f \in L^1(\mathbb{R})$ be a function such that $\mathcal{F}[f](\gamma) \neq 0$ for all $\gamma \in \mathbb{R}$ (you do not need to give an explicit example of such an f). Now use

  $$\|\mathcal{F}[f - f * u]\|_{L^\infty(\mathbb{R})} \leqslant \|f - f * u\|_{L^1(\mathbb{R})} = 0$$

  to deduce a contradiction.

Exercise 10. (On weak convergence) Let (f_n) be a sequence of functions in $L^2([0,1])$ that converges weakly to $f \in L^2([0,1])$, meaning that

$$\lim_{n\to\infty} \int_0^1 f_n g \,\mathrm{d}m = \int_0^1 f g \,\mathrm{d}m$$

for every $g \in L^2([0,1])$. Prove that there exists $K > 0$ such that $\|f_n\|_{L^2([0,1])} \leqslant K < \infty$ for every $n \in \mathbb{N}$. Hint: Uniform boundedness principle.