THE CONDITIONAL

Definition. When each of P and Q is a proposition, the **conditional** with **antecedent** P and **consequent** Q is denoted by $P \Rightarrow Q$ and is read "P implies Q" or "if P then Q."

By definition, the conditional statement $P \Rightarrow Q$ is false when P is true and Q is false. Otherwise, $P \Rightarrow Q$ is true.

TRUTH TABLE FOR THE CONDITIONAL

Truth Table for the Conditional Statement $P \Rightarrow Q$

P	Q	$P \Rightarrow Q$
T	T	T
F	T	T
T	F	F
F	F	T

Note. The truth or falsity of a $P \Rightarrow Q$ statement is determined by the truth table - not by your intuition. Note that $P \Rightarrow Q$ is always true when P is false.

Note. If P is true and $P \Rightarrow Q$ is true, then Q is true. This is important mathematics.

Note. Unlike in common English usage, in propositional logic, there need not be any connection between the antecedent and consequent in a conditional statement.

EXAMPLES

Example. "If $2 + 3 = 5$, then the sky is blue." is a true conditional statement. The antecedent is true and the consequent is true.

Example. "If $2 + 3 = 6$, then the sky is green." is a true conditional statement. The antecedent is false and the consequent is false.

Example."If $2 + 3 = 5$, then the sky is green." is a false conditional statement. The antecedent is true and the consequent is false.

Example. "If $2 + 3 = 6$, then the sky is blue." is a true conditional statement. The antecedent is false and the consequent is true.

OTHER WAYS TO EXPRESS THE CONDITIONAL

Note. In addition to "if P, then Q'' and "P implies Q'', here are some other ways to express $P \Rightarrow Q$.

"if P, Q"	"P only if Q"
"P is sufficient for Q"	"a sufficient condition for Q is P"
"Q if P"	"Q whenever P"
"Q when P"	"Q is necessary for P"
"a necessary condition for P is Q"	"Q follows from P''
"Q unless $\sim P$"	"Q provided that P"

CONVERSE, INVERSE, AND CONTRAPOSITIVE

Definition. Related to the conditional statement $P \Rightarrow Q$ is its **converse**, $Q \Rightarrow P$, its **inverse**, $\sim P \Rightarrow \sim Q$, and its **contrapositive**, $\sim Q \Rightarrow \sim P$.

Example. For the conditional **statement**

"If the sun is shining, then he wears a hat."

the **converse** is

"If he wears a hat, then the sun is shining."

the **inverse** is

"If the sun is not shining, then he does not wear a hat."

and the **contrapositive** is

"If he does not wear a hat, then the sun is not shining."

CONVERSE, INVERSE, CONTRAPOSITIVE, AND PROPOSITIONAL EQUIVALENCE

Definition. Saying that two propositions are **logically equivalent** means that they have the same truth value for each possible assignment of truth values to their elementary constituents.

Note. Here are the truth tables for $P \Rightarrow Q$ and $\sim Q \Rightarrow \sim P$.

P	Q	$P \Rightarrow Q$
T	T	T
F	T	T
T	F	F
F	F	T

P	Q	$\sim Q$	$\sim P$	$\sim Q \Rightarrow \sim P$
T	T	F	F	T
F	T	F	T	T
T	F	T	F	F
F	F	T	T	T

As you can see from the last columns in the two tables, $P \Rightarrow Q$ and $\sim Q \Rightarrow \sim P$ are logically equivalent.

CONVERSE, INVERSE, CONTRAPOSITIVE, AND PROPOSITIONAL EQUIVALENCE

Note. Here are the truth tables for $Q \Rightarrow P$ and $\sim P \Rightarrow \sim Q$.

P	Q	$Q \Rightarrow P$
T	T	T
F	T	F
T	F	T
F	F	T

P	Q	$\sim P$	$\sim Q$	$\sim P \Rightarrow \sim Q$
T	T	F	F	T
F	T	T	F	F
T	F	F	T	T
F	F	T	T	T

As you can see from the last columns in the two tables, $Q \Rightarrow P$ and $\sim P \Rightarrow \sim Q$ are logically equivalent.

CONVERSE, INVERSE, CONTRAPOSITIVE, AND PROPOSITIONAL EQUIVALENCE

Note. Here are the truth tables for $P \Rightarrow Q$ and $Q \Rightarrow P$.

P	Q	$P \Rightarrow Q$
T	T	T
F	T	T
T	F	F
F	F	T

P	Q	$Q \Rightarrow P$
T	T	T
F	T	F
T	F	T
F	F	T

As you can see from the last columns in the two tables, $P \Rightarrow Q$ and $Q \Rightarrow P$ are **NOT** logically equivalent. It is a common error to confuse a conditional statement with its converse.

THE BICONDITIONAL

Definition. When each of P and Q is a proposition, the **biconditional** between P and Q is denoted by $P \Leftrightarrow Q$ or P iff Q and is read "P if and only if Q."

