Cartesian Products and Relations

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Cartesian Productions and Relations

Section 9.1

Definition. When each of A and B is a set, the Cartesian product of A and B is denoted by

$$A \times B$$

and is defined to be the set of all ordered pairs (a, b) where the first term a is a member of A and the second term b is a member of B.

Definition. When each of A and B is a set, a **relation from** A **to** B is a subset of $A \times B$.

When R is a relation from A to B , the notation aRb means $(a, b) \in R$.

The **domain** of R is $\{a \in A : (a, b) \in R \text{ for some } b \in B\}$. So the **domain** is the set of **first terms** in the pairs that make up the relation. The domain of R is denoted by $\mathrm{Dom}(R)$.

The **range** of R is $\{b \in B : (a, b) \in R \text{ for some } a \in A\}$. So the **range** is the set of **second terms** in the pairs that make up the relation. The range of R is denoted by $\mathrm{Rng}(R)$.

A relation **on** a set A is a relation from A to A.

Examples. Let $A = \{-1, 2, 3, 4]$, $B = \{1, 2, 4, 5, 6]$,

$$R = \{(-1, 5), (2, 4), (2, 1), (4, 2)\},$$

$$S = \{(5, 2), (4, 3), (1, 3)\}, \text{ and}$$

$$T = \{(-1, 3), (2, 3), (4, 4)\}.$$

R is a relation from A to B. $\text{Dom}(R) = \{-1, 2, 4\}$, and $\text{Rng}(R) = \{1, 2, 4, 5\}$.

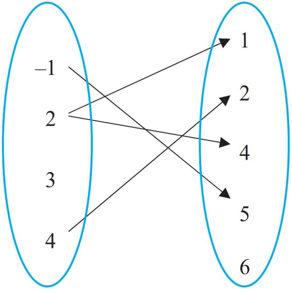
S is a relation from B to A. $\text{Dom}(S) = \{1, 4, 5\}$, and $\text{Rng}(S) = \{2, 3\}$.

T is a relation on A (meaning from A to A). $\text{Dom}(T) = \{-1, 2, 4\}$, and $\text{Rng}(T) = \{3, 4\}$

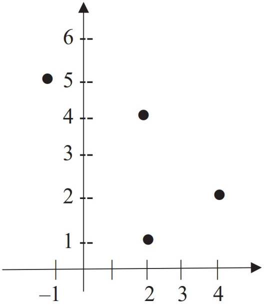
This slides shows three ways to picture a relation when the domain and range are finite and small.

–1	5
2	4
2	1
4	2

(a) Table for R



(b) Arrow diagram for R



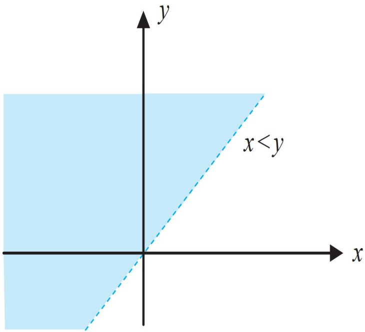
(c) Graph of R

$$R = \{(-1, 5), (2, 4), (2, 1), (4, 2)\}$$

Example.

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : \; x < y\}$$

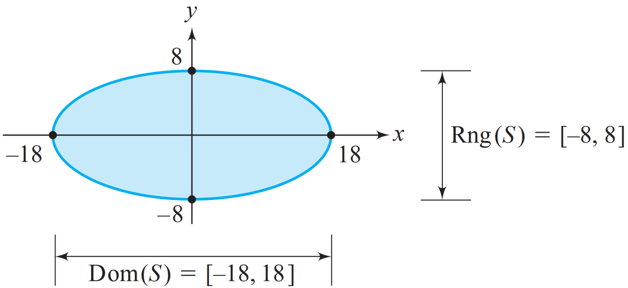
Is a relation on $\mathbb{R}$.It can be pictured as a region in the Cartesian plane.



