Cardinality of Sets
Section 2.5

Dr. Philip Walker

Note. In the slides and notes, we have used $\mathbb{N}$ for the **natural numbers**. In the text, the symbol $\mathbf{Z}^+$ is used for this set.

$$\mathbb{N} = \mathbf{Z}^+ = \{1, 2, 3, \ldots\}$$

The natural numbers are also called the **positive integers**.

Definition. When n is a natural number, $\mathbb{N}_n$ is the set consisting of the first n natural numbers.

$$\mathbb{N}_n = \{1, 2, \ldots, n\}$$

$$\mathbb{N}_n = \{k \in \mathbb{N} : 1 \leq k \leq n\}$$

Note. The following theorem can be proved by the principle of mathematical induction.

Theorem. If each of m and n is a natural number and there is a one-to-one correspondence between $\mathbb{N}_n$ and $\mathbb{N}_m$, i.e., between the first n natural numbers and the first m natural numbers, then $m = n$.

Definition. Saying that a set S is **finite** means that there is a one-to-one correspondence between S and $\mathbb{N}_n$ for some $n \in \mathbb{N}$. (When this is the case, it follows from the previous theorem that there is only one such n)

When S is finite, the **cardinality** of S or the number of elements in S is natural number n such that there is a one-to-one correspondence between S and $\mathbb{N}_n$. The cardinality of S is denoted by

$$|S|$$

Cardinality[1]

Definition: The *cardinality* of a set A is equal to the cardinality of a set B, denoted

$$|A| = |B|,$$

if and only if there is a one-to-one correspondence (*i.e.*, a bijection) from A onto B.

If there is a one-to-one function (*i.e.*, an injection) from A into B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$.

When $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and write $|A| < |B|$.

Cardinality 2

Definition: A set that is either finite or has the same cardinality as the set of positive integers ($\mathbf{Z}^+$) is called *countable*. A set that is not finite and not countable is *uncountable*.

The set of real numbers **R** is an uncountable set.

When an infinite set is countable (*countably infinite*) its cardinality is $\aleph_0$

Showing that a Set is Countable

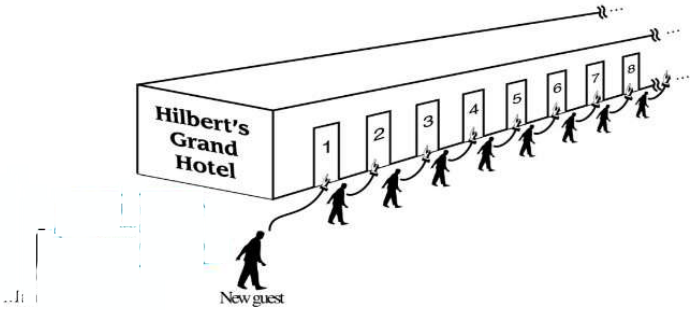
An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).

The reason for this is that a one-to-one correspondence f from the set of positive integers to a set S can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$ where $a_1 = f(1)$, $a_2 = f(2), \ldots, a_n = f(n), \ldots$

Hilbert's Grand Hotel

The Grand Hotel (example due to David Hilbert) has a countably infinite number of rooms, each occupied by a guest. We can always accommodate a new guest at this hotel. How is this possible?

Explanation: Because the rooms of Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on. When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general the guest in Room n to Room $n + 1$, for all positive integers n. This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms.



The hotel can also accommodate a countable number of new guests, all the guests on a countable number of buses where each bus contains a countable number of guests (see exercises).

Showing that a Set is Countable[1]

Example 1: Show that the set of positive even integers *E* is countable set.

Solution: Let $f(x) = 2x$.

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 2 & 4 & 6 & 8 & 10 & 12 \end{array}$$

Then *f* is a bijection from **N** to *E* since *f* is both one-to-one and onto. To show that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n = 2m$, and so $n = m$. To see that it is onto, suppose that *t* is an even positive integer. Then $t = 2k$ for some positive integer *k* and $f(k) = t$.

Showing that a Set is Countable 2

Example 2: Show that the set of integers **Z** is countable.

Solution: Can list in a sequence:

0, 1, − 1, 2, − 2, 3, − 3 ,…….

