1. Find the solution to

\[ u''(x) - x^2 = 0 \text{ for } 0 \leq x \leq 2, \]
\[ u'(0) = 2, \text{ and} \]
\[ u(2) = 1. \]

2. Let

\[ Q(x) = \beta x(x - 3) \]

and let

\[ f(x) = 1 \text{ for } 0 \leq x \leq 3. \]

Find the value of \( \beta \) so that the following problem has an equilibrium solution.

\[ \frac{\partial w}{\partial t}(x, t) = \frac{\partial^2 w}{\partial x^2}(x, t) + Q(x) \text{ for } 0 \leq x \leq 3 \text{ and } t \geq 0, \]

\[ \frac{\partial w}{\partial x}(0, t) = 1 \text{ for } t \geq 0, \]

\[ \frac{\partial w}{\partial x}(3, t) = 2 \text{ for } t \geq 0, \text{ and} \]

\[ w(x, 0) = f(x) \text{ for } 0 \leq x \leq 3. \]

3. Find the equilibrium solution \( u \) for Problem 2.
Consider the following two-point boundary value problem.

\[-\varphi''(x) = \lambda \varphi(x) \text{ for } 0 \leq x \leq L, \quad (1)\]
\[\varphi(0) = 0, \text{ and} \quad (2)\]
\[\varphi'(L) = 0. \quad (3)\]

4. Find $2 \times 2$ matrices $M$ and $N$ such that the boundary conditions (2) and (3) are equivalent to

\[M \begin{bmatrix} \varphi(0) \\ \varphi'(0) \end{bmatrix} + N \begin{bmatrix} \varphi(L) \\ \varphi'(L) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.\]

5. Use the Rayleigh Quotient to show that all eigenvalues are non-negative. Explain why 0 is not an eigenvalue.

6. Find the matrix $D(\lambda)$ and the function $\Delta(\lambda)$ when $\lambda > 0$.

7. Find a proper listing $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ of eigenvalues and eigenfunctions.
1. \[ u''(x) - x^2 = 0 \]
   \[ u'(10) = 2 \]
   \[ u(2) = 1 \]

\[ u''(x) = x^2 \]

\[ u'(x) = \frac{1}{3} x^3 + c_1 \]

\[ u'(10) = 2 \quad \Rightarrow \quad c_1 = 2 \]

\[ u'(x) = \frac{1}{3} x^3 + 2 \]

\[ u(x) = \frac{1}{12} x^4 + 2x + c_2 \]

\[ u(2) = 1 \]

\[ 1 = \frac{16}{12} + 4 + c_2 \]

\[ c_2 = -\frac{13}{3} \]

\[ u(x) = \frac{1}{12} x^4 + 2x - \frac{13}{3} \]
2. The problem for the equilibrium

Solution \( u \) is

\[ 0 = u'' + \beta x (x - 3) \]

\( u'(10) = 1 \) and

\( u'(3) = 2 \)

From the d.e.

\[ u''(x) = \beta x (3 - x) \]

\[ \int_{0}^{3} u''(x) \, dx = \beta \int_{0}^{3} (3x - x^2) \, dx \]

\[ u'(3) - u'(0) = \beta \left[ \frac{3}{2} \cdot 3^2 - \frac{1}{3} \cdot 3^3 \right] \]

\[ 2 - 1 = \beta \left[ \frac{3^2}{2} - 9 \right] = \beta \cdot \frac{9}{2} \]

\[ \boxed{\beta = \frac{2}{9}} \]
3.

\[ u''(x) = \frac{2}{9} x (3-x) = \frac{2}{3}x - \frac{2}{9}x^2 \]

\[ u'(x) = \frac{1}{3}x^2 - \frac{2}{27}x^3 + c_1 \]

\[ u'(0) = 1 \Rightarrow c_1 = 1 \]

\[ u'(1) = \frac{1}{3} - \frac{2}{27} + 1 \]

\[ u(x) = \frac{1}{9}x^3 - \frac{1}{54}x^4 + x + c_2 \]

\[ c_2 = ? \]

\[ \frac{d}{dt} \int_0^3 w(x,t) \, dx = \int_0^3 \frac{\partial w}{\partial t} (x,t) \, dx \]

\[ = \int_0^3 \left[ \frac{\partial^2 w}{\partial x^2} (x,t) + \frac{2}{9} (x)(x-3) \right] \, dx \]

