10.1 Functions and Their Graphs
10.2 The Algebra of Functions
10.3 Functions and Mathematical Models
10.4 Limits
10.5 One-Sided Limits and Continuity
10.6 The Derivative
In this chapter we review the concept of a *function* and begin the study of differential calculus. Historically, differential calculus was developed in response to the problem of finding the tangent line to an arbitrary curve. But it quickly became apparent that solving this problem provided mathematicians with a method for solving many practical problems involving the rate of change of one quantity with respect to another. The basic tool used in differential calculus is the *derivative* of a function. The concept of the derivative is based, in turn, on a more fundamental notion—that of the *limit* of a function.

How does the change in the demand for a certain make of tires affect the unit price of the tires? The management of the Titan Tire Company has determined the demand function that relates the unit price of its Super Titan tires to the quantity demanded. In Example 7, page 675, you will see how this function can be used to compute the rate of change of the unit price of the Super Titan tires with respect to the quantity demanded.
10.1 Functions and Their Graphs

FUNCTIONS

The notion of a function was introduced in Section 1.3, where we were concerned primarily with the special class of functions known as linear functions. In the study of calculus, we will be dealing with a more general class of functions called nonlinear functions. First, however, let us restate the definition of a function.

A function is a rule that assigns to each element in a set $A$ one and only one element in a set $B$.

The set $A$ is called the domain of the function. It is customary to denote a function by a letter of the alphabet, such as the letter $f$. If $x$ is an element in the domain of a function $f$, then the element in $B$ that $f$ associates with $x$ is written $f(x)$ (read “$f$ of $x$”) and is called the value of $f$ at $x$. The set comprising all the values assumed by $y = f(x)$ as $x$ takes on all possible values in its domain is called the range of the function $f$.

We can think of a function $f$ as a machine. The domain is the set of inputs (raw material) for the machine, the rule describes how the input is to be processed, and the value(s) of the function are the outputs of the machine (Figure 10.1).

We can also think of a function $f$ as a mapping in which an element $x$ in the domain of $f$ is mapped onto a unique element $f(x)$ in $B$ (Figure 10.2).
1. It is important to understand that the output $f(x)$ associated with an input $x$ is unique. To appreciate the importance of this uniqueness property, consider a rule that associates with each item $x$ in a department store its selling price $y$. Then, each $x$ must correspond to one and only one $y$. Notice, however, that different $x$’s may be associated with the same $y$. In the context of the present example, this says that different items may have the same price.

2. Although the sets $A$ and $B$ that appear in the definition of a function may be quite arbitrary, in this book they will denote sets of real numbers.

In general, to evaluate a function at a specific value of $x$, we replace $x$ with that value, as illustrated in Examples 1 and 2.

**Example 1**

Let the function $f$ be defined by the rule $f(x) = 2x^2 - x + 1$. Compute:

- a. $f(1)$
- b. $f(-2)$
- c. $f(a)$
- d. $f(a + h)$

**Solution**

- a. $f(1) = 2(1)^2 - (1) + 1 = 2 - 1 + 1 = 2$
- b. $f(-2) = 2(-2)^2 - (-2) + 1 = 8 + 2 + 1 = 11$
- c. $f(a) = 2(a)^2 - (a) + 1 = 2a^2 - a + 1$
- d. $f(a + h) = 2(a + h)^2 - (a + h) + 1 = 2a^2 + 4ah + 2h^2 - a - h + 1$

**Example 2**

The Thermo-Master Company manufactures an indoor–outdoor thermometer at its Mexican subsidiary. Management estimates that the profit (in dollars) realizable by Thermo-Master in the manufacture and sale of $x$ thermometers per week is

$$P(x) = -0.001x^2 + 8x - 5000$$

Find Thermo-Master’s weekly profit if its level of production is (a) 1000 thermometers per week and (b) 2000 thermometers per week.

- a. The weekly profit realizable by Thermo-Master when the level of production is 1000 units per week is found by evaluating the profit function $P$ at $x = 1000$. Thus,

$$P(1000) = -0.001(1000)^2 + 8(1000) - 5000 = 2000$$

or $2000$.
- b. When the level of production is 2000 units per week, the weekly profit is given by


or $7000$. 
**Determining the Domain of a Function**

Suppose we are given the function \( y = f(x) \). Then, the variable \( x \) is called the **independent variable**. The variable \( y \), whose value depends on \( x \), is called the **dependent variable**.

In determining the domain of a function, we need to find what restrictions, if any, are to be placed on the independent variable \( x \). In many practical applications the domain of a function is dictated by the nature of the problem, as illustrated in Example 3.

**Example 3**

An open box is to be made from a rectangular piece of cardboard 16 inches long and 10 inches wide by cutting away identical squares \((x \text{ by } x \text{ inches})\) from each corner and folding up the resulting flaps (Figure 10.3). Find an expression that gives the volume \( V \) of the box as a function of \( x \). What is the domain of the function?

**Solution**

The dimensions of the box are \((16 - 2x) \text{ inches long}, (10 - 2x) \text{ inches wide}, \text{ and } x \text{ inches high}, \) so its volume (in cubic inches) is given by

\[
V = f(x) = (16 - 2x)(10 - 2x)x = (160 - 52x + 4x^2)x = 4x^3 - 52x^2 + 160x
\]

Since the length of each side of the box must be greater than or equal to zero, we see that

\[
16 - 2x \geq 0, \quad 10 - 2x \geq 0, \quad x \geq 0
\]

simultaneously; that is,

\[
x \leq 8, \quad x \leq 5, \quad x \geq 0
\]

All three inequalities are satisfied simultaneously provided that \(0 \leq x \leq 5\). Thus, the domain of the function \( f \) is the interval \([0, 5]\).  

*It is customary to refer to a function \( f \) as \( f(x) \) or by the equation \( y = f(x) \) defining it.*
In general, if a function is defined by a rule relating $x$ to $f(x)$ without specific mention of its domain, it is understood that the domain will consist of all values of $x$ for which $f(x)$ is a real number. In this connection, you should keep in mind that (1) division by zero is not permitted and (2) the square root of a negative number is not defined.

**Example 4**

Find the domain of each of the functions defined by the following equations:

a. $f(x) = \sqrt{x - 1}$

b. $f(x) = \frac{1}{x^2 - 4}$

c. $f(x) = x^2 + 3$

**Solution**

a. Since the square root of a negative number is undefined, it is necessary that $x - 1 \geq 0$. The inequality is satisfied by the set of real numbers $x \geq 1$. Thus, the domain of $f$ is the interval $[1, \infty)$.

b. The only restriction on $x$ is that $x^2 - 4$ be different from zero since division by zero is not allowed. But $(x^2 - 4) = (x + 2)(x - 2) = 0$ if $x = -2$ or $x = 2$. Thus, the domain of $f$ in this case consists of the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

c. Here, any real number satisfies the equation, so the domain of $f$ is the set of all real numbers.

**Graphs of Functions**

If $f$ is a function with domain $A$, then corresponding to each real number $x$ in $A$ there is precisely one real number $f(x)$. We can also express this fact by using ordered pairs of real numbers. Write each number $x$ in $A$ as the first member of an ordered pair and each number $f(x)$ corresponding to $x$ as the second member of the ordered pair. This gives exactly one ordered pair $(x, f(x))$ for each $x$ in $A$. This observation leads to an alternative definition of a function $f$:

A function $f$ with domain $A$ is the set of all ordered pairs $(x, f(x))$ where $x$ belongs to $A$.

Observe that the condition that there be one and only one number $f(x)$ corresponding to each number $x$ in $A$ translates into the requirement that no two ordered pairs have the same first number.

Since ordered pairs of real numbers correspond to points in the plane, we have found a way to exhibit a function graphically.

The graph of a function $f$ is the set of all points $(x, y)$ in the $xy$-plane such that $x$ is in the domain of $f$ and $y = f(x)$.

Figure 10.4 shows the graph of a function $f$. Observe that the $y$-coordinate of the point $(x, y)$ on the graph of $f$ gives the height of that point (the distance
above the $x$-axis), if $f(x)$ is positive. If $f(x)$ is negative, then $-f(x)$ gives the depth of the point $(x, y)$ (the distance below the $x$-axis). Also, observe that the domain of $f$ is a set of real numbers lying on the $x$-axis, whereas the range of $f$ lies on the $y$-axis.

**EXAMPLE 5**

The graph of a function $f$ is shown in Figure 10.5.

a. What is the value of $f(3)$? The value of $f(5)$?  
b. What is the height or depth of the point $(3, f(3))$ from the $x$-axis? The point $(5, f(5))$ from the $x$-axis?  
c. What is the domain of $f$? The range of $f$?

**FIGURE 10.5**

![Graph of a function $f$](image)

**SOLUTION**

a. From the graph of $f$, we see that $y = -2$ when $x = 3$ and conclude that $f(3) = -2$. Similarly, we see that $f(5) = 3$.  
b. Since the point $(3, -2)$ lies below the $x$-axis, we see that the depth of the point $(3, f(3))$ is $-f(3) = -(-2) = 2$ units below the $x$-axis. The point
(5, f(5)) lies above the x-axis and is located at a height of f(5), or 3 units above the x-axis.

c. Observe that x may take on all values between x = −1 and x = 7, inclusive, and so the domain of f is [−1, 7]. Next, observe that as x takes on all values in the domain of f, f(x) takes on all values between −2 and 7, inclusive. (You can easily see this by running your index finger along the x-axis from x = −1 to x = 7 and observing the corresponding values assumed by the y-coordinate of each point of the graph of f.) Therefore, the range of f is [−2, 7].

Much information about the graph of a function can be gained by plotting a few points on its graph. Later on we will develop more systematic and sophisticated techniques for graphing functions.

**EXAMPLE 6**

Sketch the graph of the function defined by the equation $y = x^2 + 1$. What is the range of f?

The domain of the function is the set of all real numbers. By assigning several values to the variable x and computing the corresponding values for y, we obtain the following solutions to the equation $y = x^2 + 1$:

<table>
<thead>
<tr>
<th>x</th>
<th>−3</th>
<th>−2</th>
<th>−1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>10</td>
<td>5</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>10</td>
</tr>
</tbody>
</table>

By plotting these points and then connecting them with a smooth curve, we obtain the graph of $y = f(x)$, which is a parabola (Figure 10.6). To determine the range of f, we observe that $x^2 ≥ 0$, if x is any real number and so $x^2 + 1 ≥ 1$ for all real numbers x. We conclude that the range of f is $[1, ∞)$. The graph of f confirms this result visually.

Some functions are defined in a piecewise fashion, as Examples 7 and 8 show.

**Group Discussion**

Let $f(x) = x^2$.

1. Plot the graphs of $F(x) = x^2 + c$ on the same set of axes for $c = −2, −1, −\frac{1}{2}, 0, \frac{1}{2}, 1, 2$.
2. Plot the graphs of $G(x) = (x + c)^2$ on the same set of axes for $c = −2, −1, −\frac{1}{2}, 0, \frac{1}{2}, 1, 2$.
3. Plot the graphs of $H(x) = cx^2$ on the same set of axes for $c = −2, −1, −\frac{1}{2}, −\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, 1, 2$.
4. Study the family of graphs in parts 1–3 and describe the relationship between the graph of a function f and the graphs of the functions defined by (i) $y = f(x) + c$, (ii) $y = f(x + c)$, and (iii) $y = cf(x)$, where c is a constant.
The Madison Finance Company plans to open two branch offices 2 years from now in two separate locations: an industrial complex and a newly developed commercial center in the city. As a result of these expansion plans, Madison’s total deposits during the next 5 years are expected to grow in accordance with the rule

\[ f(x) = \begin{cases} \sqrt{2x} + 20 & \text{if } 0 \leq x \leq 2 \\ \frac{1}{2}x^2 + 20 & \text{if } 2 < x \leq 5 \end{cases} \]

where \( y = f(x) \) gives the total amount of money (in millions of dollars) on deposit with Madison in year \( x \) (\( x = 0 \) corresponds to the present). Sketch the graph of the function \( f \).

The function \( f \) is defined in a piecewise fashion on the interval \([0, 5]\). In the subdomain \([0, 2]\), the rule for \( f \) is given by \( f(x) = \sqrt{2x} + 20 \). The values of \( f(x) \) corresponding to \( x = 0, 1, \) and 2 may be tabulated as follows:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>20</td>
<td>21.4</td>
<td>22</td>
</tr>
</tbody>
</table>

Next, in the subdomain \((2, 5]\), the rule for \( f \) is given by \( f(x) = \frac{1}{2}x^2 + 20 \). The values of \( f(x) \) corresponding to \( x = 3, 4, \) and 5 are shown in the following table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>24.5</td>
<td>28</td>
<td>32.5</td>
</tr>
</tbody>
</table>

Using the values of \( f(x) \) in this table, we sketch the graph of the function \( f \) as shown in Figure 10.7.
Sketch the graph of the function $f$ defined by

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$$

The function $f$ is defined in a piecewise fashion on the set of all real numbers. In the subdomain $(-\infty, 0)$, the rule for $f$ is given by $f(x) = -x$. The equation $y = -x$ is a linear equation in the slope-intercept form (with slope $-1$ and intercept $0$). Therefore, the graph of $f$ corresponding to the subdomain $(-\infty, 0)$ is the half line shown in Figure 10.8. Next, in the subdomain $[0, \infty)$, the rule for $f$ is given by $f(x) = \sqrt{x}$. The values of $f(x)$ corresponding to $x = 0, 1, 2, 3, 4, 9, \text{ and } 16$ are shown in the following table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$9$</th>
<th>$16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>$0$</td>
<td>$1$</td>
<td>$\sqrt{2}$</td>
<td>$\sqrt{3}$</td>
<td>$2$</td>
<td>$3$</td>
<td>$4$</td>
</tr>
</tbody>
</table>

Using these values, we sketch the graph of the function $f$ as shown in Figure 10.8.

**The Vertical-Line Test**

Although it is true that every function $f$ of a variable $x$ has a graph in the $xy$-plane, it is important to realize that not every curve in the $xy$-plane is the graph of a function. For example, consider the curve depicted in Figure 10.9. This is the graph of the equation $y^2 = x$. In general, the graph of an equation is the set of all ordered pairs $(x, y)$ that satisfy the given equation. Observe that the points $(9, -3) \text{ and } (9, 3)$ both lie on the curve. This implies that the number $x = 9$ is associated with two numbers: $y = -3 \text{ and } y = 3$. But this clearly violates the uniqueness property of a function. Thus, we conclude that the curve under consideration cannot be the graph of a function.
This example suggests the following test for determining when a curve is the graph of a function.

**Vertical-Line Test**

A curve in the $xy$-plane is the graph of a function $y = f(x)$ if and only if each vertical line intersects it in at most one point.

**EXAMPLE 9**

Determine which of the curves shown in Figure 10.10 are the graphs of functions of $x$.

**FIGURE 10.10**

The vertical-line test can be used to determine which of these curves are graphs of functions.
The curves depicted in Figure 10.10a, c, and d are graphs of functions because each curve satisfies the requirement that each vertical line intersects the curve in at most one point. Note that the vertical line shown in Figure 10.10c does not intersect the graph because the point on the x-axis through which this line passes does not lie in the domain of the function. The curve depicted in Figure 10.10b is *not* the graph of a function because the vertical line shown there intersects the graph at three points.

Self-Check Exercises 10.1

1. Let \( f \) be the function defined by

\[
  f(x) = \frac{\sqrt{x + 1}}{x}
\]

   a. Find the domain of \( f \).
   b. Compute \( f(3) \).
   c. Compute \( f\left(\frac{a}{2}\right) \).

2. Statistics obtained by Amoco Corporation show that more and more motorists are pumping their own gas. The following function gives self-serve sales as a percentage of all U.S. gas sales:

\[
  f(t) = \begin{cases} 
  6t + 17 & \text{if } 0 \leq t \leq 6 \\
  15.98(t - 6)^{1/4} + 53 & \text{if } 6 < t \leq 20 
  \end{cases}
\]

Here \( t \) is measured in years, with \( t = 0 \) corresponding to the beginning of 1974.
   a. Sketch the graph of the function \( f \).
   b. What percentage of all gas sales at the beginning of 1978 were self-serve? At the beginning of 1994?

   *Source: Amoco Corporation*

3. Let \( f(x) = \sqrt{2x + 1} + 2 \). Determine whether the point \((4, 6)\) lies on the graph of \( f \).

Solutions to Self-Check Exercises 10.1 can be found on page 586.

10.1 Exercises

1. Let \( f \) be the function defined by \( f(x) = 5x + 6 \). Find \( f(3), f(-3), f(a), f(-a), \) and \( f(a + 3) \).
2. Let \( f \) be the function defined by \( f(x) = 4x - 3 \). Find \( f(4), f(0), f(0), f(a), \) and \( f(a + 1) \).
3. Let \( g \) be the function defined by \( g(x) = 3x^2 - 6x - 3 \). Find \( g(0), g(-1), g(a), g(-a), \) and \( g(x + 1) \).
4. Let \( h \) be the function defined by \( h(x) = x^3 - x^2 + x + 1 \). Find \( h(-5), h(0), h(a), \) and \( h(-a) \).

5. Let \( s \) be the function defined by \( s(t) = \frac{2t}{t^2 - 1} \). Find \( s(4), s(0), s(a), s(2 + a), \) and \( s(t + 1) \).
6. Let \( g \) be the function defined by \( g(u) = (3u - 2)^{1/2} \). Find \( g(1), g(6), g(\frac{9}{2}), \) and \( g(a + 1) \).
7. Let \( f \) be the function defined by \( f(t) = \frac{2t^2}{\sqrt{t - 1}} \). Find \( f(2), f(a), f(x + 1), \) and \( f(x - 1) \).
8. Let \( f \) be the function defined by \( f(x) = 2 + 2\sqrt{5 - x} \). Find \( f(-4), f(1), f(4) \), and \( f(x + 5) \).
9. Let \( f \) be the function defined by
\[
  f(x) = \begin{cases} 
    x^2 + 1 & \text{if } x \leq 0 \\
    \sqrt{x} & \text{if } x > 0 
  \end{cases}
\]
Find \( f(-2), f(0), \) and \( f(1) \).
10. Let \( g \) be the function defined by
\[
  g(x) = \begin{cases} 
    -\frac{1}{2}x + 1 & \text{if } x < 2 \\
    \sqrt{x - 2} & \text{if } x \geq 2 
  \end{cases}
\]
Find \( g(-2), g(0), g(2), \) and \( g(4) \).
11. Let \( f \) be the function defined by
\[
  f(x) = \begin{cases} 
    -\frac{1}{2}x^2 + 3 & \text{if } x < 1 \\
    2x^2 + 1 & \text{if } x \geq 1 
  \end{cases}
\]
Find \( f(-1), f(0), f(1), \) and \( f(2) \).
12. Let \( f \) be the function defined by
\[
  f(x) = \begin{cases} 
    2 + \sqrt{1 - x} & \text{if } x \leq 1 \\
    \frac{1}{1 - x} & \text{if } x > 1 
  \end{cases}
\]
Find \( f(0), f(1), \) and \( f(2) \).
13. Refer to the graph of the function \( f \) in the following figure.

a. Find the value of \( f(0) \).
b. Find the value of \( x \) for which (i) \( f(x) = 3 \) and (ii) \( f(x) = 0 \).
c. Find the domain of \( f \).
d. Find the range of \( f \).

14. Refer to the graph of the function \( f \) in the following figure.

a. Find the value of \( f(7) \).
b. Find the values of \( x \) corresponding to the point on the graph of \( f \) located at a height of 5 units from the \( x \)-axis.
c. Find the points on the \( x \)-axis at which the graph of \( f \) crosses it. What are the values of \( f(x) \) at those points?
d. Find the domain and range of \( f \).

In Exercises 15–18, determine whether the point lies on the graph of the function.
15. \((2, \sqrt{3}); g(x) = \sqrt{x^2 - 1}\)
16. \((3, 3); f(x) = \frac{x + 1}{\sqrt{x^2 + 7}} + 2\)
17. \((-2, -3); f(t) = \frac{|t - 1|}{t + 1}\)
18. \((-3, -1/3); h(t) = \frac{|t + 1|}{t^2 + 1}\)

In Exercises 19–32, find the domain of the function.
19. \( f(x) = x^2 + 3 \)
20. \( f(x) = 7 - x^2 \)
21. \( f(x) = \frac{3x + 1}{x^2} \)
22. \( g(x) = \frac{2x + 1}{x - 1} \)
23. \( f(x) = \sqrt{x^2 + 1} \)
24. \( f(x) = \sqrt{x - 5} \)
25. \( f(x) = \sqrt{5 - x} \)
26. \( g(x) = \sqrt{2x^2 + 3} \)
27. \( f(x) = \frac{x}{x^2 - 1} \)
28. \( f(x) = \frac{1}{x^2 + x - 2} \)
29. \( f(x) = (x + 3)^{3/2} \)  
30. \( g(x) = 2(x - 1)^{3/2} \)
31. \( f(x) = \frac{\sqrt{1 - x}}{x^2 - 4} \)  
32. \( f(x) = \frac{\sqrt{x - 1}}{(x + 2)(x - 3)} \)

33. Let \( f \) be a function defined by the rule \( f(x) = x^3 - x - 6 \).
   a. Find the domain of \( f \).
   b. Compute \( f(x) \) for \( x = -3, -2, -1, 0, 1, 2, 3 \).
   c. Use the results obtained in parts (a) and (b) to sketch the graph of \( f \).

