Consider the following two-point boundary value problem.

\[-\varphi''(x) = \lambda \varphi(x) \text{ for } 0 \leq x \leq 1,\]
\[\varphi(0) - 2\varphi'(0) = 0, \text{ and} \]
\[2\varphi(1) + \varphi'(1) = 0.\]

1. Find $2 \times 2$ matrices $M$ and $N$ such that the boundary conditions (2) and (3) are equivalent to

\[M \begin{bmatrix} \varphi(0) \\ \varphi'(0) \end{bmatrix} + N \begin{bmatrix} \varphi(1) \\ \varphi'(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.\]

2. Use the Rayleigh Quotient to show that all eigenvalues are non-negative. Explain why $0$ is not an eigenvalue.

So all eigenvalues are nonnegative. The nonzero constant functions do not satisfy (2) (or (3)), so all eigenvalues are positive.

3. Find the matrix $D(\lambda)$ and the function $\Delta(\lambda)$ when $\lambda > 0$.

4. Show graphically how to determine the eigenvalues.

5. Find numerical approximations for the first three eigenvalues.

6. Find a proper listing $\{\lambda_k\}_{k=1}^\infty$ and $\{\varphi_k\}_{k=1}^\infty$ of eigenvalues and eigenfunctions.