Multiple Integrals

1. Evaluate $\displaystyle\iint_R 2y\sqrt{x}\,dxdy$ where R is the region in the first quadrant bounded by $y = x^2$, $y = 2 - x^2$, and the y-axis.

   Solution: $\frac{32}{21}$

2. Evaluate $\displaystyle\int_0^9 \int_{\sqrt{y}}^3 \sin\left(\pi x^3\right) dx\, dy$

   Solution: $\frac{2}{3\pi}$

3. Find the volume of the solid in the first octant that is bounded above by the paraboloid $z = 8 - x^2 - y^2$, below by the x, y-plane, and on the sides by the cylinder $x^2 + y^2 = 4$, the plane $\sqrt{3}\,y = x$, and the y, z-plane.

   Solution: 4π

4. Given $\displaystyle\iint_\Omega (xy^2 - 2xy)\,dxdy$ where Ω is the region in the first quadrant bounded by $y = \sqrt{x}$, $x + y = 2$, and the x-axis.

   (a) Set up a repeated integral (or integrals) integrating first with respect to x.

   (b) Set up a repeated integral (or integrals) integrating first with respect to y.

   Solution: (a) $\displaystyle\int_0^1 \int_{y^2}^{2-y} (xy^2 - 2xy)\,dx\,dy$

   (b) $\displaystyle\int_0^1 \int_0^{\sqrt{x}} (xy^2 - 2xy)\,dy\,dx + \int_1^2 \int_0^{2-x} (xy^2 - 2xy)\,dy\,dx$

5. Evaluate $\displaystyle\int_{-1}^1 \int_0^x \int_0^{xy} (6z + 3y)\,dz\,dy\,dx.$

   Solution: $\frac{2}{5}$

6. Let S be the solid in the first octant bounded by the cylinder $y^2 + z^2 = 1$ and the planes $y = x$, $x = 0$, and $z = 0$. Evaluate $\displaystyle\iiint_S 6z\,dx\,dy\,dz.$

   Solution: $\frac{3}{4}$

7. Set up a repeated integral that gives the volume of the solid in the first octant whose top is the plane $x - y + 2z = 2$, and whose base in the x, y-plane is the region bounded by $y = \frac{1}{4}x^2$, $y = 4$, and the y-axis.

Solution:

$$\iint_{\Omega} \frac{1}{2}[2 - x + y]\, dx\, dy = \int_0^4 \int_0^{2\sqrt{y}} \frac{1}{2}[2 - x + y]\, dx\, dy = \int_0^4 \int_{x^2/4}^4 \frac{1}{2}[2 - x + y]\, dy\, dx.$$

8. Set up a repeated integral in cylindrical coordinates that gives the volume of the solid in the first octant which is bounded above by the surface $z = e^{-x^2 - y^2}$, below by the x, y-plane, and on the sides by the cylinder $x^2 + y^2 = 1$.

Solution: $\displaystyle\int_0^{\pi/2} \int_0^1 \int_0^{e^{-r^2}} r\, dz\, dr\, d\theta$

9. A solid is bounded above by the surfaces $z = \sqrt{y}$ and $y + z = 2$, below by the x, y-plane, and on the sides by the planes $x = 0$ and $x = 2$. Find the volume of the solid.

Solution: $\frac{7}{3}$

10. The repeated integral $\displaystyle\int_0^{\pi/4} \int_0^1 \int_0^{\sqrt{1-r^2}} r\, dz\, dr\, d\theta$ gives the volume of a solid T. Sketch T and evaluate the repeated integral.

Solution: $\frac{\pi}{12}$

11. The repeated integral $\displaystyle\int_{\pi/6}^{\pi/2} \int_0^{\pi/2} \int_0^3 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$ gives the volume of a solid T. Sketch T and evaluate the repeated integral.

Solution: 3π

Sec. 14.9

1. In each of the following, determine whether $\mathbf{F}$ is the gradient of a function f. If it is, find all such functions f.

   (a) $\mathbf{F}(x,y) = (3x^2y^2 + 3y + x)\,\mathbf{i} + (2x^3y + 3xy - \sqrt{y})\,\mathbf{j}$.