34. Let \( f \) be a function defined by the rule \( f(x) = 2x^3 + x - 3 \).
   a. Find the domain of \( f \).
   b. Compute \( f(x) \) for \( x = -3, -2, -1, 1, 2, 3 \).
   c. Use the results obtained in parts (a) and (b) to sketch the graph of \( f \).

In Exercises 35–46, sketch the graph of the function with the given rule. Find the domain and range of the function.

35. \( f(x) = 2x^3 + 1 \)  
36. \( f(x) = 9 - x^2 \)
37. \( f(x) = 2 + \sqrt{x} \)  
38. \( g(x) = 4 - \sqrt{2}x \)
39. \( f(x) = \sqrt{1 - x} \)  
40. \( f(x) = \sqrt{x - 1} \)
41. \( f(x) = |x| - 1 \)  
42. \( f(x) = |x| + 1 \)
43. \( f(x) = \begin{cases} x & \text{if } x < 0 \\ 2x + 1 & \text{if } x \geq 0 \end{cases} \)
44. \( f(x) = \begin{cases} 4 - x & \text{if } x < 2 \\ 2x - 2 & \text{if } x \geq 2 \end{cases} \)
45. \( f(x) = \begin{cases} -x + 1 & \text{if } x \leq 0 \\ x^2 - 1 & \text{if } x > 1 \\ -x - 1 & \text{if } x < -1 \end{cases} \)
46. \( f(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 1 \\ x + 1 & \text{if } x > 1 \end{cases} \)

In Exercises 47–54, use the vertical-line test to determine whether the graph represents \( y \) as a function of \( x \).

47. \( \)  
48. \( \)

49. \( y \)  
50. \( y \)
51. \( y \)  
52. \( y \)
53. \( y \)  
54. \( y \)

55. The circumference of a circle is given by \( C(r) = 2\pi r \), where \( r \) is the radius of the circle. What is the circumference of a circle with a 5-in. radius?

56. The volume of a sphere of radius \( r \) is given by \( V(r) = \frac{4}{3}\pi r^3 \). Compute \( V(2.1) \) and \( V(2) \). What does the quantity \( V(2.1) - V(2) \) measure?

57. Growth of a Cancerous Tumor  
   The volume of a spherical cancer tumor is given by the function
   \[ V(r) = \frac{4}{3}\pi r^3 \]
   where \( r \) is the radius of the tumor in centimeters. By what factor is the volume of the tumor increased if its radius is doubled?

58. Growth of a Cancerous Tumor  
   The surface area of a spherical cancer tumor is given by the function
   \[ S(r) = 4\pi r^2 \]
where \( r \) is the radius of the tumor in centimeters. After extensive chemotherapy treatment, the surface area of the tumor is reduced by 75\%. What is the radius of the tumor after treatment?

59. Sales of Prerecorded Music The following graphs show the sales \( y \) of prerecorded music (in billions of dollars) by format as a function of time \( t \) (in years) with \( t = 0 \) corresponding to 1985.

a. In what years were the sales of prerecorded cassettes greater than those of prerecorded CDs?
b. In what years were the sales of prerecorded CDs greater than those of prerecorded cassettes?
c. In what year were the sales of prerecorded cassettes the same as those of prerecorded CDs? Estimate the level of sales in each format at that time.

Source: Recording Industry Association of America

60. The Gender Gap The following graph shows the ratio of women’s earnings to men’s from 1960 through 1990.

a. Write the rule for the function \( f \) giving the ratio of women’s earnings to men’s in year \( t \), with \( t = 0 \) corresponding to 1960.

Hint: The function \( f \) is defined piecewise and is linear over each of three subintervals.

b. How fast was the ratio changing in the period from 1960 through 1980? The decade from 1980 to 1990?
c. In what year (approximately) was the number of bachelor’s degrees earned by women equal to that earned by men?

Source: Department of Education

61. Closing the Gender Gap in Education The following graph shows the ratio of bachelor’s degrees earned by women to men from 1960 through 1990.

a. Write the rule for the function \( f \) giving the ratio of bachelor’s degrees earned by women to men in year \( t \), with \( t = 0 \) corresponding to 1960.

b. In what decade(s) was the gender gap expanding? Shrinking?
c. Refer to part (b). How fast was the gender gap (the ratio/year) expanding or shrinking in each of these decades?

Source: U.S. Bureau of Labor Statistics

62. Consumption Function The consumption function in a certain economy is given by the equation

\[
C(y) = 0.75y + 6
\]

where \( C(y) \) is the personal consumption expenditure, \( y \) is the disposable personal income, and both \( C(y) \) and \( y \) are measured in billions of dollars. Find \( C(0) \), \( C(50) \), and \( C(100) \).

63. Sales Taxes In a certain state the sales tax \( T \) on the amount of taxable goods is 6\% of the value of the goods purchased \( (x) \), where both \( T \) and \( x \) are measured in dollars.

a. Express \( T \) as a function of \( x \).
b. Find \( T(200) \) and \( T(5.65) \).
64. **Surface Area of a Single-Celled Organism** The surface area $S$ of a single-celled organism may be found by multiplying $4\pi$ times the square of the radius $r$ of the cell. Express $S$ as a function of $r$.

65. **Friend’s Rule** Friend’s rule, a method for calculating pediatric drug dosages, is based on a child’s age. If $a$ denotes the adult dosage (in milligrams) and if $t$ is the age of the child (in years), then the child’s dosage is given by

$$D(t) = \frac{2}{25}a$$

If the adult dose of a substance is 500 mg, how much should a 4-yr-old child receive?

66. **COLA** Social Security recipients receive an automatic cost-of-living adjustment (COLA) once each year. Their monthly benefit is increased by the amount that consumer prices increased during the preceding year. Suppose that consumer prices increased by 5.3% during the preceding year.

a. Express the adjusted monthly benefit of a Social Security recipient as a function of his or her current monthly benefit.

b. If Harrington’s monthly Social Security benefit is now $620, what will be his adjusted monthly benefit?

67. **Cost of Renting a Truck** The Ace Truck Leasing Company leases a certain size truck at $30/day and $.20/mi, whereas the Acme Truck Leasing Company leases the same size truck at $25/day and $.20/mi.

a. Find the daily cost of leasing from each company as a function of the number of miles driven.

b. Sketch the graphs of the two functions on the same set of axes.

c. Which company should a customer rent a truck from for 1 day if she plans to drive at most 70 mi and wishes to minimize her cost?

68. **Linear Depreciation** A new machine was purchased by the National Textile Company for $120,000. For income tax purposes, the machine is depreciated linearly over 10 yr; that is, the book value of the machine decreases at a constant rate, so that at the end of 10 yr the book value is zero.

a. Express the book value of the machine ($V$) as a function of the age, in years, of the machine ($t$).

b. Sketch the graph of the function in part (a).

c. Find the book value of the machine at the end of the sixth year.

d. Find the rate at which the machine is being depreciated each year.

69. **Linear Depreciation** Refer to Exercise 68. An office building worth $1 million when completed in 1984 was depreciated linearly over 50 years. What was the book value of the building in 1999? What will the book value be in 2003? In 2007? (Assume that the book value of the building will be zero at the end of the 50th year.)

70. **Boyle’s Law** As a consequence of Boyle’s law, the pressure $P$ of a fixed sample of gas held at a constant temperature is related to the volume $V$ of the gas by the rule

$$P = f(V) = \frac{k}{V}$$

where $k$ is a constant. What is the domain of the function $f$? Sketch the graph of the function $f$.

71. **Poiseuille’s Law** According to a law discovered by the nineteenth-century physician Poiseuille, the velocity (in centimeters/second) of blood $r$ centimeters from the central axis of an artery is given by

$$v(r) = k(R^2 - r^2)$$

where $k$ is a constant and $R$ is the radius of the artery. Suppose that for a certain artery, $k = 1000$ and $R = 0.2$ so that $v(r) = 1000(0.04 - r^2)$.

a. What is the domain of the function $v(r)$?

b. Compute $v(0)$, $v(0.1)$, and $v(0.2)$ and interpret your results.

72. **Population Growth** A study prepared for a certain Sunbelt town’s Chamber of Commerce projected that the population of the town in the next 3 yr will grow according to the rule

$$P(x) = 50,000 + 30x^{3/2} + 20x$$

where $P(x)$ denotes the population $x$ mo from now. By how much will the population increase during the next 9 mo? During the next 16 mo?

73. **Worker Efficiency** An efficiency study conducted for the Elektra Electronics Company showed that the number of “Space Commander” walkie-talkies assembled by the average worker $t$ hr after starting work at 8:00 A.M. is given by

$$N(t) = -t^3 + 6t^2 + 15t \quad (0 \leq t \leq 4)$$

How many walkie-talkies can an average worker be expected to assemble between 8:00 and 9:00 A.M.? Between 9:00 and 10:00 A.M.?

74. **Learning Curves** The Emory Secretarial School finds from experience that the average student taking ad-
vanced typing will progress according to the rule

\[ N(t) = \frac{60t + 180}{t + 6} \]

where \( N(t) \) measures the number of words/minute the student can type after \( t \) wk in the course. How fast can the average student be expected to type after 2 wk in the course? After 4 wk in the course?

**75. Politics** Political scientists have discovered the following empirical rule, known as the “cube rule,” which gives the relationship between the proportion of seats in the House of Representatives won by Democratic candidates \( s(x) \) and the proportion of popular votes \( x \) received by the Democratic presidential candidate:

\[ s(x) = \frac{x^3}{x^3 + (1 - x)^3} \quad (0 \leq x \leq 1) \]

Compute \( s(0.6) \) and interpret your result.

**76. Home Shopping Industry** According to industry sources, revenue from the home shopping industry for the years since its inception may be approximated by the function

\[ R(t) = \begin{cases} 
-0.03t^3 + 0.25t^2 - 0.12t & \text{if } 0 \leq t \leq 3 \\
0.57t - 0.63 & \text{if } 3 < t \leq 11 
\end{cases} \]

where \( R(t) \) measures the revenue in billions of dollars and \( t \) is measured in years, with \( t = 0 \) corresponding to the beginning of 1984. What was the revenue at the beginning of 1985? At the beginning of 1993? Source: Paul Kagan Associates

**77. Postal Regulations** The postage for first-class mail is 34 cents for the first ounce or fraction thereof and 21 cents for each additional ounce or fraction thereof. Any parcel not exceeding 12 ounces may be sent by first-class mail. Letting \( x \) denote the weight of a parcel in ounces and \( f(x) \) the postage in cents, complete the following description of the “postage function” \( f \):

\[ f(x) = \begin{cases} 
34 & \text{if } 0 < x \leq 1 \\
55 & \text{if } 1 < x \leq 2 \\
\vdots & \text{if } 11 < x \leq 12 
\end{cases} \]

a. What is the domain of \( f \)?

b. Sketch the graph of \( f \).

**78. Harbor Cleanup** The amount of solids discharged from the MWRA (Massachusetts Water Resources Authority) sewage treatment plant on Deer Island (near Boston Harbor) is given by the function

\[ f(t) = \begin{cases} 
130 & \text{if } 0 \leq t \leq 1 \\
-30t + 160 & \text{if } 1 < t \leq 2 \\
100 & \text{if } 2 < t \leq 4 \\
-5t^2 + 25t + 80 & \text{if } 4 < t \leq 6 \\
1.25t^3 - 26.25t + 162.5 & \text{if } 6 < t \leq 10 
\end{cases} \]

where \( f(t) \) is measured in tons/day and \( t \) is measured in years, with \( t = 0 \) corresponding to 1989. Source: Metropolitan District Commission

a. What amount of solids were discharged per day in 1989? In 1992? In 1996?

b. Sketch the graph of \( f \).

In Exercises 79–82, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

79. If \( a = b \), then \( f(a) = f(b) \).

80. If \( f(a) = f(b) \), then \( a = b \).

81. If \( f \) is a function, then \( f(a + b) = f(a) + f(b) \).

82. A vertical line must intersect the graph of \( y = f(x) \) at exactly one point.

### Solutions to Self-Check Exercises 10.1

1. **a.** The expression under the radical sign must be nonnegative, so \( x + 1 \geq 0 \) or \( x \geq -1 \). Also, \( x \neq 0 \) because division by zero is not permitted. Therefore, the domain of \( f \) is \([-1, 0) \cup (0, \infty) \).

   **b.** \( f(3) = \frac{\sqrt{3} + 1}{3} = \frac{\sqrt{4}}{3} = \frac{2}{3} \)

   **c.** \( f(a + h) = \frac{\sqrt{a + h} + 1}{a + h} = \frac{\sqrt{a + h} + 1}{a + h} \)
2. a. For $t$ in the subdomain $[0, 6]$, the rule for $f$ is given by $f(t) = 6t + 17$. The equation $y = 6t + 17$ is a linear equation, so that portion of the graph of $f$ is the line segment joining the points $(0, 17)$ and $(6, 53)$. Next, in the subdomain $(6, 20]$, the rule for $f$ is given by $f(t) = 15.98(t - 6)^{1/4} + 53$. Using a calculator, we construct the following table of values of $f(t)$ for selected values of $t$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(t)$</td>
<td>53</td>
<td>72</td>
<td>75.6</td>
<td>78</td>
<td>79.9</td>
<td>81.4</td>
<td>82.7</td>
<td>83.9</td>
</tr>
</tbody>
</table>

We have included $t = 6$ in the table, although it does not lie in the subdomain of the function under consideration, in order to help us obtain a better sketch of that portion of the graph of $f$ in the subdomain $(6, 20]$. The graph of $f$ is as follows:

b. The percentage of all self-serve gas sales at the beginning of 1978 is found by evaluating $f$ at $t = 4$. Since this point lies in the interval $[0, 6]$, we use the rule $f(t) = 6t + 17$ and find

$$f(4) = 6(4) + 17$$

giving 41% as the required figure. The percentage of all self-serve gas sales at the beginning of 1994 is given by

$$f(20) = 15.98(20 - 6)^{1/4} + 53$$

or approximately 83.9%.

3. A point $(x, y)$ lies on the graph of the function $f$ if and only if the coordinates satisfy the equation $y = f(x)$. Now,

$$f(4) = \sqrt{2(4)} + 1 + 2 = \sqrt{9} + 2 = 5 \neq 6$$

and we conclude that the given point does not lie on the graph of $f$. 
GRAPHING A FUNCTION

Most of the graphs of functions in this book can be plotted with the help of a graphing utility. Furthermore, a graphing utility can be used to analyze the nature of a function. However, the amount and accuracy of the information obtained using a graphing utility depend on the experience and sophistication of the user. As you progress through this book, you will see that the more knowledge of calculus you gain, the more effective the graphing utility will prove as a tool in problem solving.

FINDING A SUITABLE VIEWING RECTANGLE

The first step in plotting the graph of a function with a graphing utility is to select a suitable viewing rectangle. We usually do this by experimenting. For example, you might first plot the graph using the standard viewing rectangle \([-10, 10] \times [-10, 10]\). If necessary, you then might adjust the viewing rectangle by enlarging it or reducing it to obtain a sufficiently complete view of the graph or at least the portion of the graph that is of interest.

EXAMPLE 1

Plot the graph of \(f(x) = 2x^2 - 4x - 5\) in the standard viewing rectangle.

SOLUTION

The graph of \(f\), shown in Figure T1, is a parabola. From our previous work (Example 6, Section 10.1), we know that the figure does give a good view of the graph.

EXAMPLE 2

Let \(f(x) = x^3(x - 3)^4\).

a. Plot the graph of \(f\) in the standard viewing rectangle.

b. Plot the graph of \(f\) in the rectangle \([-1, 5] \times [-40, 40]\).

SOLUTION

a. The graph of \(f\) in the standard viewing rectangle is shown in Figure T2a. Since the graph does not appear to be complete, we need to adjust the viewing rectangle.
b. The graph of \( f \) in the rectangle \([-1, 5] \times [-40, 40] \), shown in Figure T2b, is an improvement over the previous graph. (Later we will be able to show that the figure does in fact give a rather complete view of the graph of \( f \).)

**Evaluating a Function**

A graphing utility can be used to find the value of a function with minimal effort, as the next example shows.

**Example 3**

Let \( f(x) = x^3 - 4x^2 + 4x + 2 \).

a. Plot the graph of \( f \) in the standard viewing rectangle.
b. Find \( f(3) \) and verify your result by direct computation.
c. Find \( f(4.215) \).

**Solution**

a. The graph of \( f \) is shown in Figure T3.

b. Using the evaluation function of the graphing utility and the value 3 for \( x \), we find \( y = 5 \). This result is verified by computing

\[
f(3) = 3^3 - 4(3^2) + 4(3) + 2 = 27 - 36 + 12 + 2 = 5
\]
c. Using the evaluation function of the graphing utility and the value 4.215 for \( x \), we find \( y = 22.679738375 \). Thus, \( f(4.215) = 22.679738375 \). The efficacy of the graphing utility is clearly demonstrated here!

**EXAMPLE 4**

The anticipated rise in the number of Alzheimer’s patients in the United States is given by

\[
f(t) = -0.0277t^4 + 0.3346t^3 - 1.1261t^2 + 1.7575t + 3.7745 \quad (0 \leq t \leq 6)
\]

where \( f(t) \) is measured in millions and \( t \) is measured in decades, with \( t = 0 \) corresponding to the beginning of 1990.

**a.** Use a graphing utility to plot the graph of \( f \) in the viewing rectangle \([0, 7] \times [0, 12]\).

**b.** What was the anticipated number of Alzheimer’s patients in the United States at the beginning of the year 2000 \((t = 1)\)? At the beginning of 2030 \((t = 4)\)?