R

Example

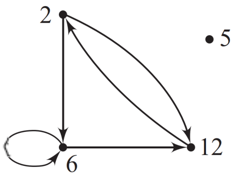
$$S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : \frac{x^2}{18^2} + \frac{y^2}{8^2} \leq 1\}$$



S

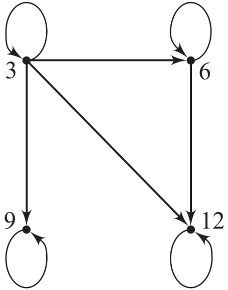
Example. Here is a digraph that shows the relation

$$R = \{(6,12), (2,6), (2,12), (6,6), (12,2)\} \text{ on the set } \{2,5,6,12\}$$



R

Example. Let $A = \{3, 6, 9, 12\}$ and let $S = \{(a, b) \in A \times A : a \text{ divides } b\}$. Here is a digraph that shows the relation S.



S

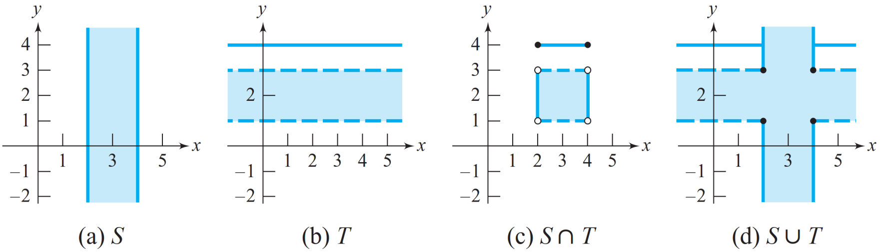
Definition. When A is a set, the identity relation on A is denoted by I_A and is given by

$$I_A = \{(x, y) \in A \times A : x = y\}.$$

This relation is also called the identity function on A.

Note. When each of S and T is a relation from A to B, $S \cup T$ is a relation from A to B, and when S and T have a member in common, $S \cap T$ is also a relation from A to B.

Example. Let $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : 2 \leq x \leq 4 \text{ and } y \in \mathbb{R}\}$ and let $T = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \in \mathbb{R}\}$ and $1 < y < 3$ or $y = 4$. Graphs of S, T, $S \cup T$, and $S \cap T$ are shown below.



Definition. If R is a relation from A to B, then R^{-1} (read "R-inverse") is the relation from B to A obtained by interchanging the first and second terms in each pair in R.

$$R^{-1} = \{(y, x) : (x, y) \in R\}$$

Note.

$$(R^{-1})^{-1} = R$$

$$\text{Dom}(R^{-1}) = \text{Rng}(R) \text{ and } \text{Rng}(R^{-1}) = \text{Dom}(R)$$

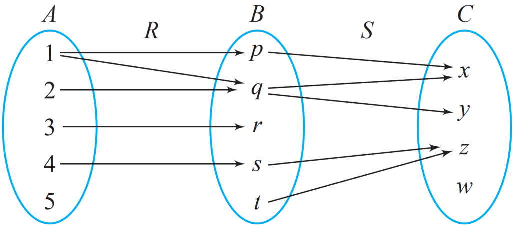
Note. If a relation is represented by a digraph, a diagraph of its inverse can be obtained by reversing the direction of each of the arrows.

If a relation is represented by a graph in $\mathbb{R}^2$, the graph of its inverse can be obtained by reflecting across the line consisting of all (x, y) where $y = x$. (This is the graph of $I_{\mathbb{R}}$.)

Definition. Suppose that R is a relation from A to B, S is a relation form B to C, and for some $b \in B$ there is an $a \in A$ and a $c \in C$ such that $(a, b) \in R$ and $(b, c) \in S$. The **composition** or **composite** of S and R is denoted by $S \circ R$ and is given by

$$S \circ R = \{(a, c) : (a, b) \in R \text{ and } (b, c) \in S \text{ for some } b \in B\}$$

Example. Let R and S be the relations shown here.



Then $S \circ R = \{(1, x), (1, y), (2, x), (2, y), (4, z)\}$