1. Review suggested homework problems and Graded Homework 2.

2. Suppose that each of $L$ and $\lambda$ is a positive number. Find the function $G$ that satisfies

$$G''(x) = \lambda G(x) \text{ for } 0 \leq x \leq L,$$

$$G(0) = 0, \text{ and}$$

$$G(L) = 1.$$

This is not an eigenvalue problem.

3. Suppose that each of $L$ and $H$ is a positive number. Derive the solution to

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0 \text{ for } 0 \leq x \leq L \text{ and } 0 \leq y \leq H,$$

$$u(0,y) = u(L,y) = 0 \text{ for } 0 \leq y \leq H,$$

$$u(x,0) = f(x), \text{ and } u(x,H) = 0 \text{ for } 0 \leq x \leq L.$$

4. Suppose that each of $L$ and $H$ is a positive number. Derive the solution to

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0 \text{ for } 0 \leq x \leq L \text{ and } 0 \leq y \leq H,$$

$$u(0,y) = u(L,y) = 0 \text{ for } 0 \leq y \leq H,$$

$$u(x,H) = f(x), \text{ and } u(x,0) = 0 \text{ for } 0 \leq x \leq L.$$

5. Suppose that each of $L$ and $H$ is a positive number. Derive the solution to

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0 \text{ for } 0 \leq x \leq L \text{ and } 0 \leq y \leq H,$$

$$u(0,y) = f(y) \text{ and } u(L,y) = 0 \text{ for } 0 \leq y \leq H,$$

$$u(x,H) = 0 \text{ and } u(x,0) = 0 \text{ for } 0 \leq x \leq L.$$
6. Suppose that each of $L$ and $H$ is a positive number. Derive the solution to
\[
\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0 \text{ for } 0 \leq x \leq L \text{ and } 0 \leq y \leq H,
\]
\[
u(0,y) = 0 \text{ and } u(L,y) = f(y) \text{ for } 0 \leq y \leq H,
\]
\[
u(x,H) = 0 \text{ and } u(x,0) = 0 \text{ for } 0 \leq x \leq L.
\]
7. Find the function $v$ of the form
\[
v(x,y) = ax + by + cxy + d
\]
such that
\[
v(0,0) = -1,
\]
\[
v(2,0) = 3,
\]
\[
v(2,4) = 4, \text{ and }
\]
\[
v(0,4) = -2.
\]
8. Suppose that each of $L$ and $H$ is a positive number. Derive the solution to
\[
\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0 \text{ for } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 1,
\]
\[
u(0,y) = y^2 + y + 4 \text{ and } u(2,y) = 8y + 6 \text{ for } 0 \leq y \leq 1,
\]
\[
u(x,H) = 4x + 6 \text{ and } u(x,0) = x^2 - x + 4 \text{ for } 0 \leq x \leq 2.
\]
In order to improve the convergence of the series solution, do this by first finding a function $v$ of the form
\[
v(x,y) = ax + by + cxy + d
\]
that agrees with the given boundary conditions at the four corners of the rectangle. Then let
\[
w(x,y) = u(x,y) - v(x,y)
\]
for all $(x,y)$ in the rectangle. Calculate the boundary conditions for $w$ ( $w$ will be zero at the four corners ) and noting that $w$ is also a solution to Laplace’s equation find the function $w$. Find $u$ by noting that $u = w + v$.

9. Suppose that each of $c$ and $L$ is a positive number. Derive the solution to
\[
\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \text{ for } 0 \leq x \leq L \text{ and all } t \text{ in } \mathbb{R},
\]
\[
u(0,t) = 0 \text{ for all } t \text{ in } \mathbb{R},
\]
\[
u(L,t) = 0 \text{ for all } t \text{ in } \mathbb{R},
\]
\[
u(x,0) = f(x) \text{ for } 0 \leq x \leq L, \text{ and }
\]
\[
\frac{\partial u}{\partial t}(x,0) = g(x) \text{ for } 0 \leq x \leq L.
\]
10. Suppose that each of \( c \) and \( L \) is a positive number. Derive the solution to
\[
\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \text{ for } 0 \leq x \leq L \text{ and all } t \in \mathbb{R},
\]
\[
u(0,t) = 0 \text{ for all } t \in \mathbb{R},
\]
\[
\frac{\partial u}{\partial x}(L,t) = 0 \text{ for all } t \in \mathbb{R},
\]
\[
u(x,0) = f(x) \text{ for } 0 \leq x \leq L, \text{ and}
\]
\[
\frac{\partial u}{\partial t}(x,0) = g(x) \text{ for } 0 \leq x \leq L.
\]