*Source: Alzheimer’s Association*

**SOLUTION**

**a.** The graph of \( f \) in the viewing rectangle \([0, 7] \times [0, 12]\) is shown in Figure T4.

**b.** Using the evaluation function of the graphing utility and the value 1 for \( x \), we see that the anticipated number of Alzheimer’s patients at the beginning of the year 2000 is given by

\[
f(1) = 4.7128
\]

or approximately 4.7 million. The anticipated number of Alzheimer’s patients at the beginning of 2030 is given by

\[
f(4) = 7.1101
\]

or approximately 7.1 million.
In Exercises 1–8, plot the graph of the function $f$ in the standard viewing window.
1. $f(x) = 2x^2 - 16x + 29$
2. $f(x) = -x^2 - 10x - 20$
3. $f(x) = x^3 - 2x^2 + x - 2$
4. $f(x) = -2.01x^3 + 1.21x^2 - 0.78x + 1$
5. $f(x) = 0.2x^4 - 2.1x^2 + 1$
6. $f(x) = -0.4x^4 + 1.2x - 1.2$
7. $f(x) = 2x\sqrt{x^2 + 1}$
8. $f(x) = \frac{\sqrt{x} + 1}{\sqrt{x} - 1}$

In Exercises 9–20, plot the graph of the function $f$ in (a) the standard viewing window and (b) the indicated window.
9. $f(x) = 2x^2 - 32x + 125; [5, 15] \times [-5, 10]$
10. $f(x) = x^3 + 20x + 95; [-20, 10] \times [-15, -5]$
11. $f(x) = x^3 - 20x^2 + 8x - 10; [-20, 20] \times [-1200, 100]$
12. $f(x) = -2x^3 + 10x^2 - 15x - 5; [-10, 10] \times [-100, 100]$
13. $f(x) = x^4 - 2x^2 + 8; [-2, 2] \times [6, 10]$
14. $f(x) = x^4 - 2x^3; [-1, 3] \times [-2, 2]$
15. $f(x) = x + \frac{1}{x}; [-1, 3] \times [-5, 5]$
16. $f(x) = \frac{4}{x^2 - 8}; [-5, 5] \times [-5, 5]$
17. $f(x) = 2 - \frac{1}{x^2 + 1}; [-3, 3] \times [0, 3]$
18. $f(x) = x - 2\sqrt{x}; [0, 20] \times [-2, 10]$
19. $f(x) = x\sqrt{4 - x^2}; [-3, 3] \times [-2, 2]$
20. $f(x) = \sqrt{x} - 1; [0, 50] \times [-0.25, 0.25]$

In Exercises 21–30, plot the graph of the function $f$ in an appropriate viewing window. (Note: The answer is not unique.)
21. $f(x) = x^2 - 4x + 16$
22. $f(x) = -x^2 + 2x - 11$
23. $f(x) = 2x^3 - 10x^2 + 5x - 10$
24. $f(x) = -x^3 + 5x^2 - 14x + 20$
25. $f(x) = 2x^4 - 3x^3 + 5x^2 - 20x + 40$
26. $f(x) = -2x^4 + 5x^2 - 4$
27. $f(x) = \frac{x^3}{x^3 + 1}$
28. $f(x) = \frac{2x^4 - 3x}{x^3 - 1}$
29. $f(x) = 0.2\sqrt{x} - 0.3x^3$
30. $f(x) = \sqrt{x}(2x - 1)^3$

In Exercises 31–34, use the evaluation function of your graphing utility to find the value of $f$ at the given value of $x$ and verify your result by direct computation.
31. $f(x) = -3x^3 + 5x^2 - 2x + 8; x = -1$
32. $f(x) = 2x^4 - 3x^3 + 2x^2 + x - 5; x = 2$
33. $f(x) = \frac{x^4 - 3x^2}{x - 2}; x = 1$
34. $f(x) = \frac{\sqrt{x^2 - 1}}{3x + 4}; x = 2$

In Exercises 35–42, use the evaluation function of your graphing utility to find the value of $f$ at the indicated value of $x$. Express your answer accurate to four decimal places.
35. $f(x) = 3x^3 - 2x^2 + x - 4; x = 2.145$
36. $f(x) = 2x^3 + 5x^2 + 3x + 1; x = -0.27$
37. $f(x) = 5x^4 - 2x^2 + 8x - 3; x = 1.28$
38. $f(x) = 4x^4 - 3x^3 + 1; x = -2.42$
39. $f(x) = \frac{2x^3 - 3x + 1}{3x - 2}; x = 2.41$
40. $f(x) = \frac{2x + 5}{3x^2 - 4x + 1}; x = -1.72$
41. $f(x) = \sqrt{2x^2 + 1} - \sqrt{3x^2 - 1}; x = 0.62$
42. $f(x) = 2x(3x^3 + 5)^{1/3}; x = -6.24$

43. **Manufacturing Capacity** Data obtained from the Federal Reserve show that the annual increase in manufacturing capacity between 1988 and 1994 is given by
where $f(t)$ is a percentage and $t$ is measured in years, with $t = 0$ corresponding to the beginning of 1988.

**44. Decline of Union Membership** The total union membership as a percentage of the private workforce is given by

$$f(t) = 0.038889t^3 - 0.283333t^2 + 0.477778t + 2.04286 \quad (0 \leq t \leq 6)$$

where $f(t)$ is a percentage and $t$ is measured in years, with $t = 0$ corresponding to the beginning of 1988.

**a.** Use a graphing utility to plot the graph of $f$ in the viewing rectangle $[0, 8] \times [0, 4]$.

**b.** What was the annual increase in manufacturing capacity at the beginning of 1990 ($t = 2$)? At the beginning of 1992 ($t = 4$)?

*Source: Federal Reserve*

**45. Keeping with the Traffic Flow** By driving at a speed to match the prevailing traffic speed, you decrease the chances of an accident. According to a University of Virginia School of Engineering and Applied Science study, the number of accidents per 100 million vehicle miles, $y$, is related to the deviation from the mean speed, $x$, in mph by the equation

$$y = 1.05x^3 - 21.95x^2 + 155.9x - 327.3 \quad (6 \leq x \leq 11)$$

**a.** Plot the graph of $y$ in the viewing rectangle $[6, 11] \times [20, 150]$.

**b.** What is the number of accidents per 100 million vehicle miles if the deviation from the mean speed is 6 mph, 8 mph, and 11 mph?

*Source: University of Virginia School of Engineering and Applied Science*
10.2 The Algebra of Functions

The Sum, Difference, Product, and Quotient of Functions

Let \( S(t) \) and \( R(t) \) denote, respectively, the federal government’s spending and revenue at any time \( t \), measured in billions of dollars. The graphs of these functions for the period between 1981 and 1991 are shown in Figure 10.11.

The budget deficit at any time \( t \) is given by \( S(t) - R(t) \) billion dollars. This observation suggests that we can define a function \( D \) whose value at any time \( t \) is given by \( D(t) = S(t) - R(t) \). The function \( D \), the difference of the two functions \( S \) and \( R \), is written \( D = S - R \), and may be called the “deficit function” since it gives the budget deficit at any time \( t \). It has the same domain as the functions \( S \) and \( R \).

Most functions are built up from other, generally simpler, functions. For example, we may view the function \( f(x) = 2x + 4 \) as the sum of the two functions \( g(x) = 2x \) and \( h(x) = 4 \). The function \( g(x) = 2x \) may in turn be viewed as the product of the functions \( p(x) = 2 \) and \( q(x) = x \).

In general, given the functions \( f \) and \( g \), we define the sum \( f + g \), the difference \( f - g \), the product \( fg \), and the quotient \( f/g \) of \( f \) and \( g \) as follows.
The Sum, Difference, Product, and Quotient of Functions

Let $f$ and $g$ be functions with domains $A$ and $B$, respectively. Then the **sum** $f + g$, **difference** $f - g$, and **product** $fg$ of $f$ and $g$ are functions with domain $A \cap B$ and rule given by

\[
(f + g)(x) = f(x) + g(x) \quad \text{(Sum)}
\]
\[
(f - g)(x) = f(x) - g(x) \quad \text{(Difference)}
\]
\[
(fg)(x) = f(x)g(x) \quad \text{(Product)}
\]

The **quotient** $f/g$ of $f$ and $g$ has domain $A \cap B$ excluding all points $x$ such that $g(x) = 0$ and rule given by

\[
\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{(Quotient)}
\]

**Example 1**

Let $f(x) = \sqrt{x + 1}$ and $g(x) = 2x + 1$. Find the sum $s$, the difference $d$, the product $p$, and the quotient $q$ of the functions $f$ and $g$.

Since the domain of $f$ is $A = [-1, \infty)$ and the domain of $g$ is $B = (-\infty, \infty)$, we see that the domain of $s$, $d$, and $p$ is $A \cap B = [-1, \infty)$. The rules follow.

\[
s(x) = (f + g)(x) = f(x) + g(x) = \sqrt{x + 1} + 2x + 1
\]
\[
d(x) = (f - g)(x) = f(x) - g(x) = \sqrt{x + 1} - (2x + 1) = \sqrt{x + 1} - 2x - 1
\]
\[
p(x) = (fg)(x) = f(x)g(x) = \sqrt{x + 1}(2x + 1) = (2x + 1)\sqrt{x + 1}
\]

The quotient function $q$ has rule

\[
q(x) = \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x + 1}}{2x + 1}
\]

Its domain is $[-1, \infty)$ together with the restriction $x \neq -\frac{1}{2}$. We denote this by $[-1, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$.

**Applications**

The mathematical formulation of a problem arising from a practical situation often leads to an expression that involves the combination of functions. Consider, for example, the costs incurred in operating a business. Costs that remain more or less constant regardless of the firm’s level of activity are called **fixed costs**. Examples of fixed costs are rental fees and executive salaries. On the other hand, costs that vary with production or sales are called **variable costs**. Examples of variable costs are wages and costs of raw materials. The **total cost** of operating a business is thus given by the **sum** of the variable costs and the fixed costs, as illustrated in the next example.
EXAMPLE 2
Suppose Puritron, a manufacturer of water filters, has a monthly fixed cost of $10,000 and a variable cost of
\[-0.0001x^2 + 10x \quad (0 \leq x \leq 40,000)\]
dollars, where \(x\) denotes the number of filters manufactured per month. Find a function \(C\) that gives the total cost incurred by Puritron in the manufacture of \(x\) filters.

SOLUTION ✔
Puritron’s monthly fixed cost is always $10,000, regardless of the level of production, and it is described by the constant function \(F(x) = 10,000\). Next, the variable cost is described by the function \(V(x) = -0.0001x^2 + 10x\). Since the total cost incurred by Puritron at any level of production is the sum of the variable cost and the fixed cost, we see that the required total cost function is given by
\[
C(x) = V(x) + F(x) \\
= -0.0001x^2 + 10x + 10,000 \quad (0 \leq x \leq 40,000)
\]
Next, the total profit realized by a firm in operating a business is the difference between the total revenue realized and the total cost incurred; that is, \(P(x) = R(x) - C(x)\).

EXAMPLE 3
Refer to Example 2. Suppose that the total revenue realized by Puritron from the sale of \(x\) water filters is given by the total revenue function
\[
R(x) = -0.0005x^2 + 20x \quad (0 \leq x \leq 40,000)\]
a. Find the total profit function—that is, the function that describes the total profit Puritron realizes in manufacturing and selling \(x\) water filters per month.
b. What is the profit when the level of production is 10,000 filters per month?

SOLUTION ✔
a. The total profit realized by Puritron in manufacturing and selling \(x\) water filters per month is the difference between the total revenue realized and the total cost incurred. Thus, the required total profit function is given by
\[
P(x) = R(x) - C(x) \\
= (-0.0005x^2 + 20x) - (-0.0001x^2 + 10x + 10,000) \\
= -0.0004x^2 + 10x - 10,000
\]
b. The profit realized by Puritron when the level of production is 10,000 filters per month is
\[
P(10,000) = -0.0004(10,000)^2 + 10(10,000) - 10,000 = 50,000
\]
or $50,000 per month.

Composition of Functions
Another way to build up a function from other functions is through a process known as the composition of functions. Consider, for example, the function
The composition of two functions is defined as follows. Let $f$ and $g$ be functions. Then the composition of $g$ and $f$ is the function $g \circ f$ defined by

$$(g \circ f)(x) = g(f(x))$$

The domain of $g \circ f$ is the set of all $x$ in the domain of $f$ such that $f(x)$ lies in the domain of $g$.

The function $g \circ f$ (read “$g$ circle $f$”) is also called a composite function. The interpretation of the function $h = g \circ f$ as a machine is illustrated in Figure 10.12, and its interpretation as a mapping is shown in Figure 10.13.
Let \( f(x) = x^2 - 1 \) and \( g(x) = \sqrt{x} + 1 \). Compute:

a. The rule for the composite function \( g \circ f \).
b. The rule for the composite function \( f \circ g \).

SOLUTION ✔

a. To find the rule for the composite function \( g \circ f \), evaluate the function \( g \) at \( f(x) \). Therefore,
\[
(g \circ f)(x) = g(f(x)) = \sqrt{f(x)} + 1 = \sqrt{x^2 - 1} + 1
\]

b. To find the rule for the composite function \( f \circ g \), evaluate the function \( f \) at \( g(x) \). Thus,
\[
(f \circ g)(x) = f(g(x)) = (g(x))^2 - 1 = (\sqrt{x} + 1)^2 - 1 = x + 2\sqrt{x} + 1 - 1 = x + 2\sqrt{x}
\]

Example 4 reminds us that in general \( g \circ f \) is different from \( f \circ g \), so care must be taken regarding the order when computing a composite function.

**Group Discussion**

Let \( f(x) = \sqrt{x} + 1 \) for \( x \geq 0 \) and let \( g(x) = (x - 1)^2 \) for \( x \geq 1 \).

1. Show that \((g \circ f)(x)\) and \((f \circ g)(x) = x\). (Remark: The function \( g \) is said to be the inverse of \( f \) and vice versa.)
2. Plot the graphs of \( f \) and \( g \) together with the straight line \( y = x \). Describe the relationship between the graphs of \( f \) and \( g \).

An environmental impact study conducted for Oxnard’s City Environmental Management Department indicates that, under existing environmental protection laws, the level of carbon monoxide present in the air due to pollution from automobile exhaust will be \( 0.01x^{2/3} \) parts per million (ppm) when the number of motor vehicles is \( x \) thousand. A separate study conducted by a state government agency estimates that \( t \) years from now the number of motor vehicles in Oxnard will be \( 0.2t^2 + 4t + 64 \) thousand.

a. Find an expression for the concentration of carbon monoxide in the air due to automobile exhaust \( t \) years from now.
b. What will be the level of concentration \( 5 \) years from now?

**SOLUTION ✔**

a. The level of carbon monoxide present in the air due to pollution from automobile exhaust is described by the function \( g(x) = 0.01x^{2/3} \), where \( x \) is the number (in thousands) of motor vehicles. But the number of motor vehicles \( x \) (in thousands) \( t \) years from now may be estimated by the rule \( f(t) = 0.2t^2 + 4t + 64 \). Therefore, the concentration of carbon monoxide due to automobile exhaust \( t \) years from now is given by
\[
C(t) = (g \circ f)(t) = g(f(t)) = 0.01(0.2t^2 + 4t + 64)^{2/3}
\]
parts per million.
Calculus wasn’t Mike Marchlik’s favorite college subject. In fact, it was not until he started his first job that the “lights went on” and he realized how using calculus allowed him to solve real problems in his everyday work.

Marchlik emphasizes that he doesn’t do “number crunching” himself. “I don’t work out integrals, but in my work I use computer models that do that.” The important issue is not computation but how the answers relate to client problems.

Marchlik’s clients typically process highly toxic or explosive materials. Ebasco evaluates a client site, such as a chemical plant, to determine how safety systems might fail and what the probable consequences might be.

To avoid a major disaster like the one that occurred in Bhopal, India, in 1984, Marchlik and his team might be asked to determine how quickly a poisonous chemical would spread if a leak occurred. Or they might help avert a disaster like the one that rocked the Houston area in 1989. In that incident, hydrocarbon vapor exploded at a Phillips 66 chemical plant, shaking office buildings in Houston, 12 miles away. Several people were killed, and hundreds were injured. Property damage totaled $1.39 billion.

In assessing risks for a fuel-storage depot, Ebasco considers variables such as weather conditions, including probable wind speed, the flow properties of a gas, and possible ignition sources. With today’s more powerful computers, the models can involve a system of very complex equations to project likely scenarios.

Mathematical models vary, however. One model might forecast how much gas will flow out of a hole and how quickly it will disperse. Another model might project where the gas will go, depending on local factors such as temperature and wind speed. Choosing the right model is essential. A model based on flat terrain when the client’s storage depot is set among hills is going to produce the wrong answer.

Marchlik and his team run several models together to come up with their projections. Each model uses “equations that have to be integrated to come up with solutions.” The bottom line? Marchlik stresses that “calculus is at the very heart” of Ebasco’s risk-assessment work.
b. The level of concentration 5 years from now will be
\[ \frac{0.01[0.2(5)^2 + 4(5) + 64]^{2/3}}{(0.01)^{892/3}} \approx 0.20 \] parts per million.

\[ \text{SELF-CHECK EXERCISES 10.2} \]

1. Let \( f \) and \( g \) be functions defined by the rules
\[ f(x) = \sqrt{x} + 1 \quad \text{and} \quad g(x) = \frac{x}{1 + x} \]
respectively. Find the rules for
a. the sum \( s \), the difference \( d \), the product \( p \), and the quotient \( q \) of \( f \) and \( g \).

2. Health-care spending per person by the private sector includes payments by individuals, corporations, and their insurance companies and is approximated by the function
\[ f(t) = 2.5t^2 + 31.3t + 406 \quad (0 \leq t \leq 20) \]
where \( f(t) \) is measured in dollars and \( t \) is measured in years, with \( t = 0 \) corresponding to the beginning of 1975. The corresponding government spending—including expenditures for Medicaid, Medicare, and other federal, state, and local government public health care—is
\[ g(t) = 1.4t^2 + 29.6t + 251 \quad (0 \leq t \leq 20) \]
where \( t \) has the same meaning as before.
a. Find a function that gives the difference between private and government health-care spending per person at any time \( t \).
b. What was the difference between private and government expenditures per person at the beginning of 1985? At the beginning of 1993?

Source: Health Care Financing Administration

Solutions to Self-Check Exercises 10.2 can be found on page 601.

10.2 Exercises

In Exercises 1–8, let \( f(x) = x^3 + 5 \), \( g(x) = x^3 - 2 \), and \( h(x) = 2x + 4 \). Find the rule for each function.

1. \( f + g \)
2. \( f - g \)
3. \( fg \)
4. \( gf \)
5. \( \frac{f}{g} \)
6. \( \frac{f - g}{h} \)
7. \( \frac{fg}{h} \)
8. \( fgh \)

In Exercises 9–18, let \( f(x) = x - 1 \), \( g(x) = \sqrt{x + 1} \), and \( h(x) = 2x^3 - 1 \). Find the rule for each function.

9. \( f + g \)
10. \( g - f \)
11. \( fg \)
12. \( gf \)
13. \( \frac{g}{h} \)
14. \( \frac{h}{g} \)
15. \( \frac{fg}{h} \)
16. \( \frac{fh}{g} \)
17. \( \frac{f - h}{g} \)
18. \( \frac{gh}{g - f} \)

In Exercises 19–24, find the functions \( f + g \), \( f - g \), \( fg \), and \( f/g \).

19. \( f(x) = x^2 + 5 \); \( g(x) = \sqrt{x} - 2 \)
20. \( f(x) = \sqrt{x - 1} \); \( g(x) = x^3 + 1 \)
21. \( f(x) = \sqrt{x + 3} \); \( g(x) = \frac{1}{x - 1} \)
22. \( f(x) = \frac{1}{x^2 + 1} \); \( g(x) = \frac{1}{x^2 - 1} \)
23. \( f(x) = \frac{x + 1}{x - 1} \); \( g(x) = \frac{x + 2}{x - 2} \)
24. \( f(x) = x^2 + 1 \); \( g(x) = \sqrt{x + 1} \)
In Exercises 25–30, find the rules for the composite functions \( f \circ g \) and \( g \circ f \).

25. \( f(x) = x^2 + x + 1; g(x) = x^2 \)
26. \( f(x) = 3x^2 + 2x + 1; g(x) = x + 3 \)
27. \( f(x) = \sqrt{x} + 1; g(x) = x^2 - 1 \)
28. \( f(x) = 2\sqrt{x} + 3; g(x) = x^2 + 1 \)
29. \( f(x) = \frac{x}{x^2 + 1}; g(x) = \frac{1}{x} \)
30. \( f(x) = \sqrt{x + 1}; g(x) = \frac{1}{x - 1} \)

In Exercises 31–34, evaluate \( h(2) \), where \( h = g \circ f \).

31. \( f(x) = x^2 + x + 1; g(x) = x^2 \)
32. \( f(x) = \sqrt{x^2 - 1}; g(x) = 3x^3 + 1 \)
33. \( f(x) = \frac{1}{2x + 1}; g(x) = \sqrt{x} \)
34. \( f(x) = \frac{1}{x - 1}; g(x) = x^2 + 1 \)

In Exercises 35–42, find functions \( f \) and \( g \) such that \( h = g \circ f \). (Note: The answer is not unique.)

35. \( h(x) = (2x^3 + x^2 + 1)^3 \)
36. \( h(x) = (3x^2 - 4)^3 \)
37. \( h(x) = \sqrt{x^2 - 1} \)
38. \( h(x) = (2x - 3)^{1/2} \)
39. \( h(x) = \frac{1}{x - 1} \)
40. \( h(x) = \frac{1}{\sqrt{x^2 - 4}} \)
41. \( h(x) = \frac{1}{(3x^2 + 2)^{1/2}} \)
42. \( h(x) = \frac{1}{\sqrt{2x + 4} + \sqrt{2x + 1}} \)

In Exercises 43–46, find \( f(a + h) - f(a) \) for each function. Simplify your answer.

43. \( f(x) = 3x + 4 \)
44. \( f(x) = -\frac{1}{2}x + 3 \)
45. \( f(x) = 4 - x^2 \)
46. \( f(x) = x^2 - 2x + 1 \)

47. If \( f(x) = x^2 + 1 \), find and simplify \( \frac{f(a + h) - f(a)}{h} \) \((h \neq 0)\)
48. If \( f(x) = 1/x \), find and simplify \( \frac{f(a + h) - f(a)}{h} \) \((h \neq 0)\)

49. **Manufacturing Costs** TMI, Inc., a manufacturer of blank audiocassette tapes, has a monthly fixed cost of $12,100 and a variable cost of $.60/tape. Find a function \( C \) that gives the total cost incurred by TMI in the manufacture of \( x \) tapes/month.

50. **Cost of Producing Electronic Organizers** Apollo, Inc., manufactures its electronic organizers at a variable cost of

\[
V(x) = 0.000003x^3 - 0.03x^2 + 200x
\]
dollars, where \( x \) denotes the number of units manufactured per month. The monthly fixed cost attributable to the division that produces these electronic organizers is $100,000. Find a function \( C \) that gives the total cost incurred by the manufacture of \( x \) electronic organizers. What is the total cost incurred in producing 2000 units/month?

