Section 1.3

Quantified Statements

Section Summary [1]

Predicates   Open Sentences

Variables

Quantifiers

- Universal Quantifier
- Existential Quantifier

Negating Quantifiers

- De Morgan’s Laws for Quantifiers

Translating English to Logic

Propositional Logic Not Enough

to analyze sentences that refer to some or all members of a certain type.

If we have:

"All men are mortal."

"Socrates is a man."

Does it follow that "Socrates is mortal?"

Can't be represented in propositional logic.

It needs a language that talks about objects, their properties, and their relations.

Introducing Predicate Logic

Predicate logic uses the following new features:

- Variables
- Predicates: or Open Sentences
- Quantifiers

Propositional functions are a generalization of propositions.

- They contain variables and a predicate, e.g., $P(x)$
- Variables can be replaced by elements from a *domain*. This is also call the universe or domain of discourse.

Propositional Functions

or Open Sentences

Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the *domain* or *bound* by a quantifier,

The statement *P(x)* is said to be the value of the propositional function *P* at *x*.

For example, let *P(x)* denote "$x > 0$" and the domain be the integers. Then:

P(-3) is false.

P(0) is false.

P(3) is true.

The truth set for an open sentence or predicate consists of those members of its domain of discourse that cause it to be a true proposition.

Often the domain is denoted by *U*. So in this example *U* is the integers.

or universe

Examples of Propositional Functions

Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find these truth values:

R(2,-1,5)

Solution: F

R(3,4,7)

Solution: T

R(x, 3, z)

Solution: Not a Proposition

Now let "$x - y = z$" be denoted by $Q(x, y, z)$, with U as the integers. Find these truth values:

Q(2,-1,3)

Solution: T

Q(3,4,7)

Solution: F

Q(x, 3, z)

Solution: Not a Proposition

Compound Expressions

Connectives from propositional logic carry over to predicate logic.

If *P(x)* denotes "$x > 0$," find these truth values:

P(3) ∨ P(-1) **Solution**: T

P(3) ∧ P(-1) **Solution**: F

P(3) → P(-1) **Solution**: F

P(3) → ¬P(-1) **Solution**: T

Expressions with variables are not propositions and therefore do not have truth values. For example,

P(3) ∧ P(*y*)

P(*x*) → P(*y*)

When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions.

Quantifiers

We need *quantifiers* to express the meaning of English words including *all* and *some*:

- "All men are Mortal."
- "Some cats do not have fur."

The two most important quantifiers are:

- *Universal Quantifier*, **"**For all," symbol: $\forall$
- *Existential Quantifier*, "There exists," symbol: $\exists$

We write as in $\forall x\, P(x)$ and $\exists x\, P(x)$.

$\forall x\, P(x)$ asserts $P(x)$ is true for <u>every</u> x in the *domain*.

$\exists x\, P(x)$ asserts $P(x)$ is true for <u>some</u> x in the *domain*.

The quantifiers are said to bind the variable x in these expressions.

Universal Quantifier

$\forall x\, P(x)$ is read as "For all x, P(x)" or "For every x, P(x)"

Examples:

1) If $P(x)$ denotes "$x > 0$" and U is the integers, then $\forall x\, P(x)$ is false.

2) If $P(x)$ denotes "$x > 0$" and U is the positive integers, then $\forall x\, P(x)$ is true.

3) If $P(x)$ denotes "x is even" and U is the integers, then $\forall x\, P(x)$ is false.

Existential Quantifier

$\exists x\, P(x)$ is read as "For some x, P(x)", or as "There is an x such that P(x)," or "For at least one x, P(x)."

Examples:

1. If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.

2. If $P(x)$ denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is false.

3. If $P(x)$ denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is true.

Thinking about Quantifiers

When the domain of discourse is finite, we can think of quantification as looping through the elements of the domain.

To evaluate $\forall x\, P(x)$ loop through all x in the domain.