   (b) $\mathbf{F}(x,y) = (2xe^y + 4xy + e^{2x})\,\mathbf{i} + (x^2e^y + 2x^2 + \cos 2y - 1)\,\mathbf{j}$.

Line Integrals

1. Calculate $\displaystyle\int_C \mathbf{h}(\mathbf{r})\cdot d\mathbf{r}$ where $\mathbf{h}$ is the vector field: $\mathbf{h}(x,y) = (x+2y)\,\mathbf{i} + (x-2y)\,\mathbf{j}$ and C is the line segment from $(1,1)$ to $(3,-1)$

   Solution: 0

2. Calculate $\displaystyle\int_C \mathbf{h}(\mathbf{r})\cdot d\mathbf{r}$ where $\mathbf{h}$ is the vector field: $\mathbf{h}(x,y,z) = xz\,\mathbf{i} + (y+z)\,\mathbf{j} + x\,\mathbf{k}$, and C is the curve: $\mathbf{r}(u) = e^u\,\mathbf{i} + e^{-u}\,\mathbf{j} + e^{2u}\,\mathbf{k}, \quad 0 \le u \le \ln 2$.

   Solution: $\frac{169}{24}$

3. Determine whether the given vector function $\mathbf{F}$ is the gradient of a function f. If it is, find all such functions f. (i) $\mathbf{F}(x,y) = \left(3x^2y^2 + 3y + x\right)\,\mathbf{i} + \left(2x^3y + 3xy - \sqrt{y}\right)\,\mathbf{j}$

   Solution: not a gradient

   (ii) $\mathbf{F}(x,y) = \left(2xe^y + e^{2x} - 3\right)\,\mathbf{i} + \left(x^2e^y + \cos 2y + 1\right)\,\mathbf{j}$

   Solution: $f(x,y) = x^2e^y + \frac{1}{2}\,e^{2x} - 3x + \frac{1}{2}\,\sin 2y + y + C$

4. Calculate $\displaystyle\int_C \mathbf{h}(\mathbf{r})\cdot d\mathbf{r}$ where $\mathbf{h}$ is the vector field: $\mathbf{h}(x,y) = (y^2+2xy)\,\mathbf{i} + (x^2+2xy)\,\mathbf{j}$ and C is the line segment from $(-1,2)$ to $(3,1)$

   Solution: 14

5. Let $f(x,y,z) = xy^3 + x^2z - x + z$. Calculate $\displaystyle\int_C \nabla f(\mathbf{r})\cdot d\mathbf{r}$ where C is the curve: $\mathbf{r}(u) = 2\cos u\,\mathbf{i} + 3\sin u\,\mathbf{j} + 4\,\mathbf{k}, \quad 0 \le u \le \pi$.

   Solution: 4

6. Find the work done by the force $\mathbf{F}(x,y) = x(y+1)\,\mathbf{i} - 2xy\,\mathbf{j}$ applied to an object that moves along the line segments that connect $(0,0)$, $(1,0)$, $(2,1)$ in that order.

   Solution: $\frac{7}{6}$

7. Evaluate $\int_C -xy^2\,dx + x^2y\,dy$ where C is the boundary of the triangular region with vertices $(0,0)$, $(1,0)$, $(0,2)$ traversed in the counterclockwise direction.

Solution: $\frac{2}{3}$

8. Evaluate $\int_C \left(2xe^y - x^2\right)\,dx + \left(x^2e^y + y\right)\,dy$ where C is the ellipse $\frac{1}{9}\,x^2 + \frac{1}{4}\,y^2 = 1$ oriented counterclockwise.

Solution: 0

9. Evaluate the line integral $\int_C -x^2y\,dx + xy^2\,dy$ where C is the boundary of the annular region in the first quadrant bounded by the quarter circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$ oriented counterclockwise.

Solution: 10π

10. Use Green's theorem to find the area enclosed by the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ Hint: parametrize the curve by $x = a\,\cos^3 u,\ y = a\,\sin^3 u,\ 0 \le u \le 2\pi$.

Solution: $A = \dfrac{3}{8}\pi\,a^2$