51. **Cost of Producing Electronic Organizers** Refer to Exercise 50. Suppose the total revenue realized by Apollo from the sale of \( x \) electronic organizers is given by the total revenue function

\[
R(x) = -0.1x^2 + 500x \quad (0 \leq x \leq 5000)
\]

where \( x \) is measured in dollars.

(a) Find the total profit function.

(b) What is the profit when 1500 units are produced and sold each month?

52. **Overcrowding of Prisons** The 1980s saw a trend toward old-fashioned punitive deterrence as opposed to the more liberal penal policies and community-based corrections popular in the 1960s and early 1970s. As a result, prisons became more crowded, and the gap between the number of people in prison and the prison capacity widened. Based on figures from the U.S. Department of Justice, the number of prisoners (in thousands) in federal and state prisons is approximated by the function

\[
N(t) = 3.5t^2 + 26.7t + 436.2 \quad (0 \leq t \leq 10)
\]

where \( t \) is measured in years and \( t = 0 \) corresponds to 1983. The number of inmates for which prisons were designed is given by

\[
C(t) = 24.3t + 365 \quad (0 \leq t \leq 10)
\]

where \( C(t) \) is measured in thousands and \( t \) has the same meaning as before.

(a) Find an expression that shows the gap between the number of prisoners and the number of inmates for which the prisons were designed at any time \( t \).

(b) Find the gap at the beginning of 1983 and at the beginning of 1986.

*Source: U.S. Department of Justice*
53. **Effect of Mortgage Rates on Housing Starts** A study prepared for the National Association of Realtors estimated that the number of housing starts per year over the next 5 yr will be

\[ N(t) = \frac{7}{1 + 0.02t^2} \]

million units, where \( r \) (percent) is the mortgage rate. Suppose the mortgage rate over the next \( t \) mo is

\[ r(t) = \frac{10t + 150}{t + 10} \quad (0 \leq t \leq 24) \]

percent/year.

a. Find an expression for the number of housing starts per year as a function of \( t \), \( t \) months from now.

b. Using the result from part (a), determine the number of housing starts at present, 12 mo from now, and 18 mo from now.

54. **Hotel Occupancy Rate** The occupancy rate of the all-suite Wonderland Hotel, located near an amusement park, is given by the function

\[ r(t) = \frac{10t^3}{81} - \frac{10}{3} t^2 + \frac{200}{9} t + 55 \quad (0 \leq t \leq 11) \]

where \( t \) is measured in months and \( t = 0 \) corresponds to the beginning of January. Management has estimated that the monthly revenue (in thousands of dollars) is approximated by the function

\[ R(r) = -\frac{3}{5000} r^3 + \frac{9}{50} r^2 \quad (0 \leq r \leq 100) \]

where \( r \) is the occupancy rate.

a. What is the hotel’s occupancy rate at the beginning of January? At the beginning of June?

b. What is the hotel’s monthly revenue at the beginning of January? At the beginning of June?

Hint: Compute \( R(r(0)) \) and \( R(r(5)) \).

55. **Housing Starts and Construction Jobs** The president of a major housing construction firm reports that the number of construction jobs (in millions) created is given by

\[ N(x) = 1.42x \]

where \( x \) denotes the number of housing starts. Suppose the number of housing starts in the next \( t \) mo is expected to be

\[ x(t) = \frac{7(t + 10)^2}{(t + 10)^2 + 2(t + 15)^2} \]

million units/year. Find an expression for the number of jobs created per month in the next \( t \) mo. How many jobs will have been created 6 months and 12 mo from now?

In Exercises 56–59, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

56. If \( f \) and \( g \) are functions with domain \( D \), then \( f + g = g + f \).

57. If \( g \circ f \) is defined at \( x = a \), then \( f \circ g \) must also be defined at \( x = a \).

58. If \( f \) and \( g \) are functions, then \( f \circ g = g \circ f \).

59. If \( f \) is a function, then \( f \circ f = f^2 \).

### Solutions to Self-Check Exercises 10.2

1. a. \( s(x) = f(x) + g(x) = \sqrt{x} + 1 + \frac{x}{1 + x} \)

\[ d(x) = f(x) - g(x) = \sqrt{x} + 1 - \frac{x}{1 + x} \]

\[ p(x) = f(x)g(x) = (\sqrt{x} + 1) \cdot \frac{x}{1 + x} = \frac{x(\sqrt{x} + 1)}{1 + x} \]

\[ q(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x} + 1}{x} = \frac{(\sqrt{x} + 1)(1 + x)}{x} \]
b. \((f \cdot g)(x) = f(g(x)) = \frac{x}{\sqrt{1 + x}} + 1\)

\((g \cdot f)(x) = g(f(x)) = \frac{\sqrt{x} + 1}{1 + (\sqrt{x} + 1)} = \frac{\sqrt{x} + 1}{\sqrt{x} + 2}\)

2. a. The difference between private and government health-care spending per person at any time \(t\) is given by the function \(d\) with the rule

\[d(t) = f(t) - g(t) = (2.5t^2 + 31.3t + 406) - (1.4t^2 + 29.6t + 251) = 1.1t^2 + 1.7t + 155\]

b. The difference between private and government expenditures per person at the beginning of 1985 is given by

\[d(10) = 1.1(10)^2 + 1.7(10) + 155\]

or $282/person.

The difference between private and government expenditures per person at the beginning of 1993 is given by

\[d(18) = 1.1(18)^2 + 1.7(18) + 155\]

or $542/person.

10.3 Functions and Mathematical Models

More on Mathematical Models

In calculus we are concerned primarily with how one (dependent) variable depends on one or more (independent) variables. Consequently, most of our mathematical models involve functions of one or more variables.* Once a function has been constructed to describe a specific real-world problem, a host of questions pertaining to the problem may be answered by analyzing the function (mathematical model). For example, if we have a function that gives the population of a certain culture of bacteria at any time \(t\), then we can determine how fast the population is increasing or decreasing at any time \(t\), and so on. Conversely, if we have a model that gives the rate of change of the cost of producing a certain item as a function of the level of production and if we know the fixed cost incurred in producing this item, then we can find the total cost incurred in producing a certain number of those items.

Before going on, let’s look at two mathematical models. The first is used to estimate spending by business on computer security, and the second is used to project the growth of the number of people enrolled in health maintenance organizations (HMOs). These models are derived from data using the least-squares technique. In the Using Technology section on page 615, you can see how mathematical models are constructed from raw data.

* Functions of more than one variable will be studied later.
EXAMPLE 1

The estimated spending (in billions of dollars) by businesses on computer security equipment and services from 1987 to 1993 is given in the following table. The figures include spending for protection against computer criminals who steal, erase, or alter data, along with protection against fires, electrical failures, and natural disasters.

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Spending</td>
<td>0.49</td>
<td>0.59</td>
<td>0.66</td>
<td>0.73</td>
<td>0.81</td>
<td>0.93</td>
<td>1.02</td>
</tr>
</tbody>
</table>

A mathematical model approximating the amount of spending over the period in question is given by

\[ S(t) = 0.0864t + 0.4879 \]

where \( t \) is measured in years, with \( t = 0 \) corresponding to 1987.

a. Sketch the graph of the function \( S \) and the given data on the same set of axes.

b. Assuming that this trend continued, what was the spending by business on computer security equipment and services in 1995 (\( t = 8 \)).

c. What is the rate of increase of the annual expenditure over the period in question?

Source: Frost & Sullivan, Inc.

SOLUTION

a. The graph of \( S \) is shown in Figure 10.14.

b. The estimated spending in 1995 is

\[ S(8) = 0.0864(8) + 0.4879 \approx 1.1791 \]

or approximately $1.18 billion.

c. The function \( S \) is linear, and so we conclude that the annual increase in the expenditure is given by the slope of the straight line represented by \( S \), which is approximately $0.09 billion per year.

FIGURE 10.14
Estimated spending by businesses on computer security equipment and services

EXAMPLE 2

The number of people (in millions) enrolled in HMOs from 1988 to 1998 is given in the following table.

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of People</td>
<td>32.7</td>
<td>36.5</td>
<td>41.4</td>
<td>51.1</td>
<td>66.5</td>
<td>78.0</td>
</tr>
</tbody>
</table>
A mathematical model approximating the number of people, \( N(t) \), enrolled in HMOs during this period is

\[
N(t) = -0.0258t^3 + 0.7465t^2 - 0.3491t + 33.1444 \quad (0 \leq t \leq 10)
\]

where \( t \) is measured in years and \( t = 0 \) corresponds to 1988.

\textbf{a.} Sketch the graph of the function \( N \) to see how the model compares with the actual data.

\textbf{b.} Assume that this trend continues and use the model to predict how many people will be enrolled in HMOs at the beginning of 2002.

\begin{align*}
\textbf{SOLUTION} \\
\textbf{a.} & \quad \text{The graph of the function } N \text{ is shown in Figure 10.15.} \\
\textbf{b.} & \quad \text{The number of people that will be enrolled in HMOs at the beginning of 2002 is given by} \\
& \quad N(14) = -0.0258(14)^3 + 0.7465(14)^2 - 0.3491(14) + 33.1444 \\
& \quad = 103.7758
\end{align*}

or approximately 103.8 million people.

We will discuss several mathematical models from the field of economics later in this section, but first we review some important functions that are the bases for many mathematical models.

**Polynomial Functions**

We begin by recalling a special class of functions, polynomial functions.

**Polynomial Function**

A polynomial function of degree \( n \) is a function of the form

\[
f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \quad (a_0 \neq 0)
\]

where \( a_0, a_1, \ldots, a_n \) are constants and \( n \) is a nonnegative integer.
For example, the functions
\[ f(x) = 4x^5 - 3x^4 + x^2 - x + 8 \]
\[ g(x) = 0.001x^5 - 2x^2 + 20x + 400 \]
are polynomial functions of degrees 5 and 3, respectively. Observe that a polynomial function is defined everywhere so that it has domain \((-\infty, \infty)\).

A polynomial function of degree 1 \((n = 1)\)
\[ f(x) = a_0x + a_1 \quad (a_0 \neq 0) \]
is the equation of a straight line in the slope-intercept form with slope \(m = a_0\) and \(y\)-intercept \(b = a_1\) (see Section 1.2). For this reason, a polynomial function of degree 1 is called a \textit{linear function}. For example, the linear function \(f(x) = 2x - 3\) may be written as a linear equation in \(x\) and \(y\)—namely, \(y = 2x + 3\) or \(2x - y + 3 = 0\). Conversely, the linear equation \(2x - 3y + 4 = 0\) can be solved for \(y\) in terms of \(x\) to yield the linear function \(y = f(x) = \frac{2}{3}x + \frac{4}{3}\).

A polynomial function of degree 2 is referred to as a \textit{quadratic function}. A polynomial function of degree 3 is called a \textit{cubic function}, and so on. The mathematical model in Example 2 involves a cubic function.

**Rational and Power Functions**

Another important class of functions is rational functions. A \textit{rational function} is simply the quotient of two polynomials. Examples of rational functions are
\[ F(x) = \frac{3x^3 + x^2 - x + 1}{x - 2} \]
\[ G(x) = \frac{x^2 + 1}{x^2 - 1} \]
In general, a rational function has the form
\[ R(x) = \frac{f(x)}{g(x)} \]
where \(f(x)\) and \(g(x)\) are polynomial functions. Since division by zero is not allowed, we conclude that the domain of a rational function is the set of all real numbers except the zeros of \(g\)—that is, the roots of the equation \(g(x) = 0\). Thus, the domain of the function \(F\) is the set of all numbers except \(x = 2\), whereas the domain of the function \(G\) is the set of all numbers except those that satisfy \(x^2 - 1 = 0\) or \(x = \pm1\).

Functions of the form
\[ f(x) = x^r \]
where \(r\) is any real number, are called \textit{power functions}. We encountered examples of power functions earlier in our work. For example, the functions
\[ f(x) = \sqrt{x} = x^{1/2} \quad \text{and} \quad g(x) = \frac{1}{x^2} = x^{-2} \]
are power functions.
Many of the functions we will encounter later will involve combinations of the functions introduced here. For example, the following functions may be viewed as suitable combinations of such functions:

\[
\begin{align*}
    f(x) &= \sqrt{\frac{1 - x^2}{1 + x^2}} \\
    g(x) &= \sqrt{x^2 - 3x + 4} \\
    h(x) &= (1 + 2x)^{1/2} + \frac{1}{\left(x^2 + 2\right)^{1/2}}
\end{align*}
\]

As with polynomials of degree 3 or greater, analyzing the properties of these functions is facilitated by using the tools of calculus, to be developed later.

**EXAMPLE 3**

A study of driving costs based on 1992 model compact (six-cylinder) cars found that the average cost (car payments, gas, insurance, upkeep, and depreciation), measured in cents per mile, is approximated by the function

\[
C(x) = \frac{2095}{x^{2.2}} + 20.08
\]

where \(x\) (in thousands) denotes the number of miles the car is driven in 1 year. Using this model, estimate the average cost of driving a compact car 10,000 miles a year and 20,000 miles a year.

*Source:* Runzheimer International Study

**SOLUTION**

The average cost of driving a compact car 10,000 miles a year is given by

\[
C(10) = \frac{2095}{10^{2.2}} + 20.08 \\
\approx 33.3
\]

or approximately 33 cents per mile. The average cost of driving 20,000 miles a year is given by

\[
C(20) = \frac{2095}{20^{2.2}} + 20.08 \\
\approx 23.0
\]

or approximately 23 cents per mile.

**Some Economic Models**

In the remainder of this section, we look at some examples of nonlinear supply and demand functions. Before proceeding, however, you might want to review the concepts of supply and demand equations and of market equilibrium (see Sections 1.3 and 1.4).

**EXAMPLE 4**

The demand function for a certain brand of videocassette is given by

\[
p = d(x) = -0.01x^2 - 0.2x + 8
\]
where $p$ is the wholesale unit price in dollars and $x$ is the quantity demanded each week, measured in units of a thousand. Sketch the corresponding demand curve. Above what price will there be no demand? What is the maximum quantity demanded per week?

The given function is quadratic, and its graph, which appears in Figure 10.16, may be sketched using the methods just developed. The $p$-intercept, 8, gives the wholesale unit price above which there will be no demand. To obtain the maximum quantity demanded, we set $p = 0$, which gives

$$-0.01x^2 - 0.2x + 8 = 0$$
$$x^2 + 20x - 800 = 0$$  \hspace{1cm} \text{(Multiplying both sides of the equation by −100)}

or

$$(x + 40)(x - 20) = 0$$

That is, $x = -40$ or $x = 20$. Since $x$ must be nonnegative, we reject the root $x = -40$. Thus, the maximum number of videocassettes demanded per week is 20,000.

The supply function for a certain brand of videocassette is given by

$$p = s(x) = 0.01x^2 + 0.1x + 3$$

where $p$ is the unit wholesale price in dollars and $x$ stands for the quantity that will be made available in the market by the supplier, measured in units of a thousand. Sketch the corresponding supply curve. What is the lowest price at which the supplier will make the videocassettes available in the market?

Figure 10.17 is a sketch of the supply curve. The $p$-intercept, 3, gives the lowest price at which the supplier will make the videocassettes available in the market.

In Examples 4 and 5, the demand function for a certain brand of videocassette is given by

$$p = d(x) = -0.01x^2 - 0.2x + 8$$
and the corresponding supply function is given by

\[ p = s(x) = 0.01x^2 + 0.1x + 3 \]

where \( p \) is expressed in dollars and \( x \) is measured in units of a thousand. Find the equilibrium quantity and price.

**SOLUTION**

We solve the following system of equations:

\[ \begin{align*}
  &0.01x^2 + 0.1x + 3 = 0.01x^2 - 0.2x + 8 \\
  &0.01x^2 + 0.1x + 3 = 0.01x^2 - 0.2x + 8
\end{align*} \]

Substituting the first equation into the second yields

\[ -0.01x^2 - 0.2x + 8 = 0.01x^2 + 0.1x + 3 \]

which is equivalent to

\[ \begin{align*}
  &0.02x^2 + 0.3x - 5 = 0 \\
  &2x^2 + 30x - 500 = 0 \\
  &x^2 + 15x - 250 = 0 \\
  &(x + 25)(x - 10) = 0
\end{align*} \]

Thus, \( x = -25 \) or \( x = 10 \). Since \( x \) must be nonnegative, the root \( x = -25 \) is rejected. Therefore, the equilibrium quantity is 10,000 videocassettes. The equilibrium price is given by

\[ p = 0.01(10)^2 + 0.1(10) + 3 = 5 \]

or $5 per videocassette (Figure 10.18).

**EXPLORE WITH TECHNOLOGY**

1. **a.** Use a graphing utility to plot the straight lines \( L_1 \) and \( L_2 \) with equations \( y = 2x - 1 \) and \( y = 2.1x + 3 \), respectively, on the same set of axes using the standard viewing rectangle. Do the lines appear to intersect?
   **b.** Plot the straight lines \( L_1 \) and \( L_2 \) using the viewing rectangle \([-100, 100] \times [-100, 100] \). Do the lines appear to intersect? Can you find the point of intersection using **TRACE** and **ZOOM**? Using the “intersection” function of your graphing utility?
   **c.** Find the point of intersection of \( L_1 \) and \( L_2 \) algebraically.
   **d.** Comment on the effectiveness of the methods of solutions in parts (b) and (c).

2. **a.** Use a graphing utility to plot the straight lines \( L_1 \) and \( L_2 \) with equations \( y = 3x - 2 \) and \( y = -2x + 3 \), respectively, on the same set of axes using the standard viewing rectangle. Then use **TRACE** and **ZOOM** to find the point of intersection of \( L_1 \) and \( L_2 \). Repeat using the “intersection” function of your graphing utility.
   **b.** Find the point of intersection of \( L_1 \) and \( L_2 \) algebraically.
   **c.** Comment on the effectiveness of the methods.

**CONSTRUCTING MATHEMATICAL MODELS**

We close this section by showing how some mathematical models can be constructed using elementary geometric and algebraic arguments.
### EXAMPLE 7

The owner of the Rancho Los Feliz has 3000 yards of fencing material with which to enclose a rectangular piece of grazing land along the straight portion of a river. Fencing is not required along the river. Letting \( x \) denote the width of the rectangle, find a function \( f \) in the variable \( x \) giving the area of the grazing land if she uses all of the fencing material (Figure 10.19).

![Figure 10.19](image)

**SOLUTION**

The area of the rectangular grazing land is \( A = xy \). Next, observe that the amount of fencing is \( 2x + y \) and this must be equal to 3000 since all the fencing material is used; that is,

\[
2x + y = 3000
\]

From the equation we see that \( y = 3000 - 2x \). Substituting this value of \( y \) into the expression for \( A \) gives

\[
A = xy = x(3000 - 2x) = 3000x - 2x^2
\]

Finally, observe that both \( x \) and \( y \) must be nonnegative since they represent the width and length of a rectangle, respectively. Thus, \( x \geq 0 \) and \( y \geq 0 \). But the latter is equivalent to \( 3000 - 2x \geq 0 \), or \( x \leq 1500 \). So the required function is \( f(x) = 3000x - 2x^2 \) with domain \( 0 \leq x \leq 1500 \).

**REMARK**

Observe that if we view the function \( f(x) = 3000x - 2x^2 \) strictly as a mathematical entity, then its domain is the set of all real numbers. But physical consideration dictates that its domain should be restricted to the interval \([0, 1500]\).

### EXAMPLE 8

If exactly 200 people sign up for a charter flight, the Leisure World Travel Agency charges $300 per person. However, if more than 200 people sign up for the flight (assume this is the case), then each fare is reduced by $1 for each additional person. Letting \( x \) denote the number of passengers above 200, find a function giving the revenue realized by the company.

**SOLUTION**

If there are \( x \) passengers above 200, then the number of passengers signing up for the flight is \( 200 + x \). Furthermore, the fare will be \$(300 - x) per passenger. Therefore, the revenue will be

\[
R = (200 + x)(300 - x) = -x^2 + 100x + 60,000
\]

Clearly, \( x \) must be nonnegative, and \( 300 - x \geq 0 \), or \( x \leq 300 \). So the required function is \( f(x) = -x^2 + 100x + 60,000 \) with domain \([0, 300]\).
1. Thomas Young has suggested the following rule for calculating the dosage of medicine for children from ages 1 to 12 years. If $a$ denotes the adult dosage (in milligrams) and $t$ is the age of the child (in years), then the child’s dosage is given by

$$D(t) = \frac{at}{t + 12}$$

If the adult dose of a substance is 500 mg, how much should a 4-yr-old child receive?