\[ = \frac{\partial w}{\partial x} (3, t) - \frac{\partial w}{\partial x} (0, t) - \frac{2}{9} \cdot 9 \]

\[ = 2 \cdot 1 - 1 = 0 \]

\[ \therefore \int_0^3 w(x,t) \, dt \text{ is constant in time.} \]

\[ \int_0^3 f(x) \, dx = \int_0^3 w(x,0) \, dx = \lim_{t \to \infty} \int_0^3 w(x,t) \, dx \]

\[ = \int_0^3 \lim_{t \to \infty} w(x,t) \, dx = \int_0^3 u(x) \, dx \]

\[ \therefore \int_0^3 1 \, dx = \int_0^3 \left[ \frac{1}{9}x^3 - \frac{1}{54}x^4 + x + c_2 \right] \, dx \]
Thus $3 = \frac{117}{20} + 3c_2$

So $c_2 = -\frac{19}{20}$

Thus $u(x) = \frac{1}{9}x^3 - \frac{1}{54}x^4 + x - \frac{19}{20}$
4. The conditions

\[ \phi(0) = 0 \quad (1) \]
\[ \phi'(L) = 0 \quad (2) \]

are equivalent to

\[ 1 \cdot \phi(0) + 0 \cdot \phi'(0) + 0 \cdot \phi(L) + 0 \cdot \phi'(L) = 0 \]
\[ 0 \cdot \phi(0) + 0 \cdot \phi'(0) + 0 \cdot \phi(L) + 1 \cdot \phi'(L) = 0 \]

so are equivalent to

\[ M \begin{bmatrix} \phi(0) \\ \phi'(0) \end{bmatrix} + N \begin{bmatrix} \phi(L) \\ \phi'(L) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

When \( M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and \( N = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \)
5. Suppose that \( z \) is an eigenvalue and \( \phi \) is a corresponding real-valued eigenfunction. Then

\[
2 = \frac{\phi'(0)\phi'(0) - \phi(L)\phi'(L) + \int_0^L (\phi'(x))^2 \, dx}{\int_0^L (\phi(x))^2 \, dx}
\]

\[
2 = \frac{\int_0^L (\phi'(x))^2 \, dx}{\int_0^L (\phi(x))^2 \, dx} > 0
\]

All eigenvalues are nonnegative.

The non-zero constant functions do not satisfy (1) so \( \int_0^L (\phi'(x))^2 \, dx > 0 \).

All eigenvalues are positive.
Let $\Phi_2(x) = \begin{pmatrix} \cos \sqrt{2}x & \sin \sqrt{2}x \\ -\sqrt{2} \sin \sqrt{2}x & \sqrt{2} \cos \sqrt{2}x \end{pmatrix}$

$D(\pi) = M \Phi_2(0) + N \Phi_2(\pi)$

$D(\pi) = \begin{pmatrix} 1 & 0 \\ -\sqrt{2} \sin \sqrt{2} \pi & \sqrt{2} \cos \sqrt{2} \pi \end{pmatrix}$

$A(\pi) = \det D(\pi) = \sqrt{2} \cos \sqrt{2} \pi$
all eigenvalues are positive

\[ \Delta(x) = 0 \iff \cos \sqrt{\lambda} x = 0 \]

\[ \iff \sqrt{\lambda} = (k - \frac{1}{2}) \pi \]

\[ \iff \sqrt{\lambda} = \frac{(2k-1) \pi}{2L} \]

\[ \iff \lambda = \left[ \frac{(2k-1) \pi}{2L} \right]^2 \]

for some positive integer \( k \)

When \( \lambda = \left[ \frac{(2k-1) \pi}{2L} \right]^2 \)

then \( \Delta(x) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 1 & 0 \\ \cos \sqrt{\lambda} & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ \iff c_1 = 0 \]

so \( \phi \) is a corresponding eigenfunction

\[ \Rightarrow \quad \phi(x) = c_2 \sin \sqrt{\lambda} x \text{ for some } c_2 \neq 0 \]

A proper listing is \( \{ 2k \}_{k=1}^{\infty} \) and \( \{ \phi_k \}_{k=1}^{\infty} \) where

\[ \lambda_k = \left( \frac{(2k-1) \pi}{2L} \right)^2 \text{ and } \phi_k(x) = \sin \left( \frac{(2k-1) \pi x}{2L} \right) \]

\( k = 1, 2, \ldots \), \( 0 \leq x \leq L \)