Or can define a bijection from **N** to **Z**:

- When n is even: $f(n) = n/2$
- When n is odd: $f(n) = -(n-1)/2$

The Positive Rational Numbers are Countable[1]

Definition: A *rational number* can be expressed as the ratio of two integers p and q such that $q \neq 0$.

- ¾ is a rational number
- √2 is not a rational number.

Example 3: Show that the positive rational numbers are countable.

Solution: The positive rational numbers are countable since they can be arranged in a sequence:

$$r_1, r_2, r_3, \ldots$$

The next slide shows how this is done.

The Positive Rational Numbers are Countable 2

Constructing the List

First list p/q with $p + q = 2$.

Next list p/q with $p + q = 3$

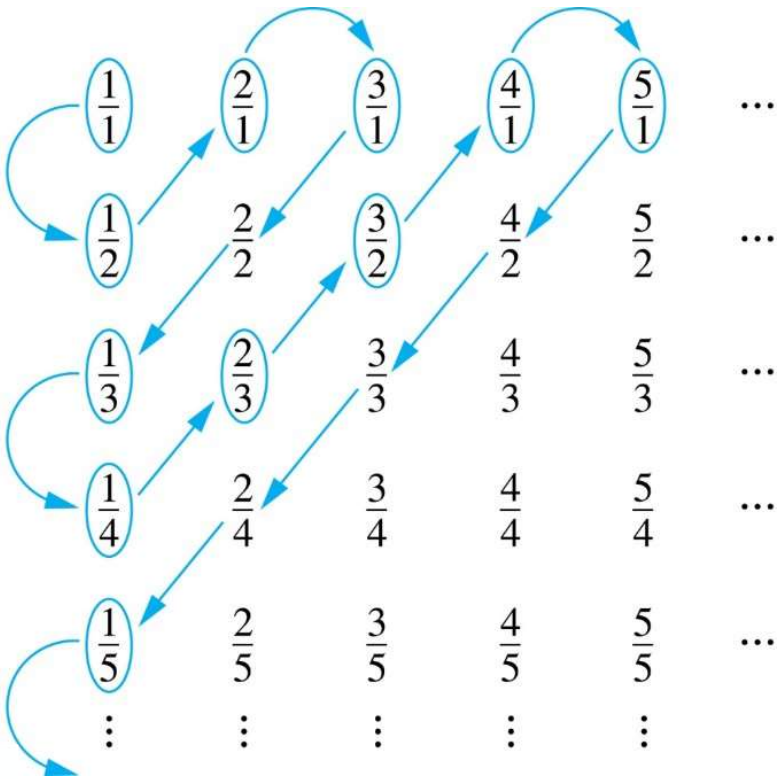
And so on.

1, ½, 2, 3, 1/3,1/4, 2/3, …

First row $q = 1$.

Second row $q = 2$.

etc.



The Real Numbers are Uncountable

Example: Show that the set of real numbers is uncountable.

Solution: The method is called the Cantor diagnalization argument, and is a proof by contradiction.

1. Suppose **R** is countable. Then the real numbers between 0 and 1 are also countable (any subset of a countable set is countable - an exercise in the text).
2. The real numbers between 0 and 1 can be listed in order $r_1, r_2, r_3, \ldots$.
3. Let the decimal representation of this listing be
$$r_1 = 0.d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}\ldots$$
$$r_2 = 0.d_{21}d_{22}d_{23}d_{24}d_{25}d_{26}\ldots$$
$$\vdots$$
4. Form a new real number with the decimal expansion $r = .r_1 r_2 r_3 r_4 \ldots$
$$\text{where } r_i = 3 \text{ if } d_{ii} \neq 3 \text{ and } r_i = 4 \text{ if } d_{ii} = 3$$
5. r is not equal to any of the $r_1, r_2, r_3, \ldots$ Because it differs from r_i in its ith position after the decimal point. Therefore there is a real number between 0 and 1 that is not on the list since every real number has a unique decimal expansion. Hence, all the real numbers between 0 and 1 cannot be listed, so the set of real numbers between 0 and 1 is uncountable.
6. Since a set with an uncountable subset is uncountable (an exercise), the set of real numbers is uncountable.