- If at every step $P(x)$ is true, then $\forall x\, P(x)$ is true.
- If at a step $P(x)$ is false, then $\forall x\, P(x)$ is false and the loop terminates.

To evaluate $\exists x\, P(x)$ loop through all x in the domain.

- If at some step, $P(x)$ is true, then $\exists x\, P(x)$ is true and the loop terminates.
- If the loop ends without finding an x for which $P(x)$ is true, then $\exists x\, P(x)$ is false.

Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

Properties of Quantifiers

The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.

Examples:

1. If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.
2. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
3. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true. But if $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false.

Precedence of Quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.

For example, $\forall x\ P(x) \vee Q(x)$ means $(\forall x\ P(x)) \vee Q(x)$

$\forall x\ (P(x) \vee Q(x))$ means something different.

Unfortunately, often people write $\forall x\ P(x) \vee Q(x)$ when they mean $\forall x\ (P(x) \vee Q(x))$.

Translating from English to Logic[1]

Example 1: Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, define a propositional function J(*x*) denoting "x has taken a course in Java" and translate as $\forall x\, J(x)$.

Solution 2: But if *U* is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow J(x))$.

$\forall x\, (S(x) \wedge J(x))$ is not correct. What does it mean?

It means all people are in the class and have taken a course in Java.

Translating from English to Logic [2]

Example 2: Translate the following sentence into predicate logic: “Some student in this class has taken a course in Java.”

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, translate as

$\exists x\, J(x)$

Solution 2: But if *U* is all people, then translate as

$\exists x\, (S(x) \land J(x))$

$p \rightarrow q \equiv \sim p \lor q$

$\exists x\, (S(x) \rightarrow J(x))$ is not correct. What does it mean?

Since the implication is equivalent to NOT S(x) OR J(x), it means there is a person who either is not a student in this class or has taken a course in Java.

Returning to the Socrates Example

Introduce the propositional functions *Man(x)* denoting "*x* is a man" and *Mortal(x)* denoting "*x* is mortal." Specify the domain as all people and ideas.

The two premises are: $\forall x\left(Man\left(x\right)\rightarrow Mortal\left(x\right)\right)$

$$Man\left(Socrates\right)$$

The conclusion is: $Mortal\left(Socrates\right)$

Later we will show how to prove that the conclusion follows from the premises.

Equivalences in Predicate Logic

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value

- for every predicate substituted into these statements and
- for every domain of discourse used for the variables in the expressions.

The notation $S \equiv T$ indicates that S and T are logically equivalent.

Example: $\forall x\ \neg\neg S(x) \equiv \forall x\ S(x)$

Thinking about Quantifiers as Conjunctions and Disjunctions

If the domain is finite, a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers and an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers.

If U consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$

$$\exists x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$

Even if the domains are infinite, you can still think of the quantifiers in this fashion

Negating Quantified Expressions [1]

Consider $\forall x\, J(x)$

"Every student in your class has taken a course in Java."

Here $J(x)$ is "x has taken a course in Java" and

the domain is students in your class.

Negating the original statement gives "It is not the case that every student in your class has taken Java." This implies that "There is a student in your class who has not taken Java."

Symbolically $\neg \forall x\, J(x)$ and $\exists x \neg J(x)$ are equivalent

Negating Quantified Expressions 2

Now Consider $\exists x\, J(x)$

"There is a student in this class who has taken a course in Java."

Where $J(x)$ is "x has taken a course in Java."

Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

Symbolically $\neg\exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent

De Morgan's Laws for Quantifiers

The rules for negating quantifiers are:

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	***Equivalent Statement***	***When Is Negation True?***	***When False?***
$\neg \exists x P(x)$	$\forall x \neg P(x)$	For every *x, P*(*x*) is false.	There is *x* for which *P*(*x*) is true.
$\neg \forall x P(x)$	$\exists x \neg P(x)$	There is an *x* for which *P* (*x*) is false.	*P*(*x*) is true for every *x*.

