ON THE NUMERICAL SOLUTION OF A NONLINEAR, NON–SMOOTH EIGENVALUE PROBLEM OR WHEN BINGHAM MEETS BRATU:
AN OPERATOR–SPLITTING APPROACH

Abstract

Some years ago, we suggested to a colleague looking for nonlinear saddle-point problems with multiple solutions (in order to test mountain-pass based solution methods) to have a look at the following elliptic one:

\[
\begin{align*}
\text{(BBPV)} & \quad \left\{ \begin{array}{l}
\text{Find } \{ u, \lambda \} \in H^1_0(\Omega) \times \mathbb{R}_+ \text{ such that }
\mu \int_{\Omega} \nabla u \cdot \nabla (v - u) dx + \tau \int_{\Omega} |\nabla v| dx - \int_{\Omega} |\nabla u| dx \geq \lambda \int_{\Omega} e^\sigma (v - u) dx, \forall v \in H^1_0(\Omega), \\
\end{array} \right.
\end{align*}
\]

where \( \Omega \) is a bounded domain of \( \mathbb{R}^2 \), \( \mu \) and \( \tau \) being both \( > 0 \).

(BBPV) is nothing, but the variational formulation of the following nonlinear, non-smooth Dirichlet problem

\[
\text{(BBPE)} \quad \left\{ \begin{array}{l}
- \mu \nabla^2 u + \tau, \partial_j(u) \ni \lambda e^\sigma \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega,
\end{array} \right.
\]

where \( \partial_j(u) \) denotes the sub-differential at \( u \) of the convex functional \( j: H^1_0(\Omega) \to \mathbb{R} \) defined by

\( j(v) = \int_{\Omega} |\nabla v| dx \). Suppose that \( \tau \sigma = 0 \) in the above formulations, then the above problem reduces to the celebrated Bratu-Gelfand problem

\[
\left\{ \begin{array}{l}
- \mu \nabla^2 u = \lambda e^\sigma \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega.
\end{array} \right.
\]

On the other hand, if, in (BBPV) and (BBPE), one replaces \( \lambda e^\sigma \) by a constant \( \sigma \), the resulting inequalities and equations model the flow of a Bingham visco-plastic medium of viscosity \( \mu \) and plasticity yield \( \tau \), in an infinitely long cylinder of cross-section \( \Omega \), with \( \sigma \), and \( u \) denoting the (algebraic) pressure drop per unit length and the flow axial velocity, respectively.

Problem (BBPV), (BBPE) has clearly the flavor of a non-smooth nonlinear eigenvalue problem for an elliptic operator. The numerical solution of such problems by minimax (mountain-pass) methods has been investigated by our colleagues Xudong Yao and Jianxin Zhou. Our goal in this lecture is to present a conceptually simpler methodology based on operator-splitting: The resulting algorithms are natural generalizations of the inverse power method for symmetric matrix eigenvalue computation.

The results of numerical experiments performed by our collaborator F. Foss will be presented.