2. The demand function for Mrs. Baker’s cookies is given by

$$d(x) = -\frac{2}{15}x + 4$$

where $d(x)$ is the wholesale price in dollars per pound and $x$ is the quantity demanded each week, measured in thousands of pounds. The supply function for the cookies is given by

$$s(x) = \frac{1}{75}x^2 + \frac{1}{10}x + \frac{3}{2}$$

where $s(x)$ is the wholesale price in dollars per pound and $x$ is the quantity, in thousands of pounds, that will be made available in the market per week by the supplier.

a. Sketch the graphs of the functions $d$ and $s$.

b. Find the equilibrium quantity and price.

Solutions to Self-Check Exercises 10.3 can be found on page 618.

In Exercises 1–6, determine whether the given function is a polynomial function, a rational function, or some other function. State the degree of each polynomial function.

1. $f(x) = 3x^6 - 2x^2 + 1$

2. $f(x) = \frac{x^2 - 9}{x - 3}$

3. $G(x) = 2(x^2 - 3)^3$

4. $H(x) = 2x^2 + 5x^{-2} + 6$

5. $f(t) = 2t^2 + 3\sqrt{t}$

6. $f(r) = \frac{6r}{(r^3 - 8)}$

7. Disposable Income Economists define the disposable annual income for an individual by the equation $D = (1 - r)T$, where $T$ is the individual’s total income and $r$ is the net rate at which he or she is taxed. What is the disposable income for an individual whose income is $40,000 and whose net tax rate is 28%?

8. Drug Dosages A method sometimes used by pediatricians to calculate the dosage of medicine for children is based on the child’s surface area. If $a$ denotes the adult dosage (in milligrams) and $S$ is the surface area of the child (in square meters), then the child’s dosage is given by

$$D(S) = \frac{Sa}{1.7}$$

If the adult dose of a substance is 500 mg, how much should a child whose surface area is 0.4 m$^2$ receive?

9. Cowling’s Rule Cowling’s rule is a method for calculating pediatric drug dosages. If $a$ denotes the adult dosage (in milligrams) and $t$ is the age of the child (in years), then the child’s dosage is given by

$$D(t) = \left(\frac{t + 1}{24}\right)a$$

If the adult dose of a substance is 500 mg, how much should a 4-yr-old child receive?
10. Worker Efficiency  An efficiency study showed that the average worker at Delphi Electronics assembled cordless telephones at the rate of

\[ f(t) = -\frac{3}{2}t^2 + 6t + 10 \quad (0 \leq t \leq 4) \]

phones/hour, \( t \) hr after starting work during the morning shift. At what rate does the average worker assemble telephones 2 hr after starting work?

11. Revenue Functions  The revenue (in dollars) realized by Cunningham Realty depends on the amount of money \( x \) spent on advertising per quarter according to the rule

\[ R(t) = -0.1x^2 + 500x \]

where \( x \) denotes the number of units manufactured per month. What is Cunningham’s profit when 1000 units are produced?

12. Effect of Advertising on Sales  The quarterly profit of Cunningham Realty depends on the amount of money \( x \) spent on advertising per quarter according to the rule

\[ P(x) = -\frac{1}{8}x^2 + 7x + 30 \quad (0 \leq x \leq 50) \]

where \( P(x) \) and \( x \) are measured in thousands of dollars. What is Cunningham’s profit when its quarterly advertising budget is $28,000?

13. E-mail Usage  The number of international e-mailings per day (in millions) is approximated by the function

\[ f(t) = 38.57t^2 - 24.29t + 79.14 \quad (0 \leq t \leq 4) \]

where \( t \) is measured in years with \( t = 0 \) corresponding to the beginning of 1998.

a. Sketch the graph of \( f \).

b. How many international e-mailings per day were there at the beginning of the year 2000? 

Source: Pioneer Consulting

14. Document Management  The size (measured in millions of dollars) of the document-management business is described by the function

\[ f(t) = 0.22t^2 + 1.4t + 3.77 \quad (0 \leq t \leq 6) \]

where \( t \) is measured in years with \( t = 0 \) corresponding to the beginning of 1996.

a. Sketch the graph of \( f \).

b. What was the size of the document-management business at the beginning of the year 2000? 

Source: Sun Trust Equitable Securities

15. Reaction of a Frog to a Drug  Experiments conducted by A. J. Clark suggest that the response \( R(x) \) of a frog’s heart muscle to the injection of \( x \) units of acetylcholine (as a percentage of the maximum possible effect of the drug) may be approximated by the rational function

\[ R(x) = \frac{100x}{b + x} \quad (x \geq 0) \]

where \( b \) is a positive constant that depends on the particular frog.

a. If a concentration of 40 units of acetylcholine produces a response of 50% for a certain frog, find the “response function” for this frog.

b. Using the model found in part (a), find the response of the frog’s heart muscle when 60 units of acetylcholine are administered.

16. Linear Depreciation  In computing income tax, business firms are allowed by law to depreciate certain assets such as buildings, machines, furniture, automobiles, and so on, over a period of time. The linear depreciation, or straight-line method, is often used for this purpose. Suppose an asset has an initial value of \( S \) and is to be depreciated linearly over \( n \) years with a scrap value of \$S\$. Show that the book value of the asset at any time \( t \) (0 \leq t \leq n) is given by the linear function

\[ V(t) = C - \frac{(C - S)}{n}t \]

Hint: Find an equation of the straight line that passes through the points \((0, C)\) and \((n, S)\). Then rewrite the equation in the slope-intercept form.

17. Linear Depreciation  Using the linear depreciation model of Exercise 16, find the book value of a printing machine at the end of the second year if its initial value is $100,000 and it is depreciated linearly over 5 years with a scrap value of $30,000.

18. Price of Ivory  According to the World Wildlife Fund, a group in the forefront of the fight against illegal ivory trade, the price of ivory (in dollars per kilo) compiled from a variety of legal and black market sources is approximated by the function

\[ f(t) = \begin{cases} 8.37t + 7.44 & \text{if } 0 \leq t \leq 8 \\ 2.84t + 51.68 & \text{if } 8 < t \leq 30 \end{cases} \]

where \( t \) is measured in years and \( t = 0 \) corresponds to the beginning of 1970.

a. Sketch the graph of the function \( f \).

b. What was the price of ivory at the beginning of 1970? At the beginning of 1990?

19. Sales of Digital TVs  The number of homes with digital TVs is expected to grow according to the function

\[ f(t) = 0.1714t^2 + 0.6657t + 0.7143 \quad (0 \leq t \leq 6) \]
where $t$ is measured in years with $t = 0$ corresponding to the beginning of the year 2000 and $f(t)$ is measured in millions of homes.

a. How many homes had digital TVs at the beginning of the year 2000?

b. How many homes will have digital TVs at the beginning of 2005?

Source: Consumer Electronics Manufacturers Association

20. **Senior Citizens’ Health Care**  According to a study conducted for the Senate Select Committee on Aging, the out-of-pocket cost to senior citizens for health care, $f(t)$ (as a percentage of income), in year $t$ where $t = 0$ corresponds to 1977, is given by

$$
y = \begin{cases} 
  \frac{2}{7}t + 12 & \text{if } 0 \leq t \leq 7 \\
  t + 7 & \text{if } 7 < t \leq 10 \\
  \frac{3}{5}t + 11 & \text{if } 10 < t < 20 
\end{cases}
$$

a. Sketch the graph of $f$.

b. What was the out-of-pocket cost to senior citizens for health care in 1982? In 1992?

Source: Senate Select Committee on Aging, AARP

21. **Price of Automobile Parts**  For years, automobile manufacturers had a monopoly on the replacement-parts market, particularly for sheet metal parts such as fenders, doors, and hoods, the parts most often damaged in a crash. Beginning in the late 1970s, however, competition appeared on the scene. In a report conducted by an insurance company to study the effects of the competition, the price of an OEM (original equipment manufacturer) fender for a particular 1983 model car was found to be

$$
f(t) = \frac{110}{1 + 2t} \quad (0 \leq t \leq 2)
$$

where $f(t)$ is measured in dollars and $t$ is in years. Over the same period of time, the price of a non-OEM fender for the car was found to be

$$
g(t) = 26\left(\frac{1}{4}t^2 - 1\right)^2 + 52 \quad (0 \leq t \leq 2)
$$

where $g(t)$ is also measured in dollars. Find a function $h(t)$ that gives the difference in price between an OEM fender and a non-OEM fender. Compute $h(0)$, $h(1)$, and $h(2)$. What does the result of your computation seem to say about the price gap between OEM and non-OEM fenders over the 2 yr?

For the demand equations in Exercises 22–25, where $x$ represents the quantity demanded in units of a thousand and $p$ is the unit price in dollars, (a) sketch the demand curve and (b) determine the quantity demanded when the unit price is set at $Sp$.

22. $p = -x^2 + 36; p = 11$

23. $p = -x^2 + 16; p = 7$

24. $p = \sqrt{9-x^2}; p = 2$

25. $p = \sqrt{18-x^2}; p = 3$

For the supply equations in Exercises 26–29, where $x$ is the quantity supplied in units of a thousand and $p$ is the unit price in dollars, (a) sketch the supply curve and (b) determine the price at which the supplier will make 2000 units of the commodity available in the market.

26. $p = 2x^2 + 18$

27. $p = x^2 + 16x + 40$

28. $p = x^3 + x + 10$

29. $p = x^3 + 2x + 3$

30. **Demand for Smoke Alarms**  The demand function for the Sentinel smoke alarm is given by

$$
p = \frac{30}{0.02x^2 + 1} \quad (0 \leq x \leq 10)
$$

where $x$ (measured in units of a thousand) is the quantity demanded per week and $p$ is the unit price in dollars. Sketch the graph of the demand function. What is the unit price that corresponds to a quantity demanded of 10,000 units?

31. **Demand for Commodities**  Assume that the demand function for a certain commodity has the form

$$
p = \sqrt{-ax^2 + b} \quad (a \geq 0, b \geq 0)
$$

where $x$ is the quantity demanded, measured in units of a thousand, and $p$ is the unit price in dollars. Suppose the quantity demanded is 6000 ($x = 6$) when the unit price is $8$ and 8000 ($x = 8$) when the unit price is $6$. Determine the demand equation. What is the quantity demanded when the unit price is set at $7.50$?

32. **Supply Functions**  The supply function for the Luminar desk lamp is given by

$$
p = 0.1x^2 + 0.5x + 15
$$

where $x$ is the quantity supplied (in thousands) and $p$ is the unit price in dollars. Sketch the graph of the supply function. What unit price will induce the supplier to make 5000 lamps available in the marketplace?

33. **Supply Functions**  Suppliers of transistor radios will market 10,000 units when the unit price is $20$ and 62,500 units when the unit price is $35$. Determine the supply
function if it is known to have the form
\[ p = a\sqrt{x} + b \quad (a \geq 0, b \geq 0) \]

where \( x \) is the quantity supplied and \( p \) is the unit price in dollars. Sketch the graph of the supply function. What unit price will induce the supplier to make 40,000 transistor radios available in the marketplace?

For each pair of supply and demand equations in Exercises 34–37, where \( x \) represents the quantity demanded in units of a thousand and \( p \) the unit price in dollars, find the equilibrium quantity and the equilibrium price.

34. \[ p = -2x^2 + 80 \quad \text{and} \quad p = 15x + 30 \]
35. \[ p = -x^2 - 2x + 100 \quad \text{and} \quad p = 8x + 25 \]
36. \[ 11p + 3x - 66 = 0 \quad \text{and} \quad 2p^2 + p - x = 10 \]
37. \[ p = 60 - 2x^2 \quad \text{and} \quad p = x^2 + 9x + 30 \]

38. Market Equilibrium The weekly demand and supply functions for Sportsman 5 \times 7 tents are given by
\[
\begin{align*}
p &= -0.1x^2 - x + 40 \\
p &= 0.1x^2 + 2x + 20
\end{align*}
\]
respectively, where \( p \) is measured in dollars and \( x \) is measured in units of a hundred. Find the equilibrium quantity and price.

39. Market Equilibrium The management of the Titan Tire Company has determined that the weekly demand and supply functions for their Super Titan tires are given by
\[
\begin{align*}
p &= 144 - x^2 \\
p &= 48 + \frac{1}{2}x^2
\end{align*}
\]
respectively, where \( p \) is measured in dollars and \( x \) is measured in units of a thousand. Find the equilibrium quantity and price.

40. Walking versus Running The oxygen consumption (in milliliter/pound/minute) for a person walking at \( x \) mph is approximated by the function
\[
f(x) = \frac{5}{3}x^3 + \frac{5}{3}x + 10 \quad (0 \leq x \leq 9)
\]
where the oxygen consumption for a runner at \( x \) mph is approximated by the function
\[
g(x) = 11x + 10 \quad (4 \leq x \leq 9)
\]
a. Sketch the graphs of \( f \) and \( g \).
b. At what speed is the oxygen consumption the same for a walker as it is for a runner? What is the level of oxygen consumption at that speed?

c. What happens to the oxygen consumption of the walker and the runner at speeds beyond that found in part (b)?

Source: Exercise Physiology, by William McArdley, Frank Katch, and Victor Katch

41. Enclosing an Area Patricia wishes to have a rectangular-shaped garden in her backyard. She has 80 ft of fencing material with which to enclose her garden. Letting \( x \) denote the width of the garden, find a function \( f \) in the variable \( x \) giving the area of the garden. What is its domain?

42. Enclosing an Area Patricia’s neighbor, Juanita, also wishes to have a rectangular-shaped garden in her backyard. But Juanita wants her garden to have an area of 250 ft². Letting \( x \) denote the width of the garden, find a function \( f \) in the variable \( x \) giving the length of the fencing material required to construct the garden. What is the domain of the function?
Hint: Refer to the figure for Exercise 41. The amount of fencing material required is equal to the perimeter of the rectangle, which is twice the width plus twice the length of the rectangle.

43. Packaging By cutting away identical squares from each corner of a rectangular piece of cardboard and folding up the resulting flaps, an open box may be made. If the cardboard is 15 in. long and 8 in. wide and the square cutaways have dimensions of \( x \) in. by \( x \) in., find a function giving the volume of the resulting box.
FINDING THE POINTS OF INTERSECTION OF TWO GRAPHS AND MODELING

A graphing utility can be used to find the point(s) of intersection of the graphs of two functions.

EXAMPLE 1
Find the points of intersection of the graphs of

\[ f(x) = 0.3x^2 - 1.4x - 3 \quad \text{and} \quad g(x) = -0.4x^2 + 0.8x + 6.4 \]

SOLUTION
The graphs of both \( f \) and \( g \) in the standard viewing rectangle are shown in Figure T1. Using \texttt{TRACE} and \texttt{ZOOM} or the function for finding the points of intersection of two graphs on your graphing utility, we find the point(s) of intersection, accurate to four decimal places, to be \((-2.4158, 2.1329)\) and \((5.5587, -1.5125)\).

EXAMPLE 2
Consider the demand and supply functions

\[ p = d(x) = -0.01x^2 - 0.2x + 8 \quad \text{and} \quad p = s(x) = 0.01x^2 + 0.1x + 3 \]

of Examples 4 and 5 in Section 10.3.

a. Plot the graphs of \( d \) and \( s \) in the viewing rectangle \([0, 15] \times [0, 10]\).

b. Verify that the equilibrium point is \((10, 5)\), as obtained in Example 4.

SOLUTION
a. The graphs of \( d \) and \( s \) are shown in Figure T2.

b. Using \texttt{TRACE} and \texttt{ZOOM} or the function for finding the point of intersection of two graphs, we see that \( x = 10 \) and \( y = 5 \), so the equilibrium point is \((10, 5)\), as obtained before.
CONSTRUCTING MATHEMATICAL MODELS FROM RAW DATA

A graphing utility can sometimes be used to construct mathematical models from sets of data. For example, if the points corresponding to the given data are scattered about a straight line, then one uses LINR (linear regression) from the STAT CALC (statistical calculation) menu of the graphing utility to obtain a function (model) that approximates the data at hand. If the points seem to be scattered along a parabola (the graph of a quadratic function), then one uses P2REG (second-order polynomial regression), and so on.

Details for using the items in the STAT CALC menu can be found at the Web site:
http://www.brookscole.com/product/0534378420

EXAMPLE 3
Indian Gaming Industry

The following data gives the estimated gross revenues (in billions of dollars) from the Indian gaming industries from 1990 \((t = 0)\) to 1997 \((t = 7)\).

<table>
<thead>
<tr>
<th>Year</th>
<th>Revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.5</td>
</tr>
<tr>
<td>1</td>
<td>0.7</td>
</tr>
<tr>
<td>2</td>
<td>1.6</td>
</tr>
<tr>
<td>3</td>
<td>2.6</td>
</tr>
<tr>
<td>4</td>
<td>3.4</td>
</tr>
<tr>
<td>5</td>
<td>4.8</td>
</tr>
<tr>
<td>6</td>
<td>5.6</td>
</tr>
<tr>
<td>7</td>
<td>6.8</td>
</tr>
</tbody>
</table>

a. Use a graphing utility to find a polynomial function \(f\) of degree 4 that models the data.

b. Plot the graph of the function \(f\), using the viewing rectangle \([0, 8] \times [0, 10]\).

c. Use the function evaluation capability of the graphing utility to compute \(f(0), f(1), \ldots, f(7)\) and compare these values with the original data.

Source: Christiansen/Cummings Associates

SOLUTION ✔

a. Choosing P4REG (fourth-order polynomial regression) from the STAT CALC (statistical calculations) menu of a graphing utility, we find

\[ f(t) = 0.00379t^4 - 0.06616t^3 + 0.41667t^2 - 0.07291t + 0.48333 \]

b. The graph of \(f\) is shown in Figure T3.

c. The required values, which compare favorably with the given data, follow:

<table>
<thead>
<tr>
<th>(t)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(t))</td>
<td>0.5</td>
<td>0.8</td>
<td>1.5</td>
<td>2.5</td>
<td>3.6</td>
<td>4.6</td>
<td>5.7</td>
<td>6.8</td>
</tr>
</tbody>
</table>
In Exercises 9–14, use the STAT CALC menu of a graphing utility to construct a mathematical model associated with given data.

9. **SALES OF DIGITAL SIGNAL PROCESSORS**

The projected sales (in billions of dollars) of digital signal processors (DSPs) follow:

<table>
<thead>
<tr>
<th>Year</th>
<th>1997</th>
<th>1998</th>
<th>1999</th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sales</td>
<td>3.1</td>
<td>4</td>
<td>5</td>
<td>6.2</td>
<td>8</td>
<td>10</td>
</tr>
</tbody>
</table>

a. Use **P2REG** to find a second-degree polynomial regression model for the data. Let $t = 0$ correspond to 1997.
b. Plot the graph of the function $f$ found in part (a), using the viewing rectangle $[0, 5] 	imes [0, 12]$.
c. Compute the values of $f(t)$ for $t = 0, 1, 2, 3, 4,$ and $5$. How does your model compare with the given data?

Source: A. G. Edwards & Sons, Inc.

10. **PRISON POPULATION**

The following data gives the past, present, and projected U.S. prison population (in millions) from 1980 through 2005.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Population</td>
<td>0.52</td>
<td>0.77</td>
<td>1.18</td>
<td>1.64</td>
<td>2.23</td>
<td>3.20</td>
</tr>
</tbody>
</table>

a. Letting $t = 0$ correspond to the beginning of 1980 and supposing $t$ is measured in 5-yr intervals, use **P2REG** to find a second-degree polynomial regression model based on the given data.
b. Plot the graph of the function $f$ found in part (a), using the viewing rectangle $[0, 5] \times [0, 3.5]$.
c. Compute $f(0), f(1), f(2), f(3), f(4),$ and $f(5)$. Compare these values with the given data.

11. **DIGITAL TV SHIPMENTS**

The estimated number of digital TV shipments between the year 2000 and 2006 (in millions of units) is given in the following table:
a. Use **P3REG** to find a third-degree polynomial regression model for the data. Let \( t = 0 \) correspond to the year 2000.

b. Plot the graph of the function \( f \) found in part (a), using the viewing rectangle \([0, 6] \times [0, 11]\).

c. Compute the values of \( f(t) \) for \( t = 0, 1, 2, 3, 4, 5, \) and 6.

Source: Consumer Electronics Manufacturers Association

12. **On-line Shopping** The following data gives the revenue per year (in billions of dollars) from Internet shopping.

<table>
<thead>
<tr>
<th>Year</th>
<th>1997</th>
<th>1998</th>
<th>1999</th>
<th>2000</th>
<th>2001</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revenue</td>
<td>2.4</td>
<td>5</td>
<td>8</td>
<td>12</td>
<td>17.4</td>
</tr>
</tbody>
</table>

a. Use **P3REG** to find a third-degree polynomial regression model for the data. Let \( t = 0 \) correspond to 1997.

b. Plot the graph of the function \( f \) found in part (a), using the viewing rectangle \([0, 4] \times [0, 20]\).

c. Compare the values of \( f \) at \( t = 1, 2, 3, 4, \) and 5 with the given data.