The reasoning in the table shows that:

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

These are important.

Translation from English to Logic

Examples:

"Some student in this class has visited Mexico."

Solution: Let *M*(*x*) denote "*x* has visited Mexico" and *S*(*x*) denote "*x* is a student in this class," and *U* be all people.

$$\exists X \left(S\left(X \right) \wedge M\left(X \right) \right)$$

"Every student in this class has visited Canada or Mexico."

Solution: Add *C*(*x*) denoting "*x* has visited Canada."

$$\forall X \left(S\left(X \right) \rightarrow \left(M\left(X \right) V C\left(X \right) \right) \right)$$

Translating from English into Logical Expressions [1]

U = {fleegles, snurds, thingamabobs}

$F(x)$: x is a fleegle

$S(x)$: x is a snurd

$T(x)$: x is a thingamabob

Translate "Everything is a fleegle"

Solution: $\forall x\ F(x)$

Translating from English into Logical Expressions 2

U = {fleegles, snurds, thingamabobs}

$F(x)$: x is a fleegle

$S(x)$: x is a snurd

$T(x)$: x is a thingamabob

"Nothing is a snurd."

Solution: $\neg\exists x\, S(x)$ What is this equivalent to?

Solution: $\forall x \neg S(x)$

Translating from English into Logical Expressions 3

U = {fleegles, snurds, thingamabobs}

F(x): *x* is a fleegle

S(x): *x* is a snurd

T(x): *x* is a thingamabob

“All fleegles are snurds.”

Solution: $\forall X\left(F(X) \rightarrow S(X)\right)$

Some Fun with Translating from English into Logical Expressions 4

U = {fleegles, snurds, thingamabobs}

F(x): *x* is a fleegle

S(x): *x* is a snurd

T(x): *x* is a thingamabob

"Some fleegles are thingamabobs."

Solution: $\exists X\left(F(X) \wedge T(X)\right)$

Translating from English into Logical Expressions 5

U = {fleegles, snurds, thingamabobs}

F(x): *x* is a fleegle

S(x): *x* is a snurd

T(x): *x* is a thingamabob

"No snurd is a thingamabob."

Solution: $\neg\exists X(S(X)\wedge T(X))$ What is this equivalent to?

Solution: $\forall X(\neg S(X)\vee\neg T(X))$

Nested Quantifiers

Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

Example: "Every real number has an additive inverse" is

$\forall x \exists y(x + y = 0)$

where the domains of x and y are the real numbers.

$\forall x \exists y(x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$ is $\exists y\, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x\ \forall y P(x,y)$ and $\forall y\ \forall x P(x,y)$ have the same truth value.
2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall x\ \exists y Q(x,y)$ is true, but $\exists y\ \forall x Q(x,y)$ is false.

There is no y such that x+y=0 for all x.

Questions on Order of Quantifiers 1

Example 1: Let *U* be the real numbers,

Define *P(x,y)* : $x \cdot y = 0$

What is the truth value of the following:

1. $\forall x \forall y P(x, y)$ $1 \cdot 1 \neq 0$

   Answer:False

2. $\forall x \exists y P(x, y)$ Let $y = 0$

   Answer:True

3. $\exists x \forall y P(x, y)$ Let $x = 0$

   Answer:True

4. $\exists x \exists y P(x, y)$ Let $x = 0$ or let $y = 0$

   Answer:True

Questions on Order of Quantifiers 2

Example 2: Let *U* be the real numbers,

Define *P(x,y)* : *x / y* = 1

What is the truth value of the following:

1. $\forall x \forall y P(x, y)$ — $1/2 \neq 1$

   Answer:False

2. $\forall x \exists y P(x, y)$ — $0/y \neq 1$ for all y

   Answer:False

3. $\exists x \forall y P(x, y)$ — $\frac{x}{x+1} \neq 1$ for all x

   Answer:False

4. $\exists x \exists y P(x, y)$ — Set $x = y = 1$.