The biconditional is also read "P is necessary and sufficient for Q."

By definition, $P \Leftrightarrow Q$ is true when P and Q have the same truth value and false when they have opposite truth values.

TRUTH TABLE FOR THE BICONDITIONAL

Truth Table for the Biconditional Statement $P \Leftrightarrow Q$

P	Q	$P \Leftrightarrow Q$
T	T	T
F	T	F
T	F	F
F	F	T

From this we see that compound propositions P and Q are logically equivalent when and only when $P \Leftrightarrow Q$ is a tautology.

THE BICONDITIONAL

Note. $P \Leftrightarrow Q$ is logically equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. This is important and is easy to verify with a truth table.

In English, the "if and only if " construction is sometimes expressed by an "if, then" or an "only if" construction with the other part being implied but not stated explicitly. In the statement "If you finish your meal, then you can have desert." the intent is "You can have desert if and only if you finish your meal."

In propositional logic and in mathematics, we will always distinguish between the implication and the biconditional. We have the symbolism and terminology that makes this easy to do.

TRUTH TABLE EXAMPLE

Example Problem. Construct a truth table for $(P \wedge Q) \Rightarrow (P \vee \sim Q)$.

Solution. There are only two elementary atomic propositions P, and Q so there will be $2^2 = 4$ rows. The first column will be headed by P, the second by Q, and the last by $(P \wedge Q) \Rightarrow (P \vee \sim Q)$. Looking at $(P \wedge Q) \Rightarrow (P \vee \sim Q)$, we see that we will need a column for $(P \wedge Q)$ and a column for $(P \vee \sim Q)$. Looking at $(P \vee \sim Q)$ we see that we will need a column for $\sim Q$. Thus the framework for the table is as follows.

P	Q	$\sim Q$	$(P \wedge Q)$	$(P \vee \sim Q)$	$(P \wedge Q) \Rightarrow (P \vee \sim Q)$

TRUTH TABLE EXAMPLE

As noted earlier, we begin with

P	Q	$\sim Q$	$(P \wedge Q)$	$(P \vee \sim Q)$	$(P \wedge Q) \Rightarrow (P \vee \sim Q)$
T	T				
F	T				
T	F				
F	F				

.

TRUTH TABLE EXAMPLE

Continuing,we have

P	Q	$\sim Q$	$(P \wedge Q)$	$(P \vee \sim Q)$	$(P \wedge Q) \Rightarrow (P \vee \sim Q)$
T	T	F	T	T	T
F	T	F	F	F	T
T	F	T	F	T	T
F	F	T	F	T	T

..

The compound proposition $(P \wedge Q) \Rightarrow (P \vee \sim Q)$ is always true. It is another example of a **tautology**.

PRECEDENCE OF LOGICAL OPERATORS

The precedence of the logical operators which have now been defined is given in the following table.

Operator	Precedence
$\sim$	1
$\wedge$	2
$\vee$	3
$\Rightarrow$	4
$\Leftrightarrow$	5

PRECEDENCE OF LOGICAL OPERATORS

Unless directed otherwise by parentheses, operators with higher precedence are applied before operators with lower precedence

Examples. $\sim P \Rightarrow Q$ means $R \Rightarrow Q$ where R is $\sim P$ while $\sim (P \Rightarrow Q)$ means $\sim R$ where R is $P \Rightarrow Q$.

$P \wedge Q \Rightarrow R$ means $S \Rightarrow R$ where S is $P \wedge Q$ while $P \wedge (Q \Rightarrow R)$ means $P \wedge S$ where S is $Q \Rightarrow R$.

PRECEDENCE OF LOGICAL OPERATORS

Note. Parentheses may be used for clarity even when they are not required by the rules of precedence

Example. $P \wedge Q \vee R \wedge S$ is easier to understand if it is written $(P \wedge Q) \vee (R \wedge S)$.

LOGICAL EQUIVALENCES

Note. Here are some logical equivalences to add to the list of those given in Section 1.1

Theorem.

$$\begin{array}{rcc} P \Rightarrow Q & \equiv & \sim P \vee Q \\ P \Leftrightarrow Q & \equiv & (P \Rightarrow Q) \wedge (Q \Rightarrow P) \\ \sim (P \Rightarrow Q) & \equiv & P \wedge \sim Q \\ \sim (P \wedge Q) & \equiv & P \Rightarrow Q \\ P \Rightarrow (Q \Rightarrow R) & \equiv & (P \wedge Q) \Rightarrow R \\ P \Rightarrow (Q \wedge R) & \equiv & (P \Rightarrow Q) \wedge (P \Rightarrow R) \\ (P \vee Q) \Rightarrow R & \equiv & (P \Rightarrow R) \vee (Q \Rightarrow R) \end{array}$$

EXAMPLES.

Note. See Section 1.2 of the text for more helpful examples.