Source: Forrester Research, Inc.

13. **Cable Ad Revenue** The past and projected revenues (in billions of dollars) from cable advertisement for the years 1995 through the year 2000 follow:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Ad Revenue</td>
<td>5.1</td>
<td>6.6</td>
<td>8.1</td>
<td>9.4</td>
<td>11.1</td>
<td>13.7</td>
</tr>
</tbody>
</table>

a. Use **P3REG** to find a third-degree polynomial regression model for the data. Let \( t = 0 \) correspond to 1995.

b. Plot the graph of the function \( f \) found in part (a), using the viewing rectangle \([0, 6] \times [0, 14]\).

c. Compare the values of \( f \) at \( t = 1, 2, 3, 4, 5 \) with the given data.

Source: National Cable Television Association


<table>
<thead>
<tr>
<th>Year</th>
<th>1997</th>
<th>1998</th>
<th>1999</th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spending</td>
<td>5.0</td>
<td>10.5</td>
<td>20.5</td>
<td>37.5</td>
<td>60</td>
<td>95</td>
</tr>
</tbody>
</table>

a. Choose **P4REG** to find a fourth-degree polynomial regression model for the data. Let \( t = 0 \) correspond to 1997.

b. Plot the graph of the function \( f \) found in part (a), using the viewing rectangle \([0, 5] \times [0, 100]\).

c. Compare the values of \( f \) at \( t = 1, 2, 3, 4, 5 \) with the given data.

Source: International Data Corporation

15. **Marijuana Arrests** The number of arrests (in thousands) for marijuana sales and possession in New York City from 1992 through 1997 is given below.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of Arrests</td>
<td>5.0</td>
<td>5.8</td>
<td>8.8</td>
<td>11.7</td>
<td>18.5</td>
<td>27.5</td>
</tr>
</tbody>
</table>

a. Use **P4REG** to find a fourth-degree polynomial regression model for the data. Let \( t = 0 \) correspond to 1992.

b. Plot the graph of the function \( f \) found in part (a), using the viewing rectangle \([0, 5] \times [0, 30]\).

c. Compare the values of \( f \) at \( t = 0, 1, 2, 3, 4, 5 \) with the given data.

Source: New York State Division of Criminal Justice Services
44. **Construction Costs** A rectangular box is to have a square base and a volume of 20 ft$^3$. The material for the base costs 30 cents/ft$^2$, the material for the sides costs 10 cents/ft$^2$, and the material for the top costs 20 cents/ft$^2$. Letting $x$ denote the length of one side of the base, find a function in the variable $x$ giving the cost of constructing the box.

![Diagram of a rectangular box]

45. **Area of a Norman Window** A Norman window has the shape of a rectangle surmounted by a semicircle (see the accompanying figure). Suppose a Norman window is to have a perimeter of 28 ft; find a function in the variable $x$ giving the area of the window.

![Diagram of a Norman window]

46. **Yield of an Apple Orchard** An apple orchard has an average yield of 36 bushels of apples/tree if tree density is 22 trees/acre. For each unit increase in tree density, the yield decreases by 2 bushels. Letting $x$ denote the number of trees beyond 22/acre, find a function in $x$ that gives the yield of apples.

In Exercises 47–50, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

47. A polynomial function is a sum of constant multiples of power functions.

48. A polynomial function is a rational function, but the converse is false.

49. If $r > 0$, then the power function $f(x) = x^r$ is defined for all nonnegative values of $x$.

50. The function $f(x) = 2^x$ is a power function.

**Solutions to Self-Check Exercises 10.3**

1. Since the adult dose of the substance is 500 mg, $a = 500$; thus, the rule in this case is

$$D(t) = \frac{500t}{t + 12}$$

A 4-yr-old should receive

$$D(4) = \frac{500(4)}{4 + 12}$$

or 125 mg of the substance.
2. a. The graphs of the functions \(d\) and \(s\) are shown in the following figure:

![Graph of functions d and s](image)

b. Solve the following system of equations:

\[
p = -\frac{2}{15}x + 4
\]

\[
p = \frac{1}{75}x^2 + \frac{1}{10}x + \frac{3}{2}
\]

Substituting the first equation into the second yields

\[
\frac{1}{75}x^2 + \frac{1}{10}x + \frac{3}{2} = -\frac{2}{15}x + 4
\]

\[
\frac{1}{75}x^2 + \left(\frac{1}{10} + \frac{2}{15}\right)x - \frac{5}{2} = 0
\]

\[
\frac{1}{75}x^2 + \frac{7}{30}x - \frac{5}{2} = 0
\]

Multiplying both sides of the last equation by 150, we have

\[
2x^2 + 35x - 375 = 0
\]

\[
(2x - 15)(x + 25) = 0
\]

Thus, \(x = -25\) or \(x = 15/2 = 7.5\). Since \(x\) must be nonnegative, we take \(x = 7.5\), and the equilibrium quantity is 7500 pounds. The equilibrium price is given by

\[
p = -\frac{2}{15} \left(\frac{15}{2}\right) + 4
\]

or $3/pound.
10.4 Limits

**Introduction to Calculus**

Historically, the development of calculus by Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) resulted from the investigation of the following problems:

1. Finding the tangent line to a curve at a given point on the curve (Figure 10.20a)
2. Finding the area of a planar region bounded by an arbitrary curve (Figure 10.20b)

![Figure 10.20](image)

(a) What is the slope of the tangent line $T$ at point $P$?  
(b) What is the area of the region $R$?

The tangent-line problem might appear to be unrelated to any practical applications of mathematics, but as you will see later, the problem of finding the rate of change of one quantity with respect to another is mathematically equivalent to the geometric problem of finding the slope of the tangent line to a curve at a given point on the curve. It is precisely the discovery of the relationship between these two problems that spurred the development of calculus in the seventeenth century and made it such an indispensable tool for solving practical problems. The following are a few examples of such problems:

- Finding the velocity of an object
- Finding the rate of change of a bacteria population with respect to time
- Finding the rate of change of a company’s profit with respect to time
- Finding the rate of change of a travel agency’s revenue with respect to the agency’s expenditure for advertising

The study of the tangent-line problem led to the creation of differential calculus, which relies on the concept of the derivative of a function. The study of the area problem led to the creation of integral calculus, which relies on the concept of the antiderivative, or integral, of a function. (The derivative of a function and the integral of a function are intimately related, as you will
see in Section 14.4.) Both the derivative of a function and the integral of a function are defined in terms of a more fundamental concept—the limit—our next topic.

**A Real-Life Example**

From data obtained in a test run conducted on a prototype of a maglev (magnetic levitation train), which moves along a straight monorail track, engineers have determined that the position of the maglev (in feet) from the origin at time \( t \) is given by

\[ s = f(t) = 4t^2 \quad (0 \leq t \leq 30) \]  

(1)

where \( f \) is called the position function of the maglev. The position of the maglev at time \( t = 0, 1, 2, 3, \ldots, 10 \), measured from its initial position, is

\[ f(0) = 0, \quad f(1) = 4, \quad f(2) = 16, \quad f(3) = 36, \ldots, \quad f(10) = 400 \]  

feet (Figure 10.21).

![Figure 10.21](image-url)  
A maglev moving along an elevated monorail track

Suppose we want to find the velocity of the maglev at \( t = 2 \). This is just the velocity of the maglev as shown on its speedometer at that precise instant of time. Offhand, calculating this quantity using only Equation (1) appears to be an impossible task; but consider what quantities we can compute using this relationship. Obviously, we can compute the position of the maglev at any time \( t \) as we did earlier for some selected values of \( t \). Using these values, we can then compute the average velocity of the maglev over an interval of time. For example, the average velocity of the train over the time interval \([2, 4]\) is given by

\[
\text{Distance covered} = \frac{f(4) - f(2)}{4 - 2} = \frac{4(4^2) - 4(2^2)}{2} = \frac{64 - 16}{2} = 24
\]

or 24 feet/second.

Although this is not quite the velocity of the maglev at \( t = 2 \), it does provide us with an approximation of its velocity at that time.
Can we do better? Intuitively, the smaller the time interval we pick (with \( t = 2 \) as the left end point), the better the average velocity over that time interval will approximate the actual velocity of the maglev at \( t = 2 \).*

Now, let’s describe this process in general terms. Let \( t > 2 \). Then, the average velocity of the maglev over the time interval \([2, t]\) is given by

\[
\frac{f(t) - f(2)}{t - 2} = \frac{4t^2 - 4(2)^2}{t - 2} = \frac{4(t^2 - 4)}{t - 2}
\]

By choosing the values of \( t \) closer and closer to 2, we obtain a sequence of numbers that gives the average velocities of the maglev over smaller and smaller time intervals. As we observed earlier, this sequence of numbers should approach the instantaneous velocity of the train at \( t = 2 \).

Let’s try some sample calculations. Using Equation (2) and taking the sequence \( t = 2.5, 2.1, 2.01, 2.001, \) and 2.0001, which approaches 2, we find

The average velocity over \([2, 2.5]\) is \( \frac{4(2.5^2 - 4)}{2.5 - 2} = 18 \), or 18 feet/second.

The average velocity over \([2, 2.1]\) is \( \frac{4(2.1^2 - 4)}{2.1 - 2} = 16.4 \), or 16.4 feet/second.

and so forth. These results are summarized in Table 10.1.

<table>
<thead>
<tr>
<th>( t )</th>
<th>2.5</th>
<th>2.1</th>
<th>2.01</th>
<th>2.001</th>
<th>2.0001</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Velocity over ([2, t])</td>
<td>18</td>
<td>16.4</td>
<td>16.04</td>
<td>16.004</td>
<td>16.0004</td>
</tr>
</tbody>
</table>

From Table 10.1, we see that the average velocity of the maglev seems to approach the number 16 as it is computed over smaller and smaller time intervals. These computations suggest that the instantaneous velocity of the train at \( t = 2 \) is 16 feet/second.

**Remark.** Notice that we cannot obtain the instantaneous velocity for the maglev at \( t = 2 \) by substituting \( t = 2 \) into Equation (2) because this value of \( t \) is not in the domain of the average velocity function.

**Intuitive Definition of a Limit**

Consider the function \( g \) defined by

\[
g(t) = \frac{4(t^2 - 4)}{t - 2}
\]

* Actually, any interval containing \( t = 2 \) will do.
which gives the average velocity of the maglev [see Equation (2)]. Suppose we are required to determine the value that \( g(t) \) approaches as \( t \) approaches the (fixed) number 2. If we take the sequence of values of \( t \) approaching 2 from the right-hand side, as we did earlier, we see that \( g(t) \) approaches the number 16. Similarly, if we take a sequence of values of \( t \) approaching 2 from the left, such as \( t = 1.5, 1.9, 1.99, 1.999, \) and 1.9999, we obtain the results shown in Table 10.2.

<table>
<thead>
<tr>
<th>( t )</th>
<th>1.5</th>
<th>1.9</th>
<th>1.99</th>
<th>1.999</th>
<th>1.9999</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(t) )</td>
<td>14</td>
<td>15.6</td>
<td>15.96</td>
<td>15.996</td>
<td>15.9996</td>
</tr>
</tbody>
</table>

\( g(t) \) approaches 16 from the left.

Observe that \( g(t) \) approaches the number 16 as \( t \) approaches 2—this time from the left-hand side. In other words, as \( t \) approaches 2 from either side of 2, \( g(t) \) approaches 16. In this situation, we say that the limit of \( g(t) \) as \( t \) approaches 2 is 16, written

\[
\lim_{t \to 2} g(t) = \lim_{t \to 2} \frac{4(t^2 - 4)}{t - 2} = 16
\]

The graph of the function \( g \), shown in Figure 10.22, confirms this observation.

![Figure 10.22](image-url)

Observe that the point \( t = 2 \) is not in the domain of the function \( g \) [for this reason, the point (2, 16) is missing from the graph of \( g \)]. This, however, is inconsequential because the value, if any, of \( g(t) \) at \( t = 2 \) plays no role in computing the limit.

This example leads to the following informal definition.
**Limit of a Function**

The function \( f \) has the **limit** \( L \) as \( x \) approaches \( a \), written

\[
\lim_{x \to a} f(x) = L
\]

if the value \( f(x) \) can be made as close to the number \( L \) as we please by taking \( x \) sufficiently close to (but not equal to) \( a \).

**Exploring with Technology**

1. Use a graphing utility to plot the graph of

\[
g(x) = \frac{4(x^2 - 4)}{x - 2}
\]

in the viewing rectangle \([0, 3] \times [0, 20]\).

2. Use `ZOOM` and `TRACE` to describe what happens to the values of \( f(x) \) as \( x \) approaches 2, first from the right and then from the left.

3. What happens to the \( y \)-value when \( x \) takes on the value 2? Explain.

4. Reconcile your results with those of the preceding example.

**Evaluating the Limit of a Function**

Let us now consider some examples involving the computation of limits.

**Example 1**

Let \( f(x) = x^3 \) and evaluate \( \lim_{x \to 2} f(x) \).

**Solution**

The graph of \( f \) is shown in Figure 10.23. You can see that \( f(x) \) can be made as close to the number 8 as we please by taking \( x \) sufficiently close to 2. Therefore,

\[
\lim_{x \to 2} x^3 = 8
\]
Referring to Figure 10.25a, we see that no matter how close \( x \) is to \( \frac{\pi}{10} \), \( f(x) \) takes on the values 1 or \( \frac{\pi}{10} \), depending on whether \( x \) is positive or negative. Thus, there is no single real number \( L \) that \( f(x) \) approaches as \( x \) approaches 0.

**EXAMPLE 2**

Let

\[
g(x) = \begin{cases} 
  x + 2 & \text{if } x \neq 1 \\
  1 & \text{if } x = 1
\end{cases}
\]

Evaluate \( \lim_{{x \to 1}} g(x) \).

**SOLUTION**

The domain of \( g \) is the set of all real numbers. From the graph of \( g \) shown in Figure 10.24, we see that \( g(x) \) can be made as close to 3 as we please by taking \( x \) sufficiently close to 1. Therefore,

\[
\lim_{{x \to 1}} g(x) = 3
\]

Observe that \( g(1) = 1 \), which is not equal to the limit of the function \( g \) as \( x \) approaches 1. [Once again, the value of \( g(x) \) at \( x = 1 \) has no bearing on the existence or value of the limit of \( g \) as \( x \) approaches 1.]

**EXAMPLE 3**

Evaluate the limit of the following functions as \( x \) approaches the indicated point.

a. \( f(x) = \begin{cases} 
  -1 & \text{if } x < 0 \\
   1 & \text{if } x \geq 0
\end{cases} \), \( x = 0 \)

b. \( g(x) = \frac{1}{x^2} \), \( x = 0 \)

**SOLUTION**

The graphs of the functions \( f \) and \( g \) are shown in Figure 10.25.

(a) \( \lim_{{x \to 0}} f(x) \) does not exist.

(b) \( \lim_{{x \to 0}} g(x) \) does not exist.

a. Referring to Figure 10.25a, we see that no matter how close \( x \) is to \( x = 0 \), \( f(x) \) takes on the values 1 or -1, depending on whether \( x \) is positive or negative. Thus, there is no single real number \( L \) that \( f(x) \) approaches as \( x \) approaches 0.
\[ x \text{ approaches zero. We conclude that the limit of } f(x) \text{ does not exist as } x \text{ approaches zero.} \]

**b.** Referring to Figure 10.25b, we see that as \( x \) approaches \( x = 0 \) (from either side), \( g(x) \) increases without bound and thus does not approach any specific real number. We conclude, accordingly, that the limit of \( g(x) \) does not exist as \( x \) approaches zero.

---

**Group Discussion**

Consider the graph of the function \( h \) whose graph is depicted in the following figure:

![Graph of Function h](image_url)

It has the property that as \( x \) approaches 0 from either the right or the left, the curve oscillates more and more frequently between the lines \( y = -1 \) and \( y = 1 \).

1. Explain why \( \lim_{x \to 0} h(x) \) does not exist.
2. Compare this function with those in Example 3. More specifically, discuss the different ways the functions fail to have a limit at \( x = 0 \).

---

Until now, we have relied on knowing the actual values of a function or the graph of a function near \( x = a \) to help us evaluate the limit of the function \( f(x) \) as \( x \) approaches \( a \). The following properties of limits, which we list without proof, enable us to evaluate limits of functions algebraically.

---

### THEOREM 1

**Properties of Limits**

Suppose

\[ \lim_{x \to a} f(x) = L \quad \text{and} \quad \lim_{x \to a} g(x) = M \]

Then,

1. \( \lim_{x \to a} [f(x)]^r = [\lim_{x \to a} f(x)]^r = L^r \) (\( r \), a real number)
2. \( \lim_{x \to a} cf(x) = c \lim_{x \to a} f(x) = cL \) (\( c \), a real number)
3. \( \lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M \)
4. \( \lim_{x \to a} [f(x)g(x)] = [\lim_{x \to a} f(x)][\lim_{x \to a} g(x)] = LM \)
5. \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M} \) (Provided \( M \neq 0 \))
Use Theorem 1 to evaluate the following limits.

a. \( \lim_{x \to 2} x^3 \)

\[ = 2^3 = 8 \]

(b) \( \lim_{x \to 2} 5x^{3/2} \)

\[ = 5(4)^{3/2} = 40 \]

(c) \( \lim_{x \to 1} (5x^4 - 2) = \lim_{x \to 1} 5x^4 - \lim_{x \to 1} 2 \)

To evaluate \( \lim_{x \to 1} 2 \), observe that the constant function \( g(x) = 2 \) has value 2 for all values of \( x \). Therefore, \( g(x) \) must approach the limit 2 as \( x \) approaches 1 (or any other point for that matter!). Therefore,

\[ \lim_{x \to 1} (5x^4 - 2) = 5(1)^4 - 2 = 3 \]

(d) \( \lim_{x \to 3} 2x^3 \sqrt{x^2 + 7} \)

\[ = 2 \lim_{x \to 3} x^3 \lim_{x \to 3} \sqrt{x^2 + 7} \]

\[ = 2(3)^3 \sqrt{3^2 + 7} \]

\[ = 2(27) \sqrt{16} = 216 \]

(e) \( \lim_{x \to 2} \frac{2x^2 + 1}{x + 1} \)

\[ = \frac{\lim_{x \to 2}(2x^2 + 1)}{\lim_{x \to 2}(x + 1)} \]

\[ = \frac{2(2)^2 + 1}{2 + 1} = \frac{9}{3} = 3 \]

### Indeterminate Forms

Let’s emphasize once again that Property 5 of limits is valid only when the limit of the function that appears in the denominator is not equal to zero at the point in question.

If the numerator has a limit different from zero and the denominator has a limit equal to zero, then the limit of the quotient does not exist at the point in question. This is the case with the function \( g(x) = \frac{1}{x^2} \) in Example 3b. Here, as \( x \) approaches zero, the numerator approaches 1 but the denominator approaches zero, so the quotient becomes arbitrarily large. Thus, as observed earlier, the limit does not exist.
Next, consider
\[
\lim_{x \to 2} \frac{4(x^2 - 4)}{x - 2}
\]
which we evaluated earlier by looking at the values of the function for \(x\) near \(x = 2\). If we attempt to evaluate this expression by applying Property 5 of limits, we see that both the numerator and denominator of the function
\[
\frac{4(x^2 - 4)}{x - 2}
\]
approach zero as \(x\) approaches 2; that is, we obtain an expression of the form 0/0. In this event, we say that the limit of the quotient \(f(x)/g(x)\) as \(x\) approaches 2 has the \textit{indeterminate form 0/0}.

We will need to evaluate limits of this type when we discuss the derivative of a function, a fundamental concept in the study of calculus. As the name suggests, the meaningless expression 0/0 does not provide us with a solution to our problem. One strategy that can be used to solve this type of problem follows.

**Strategy for Evaluating Indeterminate Forms**

1. Replace the given function with an appropriate one that takes on the same values as the original function everywhere except at \(x = a\).
2. Evaluate the limit of this function as \(x\) approaches \(a\).

Examples 5 and 6 illustrate this strategy.