   Answer:True

Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	*P*(*x*,*y*) is true for every pair *x*,*y*.	There is a pair *x*, *y* for which *P*(*x*,*y*) is false.
$\forall x \exists y P(x, y)$	For every *x* there is a *y* for which *P*(*x*,*y*) is true.	There is an x such that *P*(*x*,*y*) is false for every *y*.
$\exists x \forall y P(x, y)$	There is an *x* for which *P*(*x*,*y*) is true for every *y*.	For every *x* there is a y for which *P*(x,y) is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair *x*, *y* for which *P*(*x*,*y*) is true.	*P*(x,y) is false for every pair *x*,*y*

Translating Nested Quantifiers into English

Example 1: Translate the statement

$$\forall x\big(C(X) \vee \exists y\big(C(y) \wedge F(X,y)\big)\big)$$

where C(x) is "*x* has a computer," and *F*(*x*,*y*) is "*x* and *y* are friends," and the domain for both *x* and *y* consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 2: Translate the statement

$$\exists x \forall y \forall z\big(\big(F(x,y) \wedge F(x,z) \wedge (y \neq z)\big) \rightarrow \neg F(y,z)\big)$$

Solution: There is a student none of whose friends are also friends with each other.

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:
   "For every two integers, if these integers are both positive, then the sum of these integers is positive."
2. Introduce the variables *x* and *y*, and specify the domain, to obtain:
   "For all positive integers *x* and *y*, *x* + *y* is positive."
3. The result is:

$$\forall x \forall y \left(\left(x > 0 \right) \wedge \left(y > 0 \right) \right) \rightarrow \left(x + y > 0 \right)$$

where the domain of both variables consists of all integers

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let *P(w,f)* be "*w* has taken *f* " and *Q*(*f,a*) be "*f* is a flight on *a* ."

2. The domain of *w* is all women, the domain of *f* is all flights, and the domain of *a* is all airlines.

3. Then the statement can be expressed as:

$$\exists w \forall a \exists f \left(P\left(w, f\right) \wedge Q\left(f, a\right)\right)$$

Questions on Translation from English

Choose the obvious predicates and express in predicate logic.

Example 1: “Brothers are siblings.”

Solution: $\forall x\ \forall y\ (B(\mathrm{x,y}) \rightarrow S(\mathrm{x,y}))$

Example 2: “Siblinghood is symmetric.”

Solution: $\forall x\ \forall y\ (S(x,y) \rightarrow S(y,x))$

Example 3: “Everybody loves somebody.”

Solution: $\forall x\ \exists y\ L(x,y)$

Example 4: “There is someone who is loved by everyone.”

Solution: $\exists y\ \forall x\ L(x,y)$

Example 5: “There is someone who loves someone.”

Solution: $\exists x\ \exists y\ L(x,y)$

Example 6: “Everyone loves himself”

Solution: $\forall x\ L(x,x)$

Negating Nested Quantifiers

Example 1: Recall the logical expression developed three slides back:

$$\exists w \forall a \exists f \left(P(w,f) \wedge Q(f,a)\right)$$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg \exists w \forall a \exists f \left(P(w,f) \wedge Q(f,a)\right)$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:a

1. $\neg \exists w \forall a \exists f \left(P(w,f) \wedge Q(f,a)\right)$
2. $\forall w \neg \forall a \exists f \left(P(w,f) \wedge Q(f,a)\right)$ by De Morgan's for ∃
3. $\forall w \exists a \neg \exists f \left(P(w,f) \wedge Q(f,a)\right)$ by De Morgan's for ∀
4. $\forall w \exists a \forall f \neg \left(P(w,f) \wedge Q(f,a)\right)$ by De Morgan's for ∃
5. $\forall w \exists a \forall f \left(\neg P(w,f) \vee \neg Q(f,a)\right)$ by De Morgan's for ∧.

Part 3: Can you translate the result back into English?

Solution:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"