11. Suppose that each of \( c \) and \( L \) is a positive number. Derive the solution to
\[
\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \text{ for } 0 \leq x \leq L \text{ and all } t \in \mathbb{R},
\]
\[
\frac{\partial u}{\partial x}(0,t) = 0 \text{ for all } t \in \mathbb{R},
\]
\[
u(L,t) = 0 \text{ for all } t \in \mathbb{R},
\]
\[
u(x,0) = f(x) \text{ for } 0 \leq x \leq L, \text{ and}
\]
\[
\frac{\partial u}{\partial t}(x,0) = g(x) \text{ for } 0 \leq x \leq L.
\]

12. Suppose that each of \( c \) and \( L \) is a positive number. Derive the solution to
\[
\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \text{ for } 0 \leq x \leq L \text{ and all } t \in \mathbb{R},
\]
\[
\frac{\partial u}{\partial x}(0,t) = 0 \text{ for all } t \in \mathbb{R},
\]
\[
u(L,t) = 0 \text{ for all } t \in \mathbb{R},
\]
\[
u(x,0) = f(x) \text{ for } 0 \leq x \leq L, \text{ and}
\]
\[
\frac{\partial u}{\partial t}(x,0) = g(x) \text{ for } 0 \leq x \leq L.
\]

13. Let
\[
f(x) = \begin{cases}
1 - x & \text{when } -1 < x < 0 \\
x & \text{when } 0 < x < 1
\end{cases}
\]
Find the Fourier series for \( f \).

14. Let \( \{S_n\} \) be the Fourier series for the function \( f \) in the previous problem and let
\[
g(x) = \lim_{n \to \infty} S_n(x) \text{ for } -5 \leq x \leq 5.
\]
Sketch the graph of \( g \). Be sure to indicate the value of \( g \) at the numbers where \( g \) is discontinuous.

15. Suppose that \( L \) is a positive number.
(a) Define the trigonometric Fourier series for \( f \) when \( f \) is defined on \([-L, L]\).

(b) Define the cosine series for \( f \) when \( f \) is defined on \([0, L]\).

(c) Define the sine series for \( f \) when \( f \) is defined on \([0, L]\).

16. Let \( f(x) = 1 - x^2 \) for \( 0 < x < 1 \).

(a) Sketch the function to which the cosine series of \( f \) converges on \([-4, 4]\).

(b) Sketch the function to which the sine series of \( f \) converges on \([-4, 4]\).

17. Let
\[
f(x) = \begin{cases} -1 & \text{when } -1 < x < 0 \\ 1 & \text{when } 0 < x < 1 \end{cases},
\]
and let \( \{S_n\}_{n=1}^{\infty} \) be the trigonometric Fourier series for \( f \). Sketch the graph of \( f \) and the graph of a typical \( S_n \) on the same set of axes. Describe the Gibbs phenomenon.

18. Let \( f \) and \( \{S_n\}_{n=1}^{\infty} \) be as in the previous problem. Explain why \( \{S_n\}_{n=1}^{\infty} \) does not converge uniformly.

19. Suppose that \( L \) is a positive number and that \( j \) and \( k \) are nonnegative integers \((j \neq k)\). Evaluate
\[
\int_{-L}^{L} \cos \frac{k\pi x}{L} \cos \frac{j\pi x}{L} \, dx.
\]

20. Let
\[
f(x) = \begin{cases} x & \text{when } -1 < x < 0 \\ 2 - x & \text{when } 0 < x < 1 \end{cases}.
\]
Find the Fourier series for \( f \), the sine series for \( f \), and the cosine series for \( f \). In each case take \( L = 1 \).

21. Do the following problems from the text. When Haberman asks you to sketch a series, he should be asking you to sketch the limit of that series, and that is what you are to do.

(a) 2.5.1 and 2.5.2 pages 81 and 82.

(b) 4.4.1 page 140.

(c) 3.2.1 and 3.2.2 page 92.

(d) 3.3.1 through 3.3.5 and 3.3.7 through 3.3.11 pages 110 and 111.

(e) 3.4.1 through 3.4.7 pages 120 and 121.

(f) 3.5.1 through 3.5.7 pages 126 and 127.