**Example 5**

Evaluate:
\[
\lim_{x \to 2} \frac{4(x^2 - 4)}{x - 2}
\]

**Solution**

Since both the numerator and the denominator of this expression approach zero as \(x\) approaches 2, we have the indeterminate form 0/0. We rewrite
\[
\frac{4(x^2 - 4)}{x - 2} = \frac{4(x - 2)(x + 2)}{(x - 2)}
\]
which, upon canceling the common factors, is equivalent to \(4(x + 2)\). Next, we replace \(4(x^2 - 4)/(x - 2)\) with \(4(x + 2)\) and take the limit as \(x\) approaches 2, obtaining
\[
\lim_{x \to 2} \frac{4(x^2 - 4)}{x - 2} = \lim_{x \to 2} 4(x + 2) = 16
\]
The graphs of the functions
\[
f(x) = \frac{4(x^2 - 4)}{x - 2} \quad \text{and} \quad g(x) = 4(x + 2)
\]
are shown in Figure 10.26a–b. Observe that the graphs are identical except when \( x = 2 \). The function \( g \) is defined for all values of \( x \) and, in particular, its value at \( x = 2 \) is \( g(2) = 4(2 + 2) = 16 \). Thus, the point \((2, 16)\) is on the graph of \( g \). However, the function \( f \) is not defined at \( x = 2 \). Since \( f(x) = g(x) \) for all values of \( x \) except \( x = 2 \), it follows that the graph of \( f \) must look exactly like the graph of \( g \), with the exception that the point \((2, 16)\) is missing from the graph of \( f \). This illustrates graphically why we can evaluate the limit of \( f \) by evaluating the limit of the “equivalent” function \( g \).

**Remark** Notice that the limit in Example 5 is the same limit that we evaluated earlier when we discussed the instantaneous velocity of a maglev at a specified time.

**Exploring with Technology**

1. Use a graphing utility to plot the graph of

   \[ f(x) = \frac{4(x^2 - 4)}{x - 2} \]

   in the viewing rectangle \([0, 3] \times [0, 20]\). Then use **zoom** and **trace** to find

   \[ \lim_{x \to 2} \frac{4(x^2 - 4)}{x - 2} \]

2. Use a graphing utility to plot the graph of \( g(x) = 4(x + 2) \) in the viewing rectangle \([0, 3] \times [0, 20]\). Then use **zoom** and **trace** to find \( \lim_{x \to 2} 4(x + 2) \). What happens to the \( y \)-value when \( x \) takes on the value 2? Explain.

3. Can you distinguish between the graphs of \( f \) and \( g \)?

4. Reconcile your results with those of Example 5.
1. Use a graphing utility to plot the graph of
   \[ g(x) = \frac{\sqrt{1 + x} - 1}{x} \]
   in the viewing rectangle \([-1, 2] \times [0, 1]\). Then use ZOOM and TRACE to find
   \[ \lim_{{x \to 0}} \frac{\sqrt{1 + x} - 1}{x} \]
   by observing the values of \(g(x)\) as \(x\) approaches 0 from the left and from the right.

2. Use a graphing utility to plot the graph of
   \[ f(x) = \frac{1}{\sqrt{1 + x} + 1} \]
   in the viewing rectangle \([-1, 2] \times [0, 1]\). Then use ZOOM and TRACE to find
   \[ \lim_{{x \to 0}} \frac{1}{\sqrt{1 + x} + 1} \]
   What happens to the \(y\)-value when \(x\) takes on the value 0? Explain.

3. Can you distinguish between the graphs of \(f\) and \(g\)?

4. Reconcile your results with those of Example 6.
LIMITS AT INFINITY

Up to now we have studied the limit of a function as \( x \) approaches a (finite) number \( a \). There are occasions, however, when we want to know whether \( f(x) \) approaches a unique number as \( x \) increases without bound. Consider, for example, the function \( P \), giving the number of fruit flies (\emph{Drosophila}) in a container under controlled laboratory conditions, as a function of a time \( t \). The graph of \( P \) is shown in Figure 10.27. You can see from the graph of \( P \) that, as \( t \) increases without bound (gets larger and larger), \( P(t) \) approaches the number 400. This number, called the \emph{carrying capacity} of the environment, is determined by the amount of living space and food available, as well as other environmental factors.

As another example, suppose we are given the function

\[
f(x) = \frac{2x^2}{1 + x^2}
\]

and we want to determine what happens to \( f(x) \) as \( x \) gets larger and larger. Picking the sequence of numbers 1, 2, 5, 10, 100, and 1000 and computing the corresponding values of \( f(x) \), we obtain the following table of values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>1</td>
<td>1.6</td>
<td>1.92</td>
<td>1.98</td>
<td>1.998</td>
<td>1.99998</td>
</tr>
</tbody>
</table>

From the table, we see that as \( x \) gets larger and larger, \( f(x) \) gets closer and closer to 2. The graph of the function \( f \) shown in Figure 10.28 confirms this observation. We call the line \( y = 2 \) a \emph{horizontal asymptote}.*

* We will discuss asymptotes in greater detail in Section 12.3.
In this situation we say that the limit of the function

\[ f(x) = \frac{2x^2}{1 + x^2} \]

as \( x \) increases without bound is 2, written

\[ \lim_{x \to \infty} \frac{2x^2}{1 + x^2} = 2 \]

In the general case, the following definition is applicable:

**Limit of a Function at Infinity**

The function \( f \) has the limit \( L \) as \( x \) increases without bound (or, as \( x \) approaches infinity), written

\[ \lim_{x \to \infty} f(x) = L \]

if \( f(x) \) can be made arbitrarily close to \( L \) by taking \( x \) large enough.

Similarly, the function \( f \) has the limit \( M \) as \( x \) decreases without bound (or as \( x \) approaches negative infinity), written

\[ \lim_{x \to -\infty} f(x) = M \]

if \( f(x) \) can be made arbitrarily close to \( M \) by taking \( x \) to be negative and sufficiently large in absolute value.

**EXAMPLE 7**

Let \( f \) and \( g \) be the functions

\[ f(x) = \begin{cases} 
-1 & \text{if } x < 0 \\
1 & \text{if } x \geq 0 \end{cases} \quad \text{and} \quad g(x) = \frac{1}{x^2} \]

Evaluate:

a. \( \lim_{x \to -\infty} f(x) \) and \( \lim_{x \to \infty} f(x) \)

b. \( \lim_{x \to -\infty} g(x) \) and \( \lim_{x \to \infty} g(x) \)

**SOLUTION**

The graphs of \( f(x) \) and \( g(x) \) are shown in Figure 10.29. Referring to the graphs of the respective functions, we see that

a. \( \lim_{x \to -\infty} f(x) = 1 \) and \( \lim_{x \to \infty} f(x) = -1 \)

b. \( \lim_{x \to -\infty} \frac{1}{x^2} = 0 \) and \( \lim_{x \to \infty} \frac{1}{x^2} = 0 \)
All the properties of limits listed in Theorem 1 are valid when \( a \) is replaced by \( \infty \) or \( -\infty \). In addition, we have the following property for the limit at infinity.

**Theorem 2**

For all \( n > 0 \),

\[
\lim_{x \to \infty} \frac{1}{x^n} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x^n} = 0
\]

provided that \( \frac{1}{x^n} \) is defined.

**Exploring with Technology**

1. Use a graphing utility to plot the graphs of

\[
y_1 = \frac{1}{x^{0.5}}, \quad y_2 = \frac{1}{x}, \quad \text{and} \quad y_3 = \frac{1}{x^{1.5}}
\]

in the viewing rectangle \([0, 200] \times [0, 0.5]\). What can you say about \( \lim_{x \to \infty} \frac{1}{x^{0.5}} \) if \( n = 0.5, \ n = 1, \) and \( n = 1.5 \)? Are these results predicted by Theorem 2?

2. Use a graphing utility to plot the graphs of

\[
y_1 = \frac{1}{x} \quad \text{and} \quad y_2 = \frac{1}{x^{5/3}}
\]

in the viewing rectangle \([-50, 0] \times [-0.5, 0]\). What can you say about \( \lim_{x \to -\infty} \frac{1}{x^{5/3}} \) if \( n = 1 \) and \( n = \frac{5}{3} \)? Are these results predicted by Theorem 2?

*Hint:* To graph \( y_1 \), write it in the form \( y_2 = 1/(x^{(1/3)})^{5} \).
In evaluating the limit at infinity of a rational function, the following technique is often used: Divide the numerator and denominator of the expression by \( x^n \), where \( n \) is the highest power present in the denominator of the expression.

**EXAMPLE 8**

Evaluate:

\[
\lim_{{x \to \infty}} \frac{x^2 - x + 3}{2x^3 + 1}
\]

**SOLUTION**

Since the limits of both the numerator and the denominator do not exist as \( x \) approaches infinity, the property pertaining to the limit of a quotient (Property 5) is not applicable. Let us divide the numerator and denominator of the rational expression by \( x^3 \), obtaining

\[
\lim_{{x \to \infty}} \frac{x^2 - x + 3}{2x^3 + 1} = \lim_{{x \to \infty}} \frac{\frac{1}{x} - \frac{1}{x^2} + \frac{3}{x^3}}{\frac{2}{x^3} + \frac{1}{x^3}}
\]

\[
= \frac{0 - 0 + 0}{2 + 0} = 0
\]

(Using Theorem 2)

**EXAMPLE 9**

Let

\[ f(x) = \frac{3x^2 + 8x - 4}{2x^2 + 4x - 5} \]

Compute \( \lim_{{x \to \infty}} f(x) \) if it exists.

**SOLUTION**

Again, we see that Property 5 is not applicable. Dividing the numerator and the denominator by \( x^2 \), we obtain

\[
\lim_{{x \to \infty}} \frac{3x^2 + 8x - 4}{2x^2 + 4x - 5} = \lim_{{x \to \infty}} \frac{\frac{3}{x} + \frac{8}{x^2} - \frac{4}{x^2}}{2 + \frac{4}{x} - \frac{5}{x^2}}
\]

\[
= \frac{3 + 8 \lim_{{x \to \infty}} \frac{1}{x} - \lim_{{x \to \infty}} \frac{4}{x^2}}{2 + 4 \lim_{{x \to \infty}} \frac{1}{x} - 5 \lim_{{x \to \infty}} \frac{1}{x^2}}
\]

\[
= \frac{3 + 0 - 0}{2 + 0 - 0}
\]

\[
= \frac{3}{2}
\]

(Using Theorem 2)
**Example 10**

Let \( f(x) = \frac{2x^3 - 3x^2 + 1}{x^2 + 2x + 4} \) and evaluate:

a. \( \lim_{x \to \infty} f(x) \)  

b. \( \lim_{x \to -\infty} f(x) \)

**Solution**

a. Dividing the numerator and the denominator of the rational expression by \( x^2 \), we obtain

\[
\lim_{x \to \infty} \frac{2x^3 - 3x^2 + 1}{x^2 + 2x + 4} = \lim_{x \to \infty} \frac{2x - 3 + \frac{1}{x^2}}{1 + \frac{2}{x} + \frac{4}{x^2}}
\]

Since the numerator becomes arbitrarily large whereas the denominator approaches 1 as \( x \) approaches infinity, we see that the quotient \( f(x) \) gets larger and larger as \( x \) approaches infinity. In other words, the limit does not exist. We indicate this by writing

\[
\lim_{x \to \infty} \frac{2x^3 - 3x^2 + 1}{x^2 + 2x + 4} = \infty
\]

b. Once again, dividing both the numerator and the denominator by \( x^2 \), we obtain

\[
\lim_{x \to \infty} \frac{2x^3 - 3x^2 + 1}{x^2 + 2x + 4} = \lim_{x \to \infty} \frac{2x - 3 + \frac{1}{x^2}}{1 + \frac{2}{x} + \frac{4}{x^2}}
\]

In this case the numerator becomes arbitrarily large in magnitude but negative in sign, whereas the denominator approaches 1 as \( x \) approaches negative infinity. Therefore, the quotient \( f(x) \) decreases without bound, and the limit does not exist. We indicate this by writing

\[
\lim_{x \to \infty} \frac{2x^3 - 3x^2 + 1}{x^2 + 2x + 4} = -\infty
\]

Example 11 gives an application of the concept of the limit of a function at infinity.

**Example 11**

The Custom Office Company makes a line of executive desks. It is estimated that the total cost of making \( x \) Senior Executive Model desks is \( C(x) = 100x + 200,000 \) dollars per year, so that the average cost of making \( x \) desks is given by

\[
\overline{C}(x) = \frac{C(x)}{x} = \frac{100x + 200,000}{x} = 100 + \frac{200,000}{x}
\]

dollars per desk. Evaluate \( \lim_{x \to \infty} \overline{C}(x) \) and interpret your results.
As the level of production increases, the average cost approaches $100 per desk.

\[
\lim_{x \to \infty} \bar{C}(x) = \lim_{x \to \infty} \left( 100 + \frac{200,000}{x} \right)
\]

\[
= \lim_{x \to \infty} 100 + \lim_{x \to \infty} \frac{200,000}{x} = 100
\]

A sketch of the graph of the function \( \bar{C}(x) \) appears in Figure 10.30. The result we obtained is fully expected if we consider its economic implications. Note that as the level of production increases, the fixed cost per desk produced, represented by the term \( (200,000/x) \), drops steadily. The average cost should approach a constant unit cost of production—$100 in this case.

**Group Discussion**

Consider the graph of the function \( f \) depicted in the following figure:

It has the property that the curve oscillates between \( y = -1 \) and \( y = 1 \) indefinitely in either direction.

1. Explain why \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \) do not exist.
2. Compare this function with those of Example 10. More specifically, discuss the different ways each function fails to have a limit at infinity or minus infinity.
SELF-CHECK EXERCISES 10.4

1. Find the indicated limit if it exists.
   
a. \( \lim_{x \to -3} \frac{\sqrt{x^2 + 7} + \sqrt{3x - 5}}{x + 2} \)
   
b. \( \lim_{x \to -1} \frac{x^2 - x - 2}{2x^2 - x - 3} \)

2. The average cost per disc (in dollars) incurred by the Herald Record Company in pressing \( x \) compact audio discs is given by the average cost function
   \[ C(x) = 1.8 + \frac{3000}{x} \]

   Evaluate \( \lim_{x \to \infty} C(x) \) and interpret your result.

Solutions to Self-Check Exercises 10.4 can be found on page 645.

10.4 Exercises

In Exercises 1–8, use the graph of the given function \( f \) to determine \( \lim_{x \to a} f(x) \) at the indicated value of \( a \), if it exists.

1. \( y = f(x) \)

2. \( y = f(x) \)

3. \( y = f(x) \) \( a = 3 \)

4. \( y = f(x) \)

5. \( y = f(x) \) \( a = -2 \)

6. \( y = f(x) \)

7. \( y = f(x) \)

8. \( y = f(x) \) \( a = 0 \)

In Exercises 9–16, complete the table by computing \( f(x) \) at the given values of \( x \). Use these results to estimate the indicated limit (if it exists).

9. \( f(x) = x^2 + 1; \lim_{x \to 2} f(x) \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.9</th>
<th>1.99</th>
<th>1.999</th>
<th>2.001</th>
<th>2.01</th>
<th>2.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

10. \( f(x) = 2x^2 - 1; \lim_{x \to -1} f(x) \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.9</th>
<th>0.99</th>
<th>0.999</th>
<th>1.001</th>
<th>1.01</th>
<th>1.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
11. \( f(x) = \frac{|x|}{x}; \lim_{x \to 0} f(x) \)

\[
\begin{array}{cccccc}
\text{x} & -0.1 & -0.01 & -0.001 & 0.001 & 0.01 & 0.1 \\
\hline
\text{f(x)}
\end{array}
\]

12. \( f(x) = \frac{|x - 1|}{x - 1}; \lim_{x \to 1} f(x) \)

\[
\begin{array}{cccccc}
\text{x} & 0.9 & 0.99 & 0.999 & 1.001 & 1.01 & 1.1 \\
\hline
\text{f(x)}
\end{array}
\]

13. \( f(x) = \frac{1}{(x - 1)^2}; \lim_{x \to 1} f(x) \)

\[
\begin{array}{cccccc}
\text{x} & 0.9 & 0.99 & 0.999 & 1.001 & 1.01 & 1.1 \\
\hline
\text{f(x)}
\end{array}
\]

14. \( f(x) = \frac{1}{x - 2}; \lim_{x \to 2} f(x) \)

\[
\begin{array}{cccccc}
\text{x} & 1.9 & 1.99 & 1.999 & 2.001 & 2.01 & 2.1 \\
\hline
\text{f(x)}
\end{array}
\]

15. \( f(x) = \frac{x^2 + x - 2}{x - 1}; \lim_{x \to 1} f(x) \)

\[
\begin{array}{cccccc}
\text{x} & 0.9 & 0.99 & 0.999 & 1.001 & 1.01 & 1.1 \\
\hline
\text{f(x)}
\end{array}
\]

16. \( f(x) = \frac{x - 1}{x - 1}; \lim_{x \to 1} f(x) \)

\[
\begin{array}{cccccc}
\text{x} & 0.9 & 0.99 & 0.999 & 1.001 & 1.01 & 1.1 \\
\hline
\text{f(x)}
\end{array}
\]

**In Exercises 17–22, sketch the graph of the function \( f \) and evaluate \( \lim_{x \to a} f(x) \), if it exists, for the given values of \( a \).**

17. \( f(x) = \begin{cases} x - 1 & \text{if } x \leq 0 \\ -1 & \text{if } x > 0 \end{cases} (a = 0) \)

18. \( f(x) = \begin{cases} x - 1 & \text{if } x \leq 3 \\ -2x + 8 & \text{if } x > 3 \end{cases} (a = 3) \)

19. \( f(x) = \begin{cases} x & \text{if } x < 1 \\ 0 & \text{if } x = 1 \\ -x + 2 & \text{if } x > 1 \end{cases} (a = 1) \)

20. \( f(x) = \begin{cases} -2x + 4 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ x^2 + 1 & \text{if } x > 1 \end{cases} (a = 1) \)

21. \( f(x) = \begin{cases} |x| & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} (a = 0) \)

22. \( f(x) = \begin{cases} |x - 1| & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases} (a = 1) \)

**In Exercises 23–40, find the indicated limit.**

23. \( \lim_{x \to 3} \)

24. \( \lim_{x \to -2} \ -3 \)

25. \( \lim_{x \to x} \ x \)

26. \( \lim_{x \to -3} \ -3x \)

27. \( \lim_{x \to (1 - 2x^2)} \)

28. \( \lim_{x \to (4t^2 - 2t + 1)} \)

29. \( \lim_{x \to (2x^3 - 3x^2 + x + 2)} \)

30. \( \lim_{x \to (4x^5 - 20x^2 + 2x + 1)} \)

31. \( \lim_{x \to (2x^2 - 1)(2x + 4)} \)

32. \( \lim_{x \to (x^2 + 1)(x^2 - 4)} \)

33. \( \lim_{x \to \frac{2x + 1}{x + 2}} \)

34. \( \lim_{x \to \frac{x^3 + 1}{2x^3 + 2}} \)

35. \( \lim_{x \to \sqrt{x + 2}} \)

36. \( \lim_{x \to \sqrt[4]{5x + 2}} \)

37. \( \lim_{x \to \sqrt{2x^4 + x^2}} \)

38. \( \lim_{x \to \sqrt{\frac{2x^3 + 4}{x^2 + 1}}} \)

39. \( \lim_{x \to \sqrt{\frac{x^3 + 8}{2x + 4}}} \)

40. \( \lim_{x \to \sqrt{\frac{x^3 + 7}{2x - \sqrt{2x + 3}}} \)

**In Exercises 41–48, find the indicated limit given that \( \lim_{x \to a} f(x) = 3 \) and \( \lim_{x \to a} g(x) = 4. \)**

41. \( \lim_{x \to a} \left[ f(x) - g(x) \right] \)

42. \( \lim_{x \to a} 2f(x) \)

43. \( \lim_{x \to a} \left[ 2f(x) - 3g(x) \right] \)

44. \( \lim_{x \to a} \left[ f(x)g(x) \right] \)
45. \( \lim_{x \to 1} \sqrt{g(x)} \)  
46. \( \lim_{x \to 1} \sqrt[3]{f(x)} + 3g(x) \)  
47. \( \lim_{x \to 1} \frac{2f(x) - g(x)}{f(x)g(x)} \)  
48. \( \lim_{x \to 1} \frac{g(x) - f(x)}{f(x) + \sqrt{g(x)}} \)  

In Exercises 49–62, find the indicated limit, if it exists.

49. \( \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \)  
50. \( \lim_{x \to 1} \frac{x^2 - 4}{x + 2} \)  
51. \( \lim_{x \to 0} \frac{x^2 - x}{x} \)  
52. \( \lim_{x \to 0} \frac{2x^2 - 3x}{x} \)  
53. \( \lim_{x \to 1} \frac{x^2 - 25}{x + 5} \)  
54. \( \lim_{x \to -1} \frac{b + 1}{b + 3} \)  
55. \( \lim_{x \to 1} \frac{x}{x - 1} \)  
56. \( \lim_{x \to -2} \frac{x + 2}{x - 2} \)  
57. \( \lim_{x \to -2} \frac{x^2 - x - 6}{x^2 + x - 2} \)  
58. \( \lim_{x \to -2} \frac{x^2 - 8}{z - 2} \)  
59. \( \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} \)  
Hint: Multiply by \( \frac{\sqrt{x} + 1}{\sqrt{x} + 1} \).  
60. \( \lim_{x \to 1} \frac{x - 4}{\sqrt{x} - 2} \)  
Hint: See Exercise 59.  
61. \( \lim_{x \to 1} \frac{x - 1}{x^2 + x^2 - 2x} \)  
62. \( \lim_{x \to 1} \frac{4 - x^2}{2x^2 + x^3} \)  

In Exercises 63–68, use the graph of the function \( f \) to determine \( \lim_{x \to a} f(x) \) and \( \lim_{x \to \pm \infty} f(x) \), if they exist.

63. \( f(x) = 2x^2 - 10 \)  
64. \( f(x) = x^3 - x \)  

In Exercises 69–72, complete the table by computing \( f(x) \) at the given values of \( x \). Use the results to guess at the indicated limits, if they exist.

69. \( f(x) = \frac{1}{x^2 + 1} \); \( \lim_{x \to a} f(x) \) and \( \lim_{x \to \pm \infty} f(x) \)  

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>( f(x) )</th>
</tr>
</thead>
</table>

70. \( f(x) = \frac{2x}{x + 1} \); \( \lim_{x \to a} f(x) \) and \( \lim_{x \to \pm \infty} f(x) \)  

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>( f(x) )</th>
</tr>
</thead>
</table>

(continued on p. 643)
**Using Technology**

**Finding the Limit of a Function**

A graphing utility can be used to help us find the limit of a function, if it exists, as illustrated in the following examples.

**Example 1**

Let \( f(x) = \frac{x^3 - 1}{x - 1} \).

a. Plot the graph of \( f \) in the viewing rectangle \([-2, 2] \times [0, 4]\).

b. Use zoom to find \( \lim_{x \to 1} \frac{x^3 - 1}{x - 1} \).

c. Verify your result by evaluating the limit algebraically.

**Solution**

a. The graph of \( f \) in the viewing rectangle \([-2, 2] \times [0, 4]\) is shown in Figure T1.

b. Using zoom-in repeatedly, we see that the y-value approaches 3 as the x-value approaches 1. We conclude, accordingly, that

\[
\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3
\]


c. We compute

\[
\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{x - 1} = \lim_{x \to 1} (x^2 + x + 1) = 3
\]

**Remark**

If you attempt to find the limit in Example 1 by using the evaluation function of your graphing utility to find the value of \( f(x) \) when \( x = 1 \), you will see that the graphing utility does not display the y-value. This happens because the point \( x = 1 \) is not in the domain of \( f \).
EXAMPLE 2

Use zoom to find \( \lim_{x \to 0} (1 + x)^{1/x} \).

SOLUTION

We first plot the graph of \( f(x) = (1 + x)^{1/x} \) in a suitable viewing rectangle. Figure T2 shows a plot of \( f \) in the rectangle \([-1, 1] \times [0, 4] \). Using zoom-in repeatedly, we see that \( \lim_{x \to 0} (1 + x)^{1/x} \approx 2.71828 \).

FIGURE T2
The graph of \( f(x) = (1 + x)^{1/x} \) in the viewing rectangle \([-1, 1] \times [0, 4] \)

The limit of \( f(x) = (1 + x)^{1/x} \) as \( x \) approaches zero, denoted by the letter \( e \), plays a very important role in the study of mathematics and its applications (see Section 13.5). Thus,

\[
\lim_{x \to 0} (1 + x)^{1/x} = e
\]

where, as we have just seen, \( e \approx 2.71828 \).

EXAMPLE 3

When organic waste is dumped into a pond, the oxidation process that takes place reduces the pond’s oxygen content. However, given time, nature will restore the oxygen content to its natural level. Suppose that the oxygen content \( t \) days after the organic waste has been dumped into the pond is given by

\[
f(t) = 100 \left( \frac{t^2 + 10t + 100}{t^2 + 20t + 100} \right)
\]

percent of its normal level.

a. Plot the graph of \( f \) in the viewing rectangle \([0, 200] \times [70, 100] \).

b. What can you say about \( f(t) \) when \( t \) is very large?

c. Verify your observation in part (b) by evaluating \( \lim_{t \to \infty} f(t) \).

SOLUTION

a. The graph of \( f \) is shown in Figure T3.

b. From the graph of \( f \) it appears that \( f(t) \) approaches 100 steadily as \( t \) gets
larger and larger. This observation tells us that eventually the oxygen content of the pond will be restored to its natural level.

c. To verify the observation made in part (b), we compute

\[
\lim_{t \to \infty} f(t) = \lim_{t \to \infty} \frac{t^2 + 10t + 100}{t^2 + 20t + 100} = 100 \\
\]

\[
\lim_{t \to \infty} \left( \frac{t + 10}{t} + \frac{100}{t^2} \right) = 100
\]

---

**Exercises**

In Exercises 1–10, use a graphing utility to find the indicated limit by first plotting the graph of the function in a suitable viewing rectangle and then using the zoom-in feature of the calculator.

1. \[\lim_{x \to 1} \frac{2x^3 - 2x^2 + 3x - 3}{x - 1}\]
2. \[\lim_{x \to -2} \frac{2x^3 + 3x^2 - x + 2}{x + 2}\]
3. \[\lim_{x \to -1} \frac{x^3 + 1}{x + 1}\]
4. \[\lim_{x \to -1} \frac{x^4 - 1}{x - 1}\]
5. \[\lim_{x \to 1} \frac{x^4 - x^2 - x + 1}{x^3 - 3x + 2}\]
6. \[\lim_{x \to 2} \frac{x^3 + 2x^2 - 16}{2x^3 - x^2 + 2x - 16}\]
7. \[\lim_{x \to 0} \frac{\sqrt[3]{x} + 1}{x}\]
8. \[\lim_{x \to 0} \frac{(x + 4)^{3/2} - 8}{x}\]
9. \[\lim_{x \to 0} (1 + 2x)^{1/2}\]
10. \[\lim_{x \to 0} \frac{2x - 1}{x}\]

11. Use a graphing utility to show that \(\lim_{x \to 3} \frac{2}{x - 3}\) does not exist.

12. Use a graphing utility to show that \(\lim_{x \to -2} \frac{x^3 - 2x + 1}{x - 2}\) does not exist.

13. CITY PLANNING A major developer is building a 5000-acre complex of homes, offices, stores, schools, and churches in the rural community of Marlboro. As a result of this development, the planners have estimated that Marlboro’s population (in thousands) \(t\) yr from now will be given by

\[P(t) = \frac{25t^2 + 125t + 200}{t^2 + 5t + 40}\]

a. Plot the graph of \(P\) in the viewing rectangle \([0, 50] \times [0, 30]\).

b. What will be the population of Marlboro in the long run?

**Hint:** Find \(\lim_{t \to \infty} P(t)\).
71. \( f(x) = 3x^3 - x^2 + 10; \) \( \lim_{x \to a} f(x) \) and \( \lim_{x \to -a} f(x) \)

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>[\frac{3x^3 - x^2 + 10}{x - 5}]</td>
<td>[\frac{3x^3 - x^2 + 10}{x + 5}]</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

72. \( f(x) = \frac{|x|}{x}; \) \( \lim_{x \to a} f(x) \) and \( \lim_{x \to -a} f(x) \)

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>-1</th>
<th>-10</th>
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<th>-100</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>[\frac{</td>
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<td>x</td>
<td>}{x}]</td>
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</tr>
</tbody>
</table>

**Exercise 73–80, find the indicated limits, if they exist.**

73. \( \lim_{x \to -\infty} \frac{3x + 2}{x - 5} \)

74. \( \lim_{x \to -\infty} \frac{4x^2 - 1}{x + 2} \)

75. \( \lim_{x \to -\infty} \frac{3x^2 + x^2 + 1}{x^2 + 1} \)

76. \( \lim_{x \to -\infty} \frac{2x^2 + 3x + 1}{x^2 - x} \)

77. \( \lim_{x \to -\infty} \frac{x^2 + 1}{x^3 - 1} \)

78. \( \lim_{x \to -\infty} \frac{4x^3 - 3x^2 + 1}{2x^4 + x^3 + x^2 + x + 1} \)

79. \( \lim_{x \to -\infty} \frac{x^3 - x^3 + x - 1}{x^4 + 2x^2 + 1} \)

80. \( \lim_{x \to -\infty} \frac{2x^2 - 1}{x^3 + x^2 + 1} \)

81. **Toxic Waste** A city’s main well was recently found to be contaminated with trichloroethylene, a cancer-causing chemical, as a result of an abandoned chemical dump leaching chemicals into the water. A proposal submitted to city council members indicates that the cost, measured in millions of dollars, of removing \( x \) percent of the toxic pollutant is given by

\[
C(x) = \frac{0.5x}{100 - x} \quad (0 < x < 100)
\]

**a.** Find the cost of removing 50%, 60%, 70%, 80%, 90%, and 95% of the pollutant.

**b.** Evaluate

\[
\lim_{x \to 0} \frac{0.5x}{100 - x}
\]

and interpret your result.

82. **A Doomsday Situation** The population of a certain breed of rabbits introduced into an isolated island is given by

\[
P(t) = \frac{72}{9 - t} \quad (0 < t < 9)
\]

where \( t \) is measured in months.

**a.** Find the number of rabbits present in the island initially.

**b.** Show that the population of rabbits is increasing without bound.

**c.** Sketch the graph of the function \( P \).

(Comment: This phenomenon is referred to as a doomsday situation.)

83. **Average Cost** The average cost per disc in dollars incurred by the Herald Record Company in pressing \( x \) video discs is given by the average cost function

\[
\bar{C}(x) = 2.2 + \frac{25000}{x}
\]

Evaluate \( \lim_{x \to 0} \bar{C}(x) \) and interpret your result.

84. **Concentration of a Drug in the Bloodstream** The concentration of a certain drug in a patient’s bloodstream \( t \) hr after injection is given by

\[
C(t) = \frac{0.2t}{t^2 + 1}
\]

mg/cm³. Evaluate \( \lim_{t \to 0} C(t) \) and interpret your result.

85. **Box Office Receipts** The total worldwide box office receipts for a long-running blockbuster movie are approximated by the function

\[
T(x) = \frac{120x^2}{x^3 + 4}
\]

where \( T(x) \) is measured in millions of dollars and \( x \) is the number of months since the movie’s release.

**a.** What are the total box office receipts after the first month? The second month? The third month?

**b.** What will the movie gross in the long run?
86. **Population Growth** A major corporation is building a 4325-acre complex of homes, offices, stores, schools, and churches in the rural community of Glen Cove. As a result of this development, the planners have estimated that Glen Cove’s population (in thousands) $t$ yr from now will be given by

$$P(t) = \frac{25t^2 + 125t + 200}{t^2 + 5t + 40}$$

a. What is the current population of Glen Cove?  
b. What will the population be in the long run?

87. **Driving Costs** A study of driving costs of 1992 model subcompact (four-cylinder) cars found that the average cost (car payments, gas, insurance, upkeep, and depreciation), measured in cents per mile, is approximated by the function

$$C(x) = \frac{2010}{x^{22}} + 17.80$$

where $x$ denotes the number of miles (in thousands) the car is driven in a year.  
a. What is the average cost of driving a subcompact car 5000 mi/yr? 10,000 mi/yr? 15,000 mi/yr? 20,000 mi/yr? 25,000 mi/yr?  
b. Use part (a) to help sketch the graph of the function $C$.  
c. What happens to the average cost as the number of miles driven increases without bound?

In Exercises 89–94, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

89. If $\lim_{x \to a} f(x)$ exists, then $f$ is defined at $x = a$.  
90. If $\lim_{x \to a} f(x) = 4$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} (f(x)g(x)) = 0$.  
91. If $\lim_{x \to a} f(x) = 3$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$ does not exist.

92. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$ does not exist.  
93. $\lim_{x \to 2} \left( \frac{x}{x + 1} + \frac{3}{x - 1} \right) = \lim_{x \to 2} \frac{x}{x + 1} + \lim_{x \to 2} \frac{3}{x - 1}$  
94. $\lim_{x \to 1} \left( \frac{2x}{x - 1} - \frac{2}{x - 1} \right) = \lim_{x \to 1} \frac{2x}{x - 1} - \lim_{x \to 1} \frac{2}{x - 1}$

95. **Speed of a Chemical Reaction** Certain proteins, known as enzymes, serve as catalysts for chemical reactions in living things. In 1913 Leonor Michaelis and L. M. Menten discovered the following formula giving the initial speed $V$ (in moles/liter/second) at which the reaction begins in terms of the amount of substrate $x$ (the substance being acted upon, measured in moles/liter):

$$V = \frac{ax}{x + b}$$

where $a$ and $b$ are positive constants. Evaluate

$$\lim_{x \to a} \frac{ax}{x + b}$$

and interpret your result.

96. Show by means of an example that $\lim_{x \to a} \frac{f(x) + g(x)}{x^2}$ may exist even though neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists. Does this example contradict Theorem 1?  
97. Show by means of an example that $\lim_{x \to a} \frac{f(x)g(x)}{x^2}$ may exist even though neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists. Does this example contradict Theorem 1?
Solutions to Self-Check Exercises 10.4

1. a. \[ \lim_{x \to 0} \frac{\sqrt{x^2 + 7} + \sqrt{3x - 5}}{x + 2} = \frac{\sqrt{9 + 7} + \sqrt{3(3) - 5}}{3 + 2} \]
   \[ = \frac{\sqrt{16} + \sqrt{4}}{5} \]
   \[ = \frac{6}{5} \]

b. Letting \(x\) approach \(-1\) leads to the indeterminate form \(0/0\). Thus, we proceed as follows:
   \[ \lim_{x \to -1} \frac{x^2 - x - 2}{2x^3 - x - 3} = \lim_{x \to -1} \frac{(x + 1)(x - 2)}{(x + 1)(2x - 3)} \]
   \[ = \lim_{x \to -1} \frac{x - 2}{2x - 3} \]
   \[ = \frac{-1 - 2}{2(-1) - 3} \]
   \[ = \frac{-3}{-5} \]
   \[ = \frac{3}{5} \]

2. \[ \lim_{x \to \infty} \overline{c}(x) = \lim_{x \to \infty} \left(1.8 + \frac{3000}{x}\right) \]
   \[ = \lim_{x \to \infty} 1.8 + \lim_{x \to \infty} \frac{3000}{x} \]
   \[ = 1.8 \]

Our computation reveals that, as the production of audio discs increases “without bound,” the average cost drops and approaches a unit cost of $1.80/disc.

10.5 One-Sided Limits and Continuity

**One-Sided Limits**

Consider the function \(f\) defined by

\[ f(x) = \begin{cases} 
  x - 1 & \text{if } x < 0 \\
  x + 1 & \text{if } x \geq 0
\end{cases} \]

From the graph of \(f\) shown in Figure 10.31, we see that the function \(f\) does not have a limit as \(x\) approaches zero because no matter how close \(x\) is to zero, \(f(x)\) takes on values that are close to 1 if \(x\) is positive and values that are close to \(-1\) if \(x\) is negative. Therefore, \(f(x)\) cannot be close to a single number \(L\)—no matter how close \(x\) is to zero. Now, if we restrict \(x\) to be greater than zero (to the right of zero), then we see that \(f(x)\) can be made as close to the number 1 as we please by taking \(x\) sufficiently close to zero. In this situation we say that the right-hand limit of \(f\) as \(x\) approaches zero (from
the right) is 1, written
\[ \lim_{x \to 0^+} f(x) = 1 \]

Similarly, we see that \( f(x) \) can be made as close to the number \(-1\) as we please by taking \( x \) sufficiently close to, but to the left of, zero. In this situation we say that the left-hand limit of \( f \) as \( x \) approaches zero (from the left) is \(-1\), written
\[ \lim_{x \to 0^-} f(x) = -1 \]

These limits are called \textbf{one-sided limits}. More generally, we have the following definitions.

\underline{One-Sided Limits}

The function \( f \) has the \textbf{right-hand limit} \( L \) as \( x \) approaches \( a \) from the right, written
\[ \lim_{x \to a^+} f(x) = L \]
if the values \( f(x) \) can be made as close to \( L \) as we please by taking \( x \) sufficiently close to (but not equal to) \( a \) and to the right of \( a \).

Similarly, the function \( f \) has the \textbf{left-hand limit} \( M \) as \( x \) approaches \( a \) from the left, written
\[ \lim_{x \to a^-} f(x) = M \]
if the values \( f(x) \) can be made as close to \( M \) as we please by taking \( x \) sufficiently close to (but not equal to) \( a \) and to the left of \( a \).

The connection between one-sided limits and the two-sided limit defined earlier is given by the following theorem.

\underline{THEOREM 3}

Let \( f \) be a function that is defined for all values of \( x \) close to \( x = a \) with the possible exception of \( a \) itself. Then,
\[ \lim_{x \to a^+} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = \lim_{x \to a} f(x) = L \]

Thus, the two-sided limit exists if and only if the one-sided limits exist and are equal.

\underline{EXAMPLE 1}

Let
\[ f(x) = \begin{cases} \sqrt{x} & \text{if } x > 0 \\ -x & \text{if } x \leq 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \]
10.5 ONE-SIDED LIMITS AND CONTINUITY

a. Show that \( \lim_{x \to 0} f(x) \) exists by studying the one-sided limits of \( f \) as \( x \) approaches \( x = 0 \).
b. Show that \( \lim_{x \to 0} g(x) \) does not exist.

**SOLUTION**

a. For \( x > 0 \), we find

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sqrt{x} = 0
\]

and for \( x \leq 0 \)

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0
\]

Thus,

\[
\lim_{x \to 0} f(x) = 0
\]

(Figure 10.32a).

b. We have

\[
\lim_{x \to 0^+} g(x) = -1 \quad \text{and} \quad \lim_{x \to 0^-} g(x) = 1
\]

and since these one-sided limits are not equal, we conclude that \( \lim_{x \to 0} g(x) \) does not exist (Figure 10.32b).

**FIGURE 10.32**

Continuous functions will play an important role throughout most of our study of calculus. Loosely speaking, a function is continuous at a point if the graph of the function at that point is devoid of holes, gaps, jumps, or breaks. Consider, for example, the graph of the function \( f \) depicted in Figure 10.33.

Let’s take a closer look at the behavior of \( f \) at or near each of the points \( x = a, x = b, x = c, \) and \( x = d \). First, note that \( f \) is not defined at \( x = a \); that is, the point \( x = a \) is not in the domain of \( f \), thereby resulting in a “hole” in the graph of \( f \). Next, observe that the value of \( f \) at \( b, f(b) \), is not equal to the
Continuity at a Point

A function $f$ is **continuous at the point** $x = a$ if the following conditions are satisfied:

1. $f(a)$ is defined.
2. $\lim_{{x \to a}} f(x)$ exists.
3. $\lim_{{x \to a}} f(x) = f(a)$

Thus, a function $f$ is continuous at the point $x = a$ if the limit of $f$ at the point $x = a$ exists and has the value $f(a)$. Geometrically, $f$ is continuous at the point $x = a$ if proximity of $x$ to $a$ implies the proximity of $f(x)$ to $f(a)$.

If $f$ is not continuous at $x = a$, then $f$ is said to be **discontinuous** at $x = a$. Also, $f$ is **continuous on an interval** if $f$ is continuous at every point in the interval.

Figure 10.34 depicts the graph of a continuous function on the interval $(a, b)$. Notice that the graph of the function over the stated interval can be sketched without lifting one’s pencil from the paper.
Find the values of $x$ for which each of the following functions is continuous.

a. $f(x) = x + 2$

b. $g(x) = \frac{x^2 - 4}{x - 2}$

c. $h(x) = \begin{cases} x + 2 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

d. $F(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

e. $G(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}$

The graph of each function is shown in Figure 10.35.

SOLUTION

a. The function $f$ is continuous everywhere because the three conditions for continuity are satisfied for all values of $x$.

b. The function $g$ is discontinuous at the point $x = 2$ because $g$ is not defined at that point. It is continuous everywhere else.

c. The function $h$ is discontinuous at $x = 2$ because the third condition for continuity is violated; the limit of $h(x)$ as $x$ approaches 2 exists and has the value 4, but this limit is not equal to $h(2) = 1$. It is continuous for all other values of $x$.

d. The function $F$ is continuous everywhere except at the point $x = 0$, where the limit of $F(x)$ fails to exist as $x$ approaches zero (see Example 3a, Section 10.4).

e. Since the limit of $G(x)$ does not exist as $x$ approaches zero, we conclude that $G$ fails to be continuous at $x = 0$. The function $G$ is continuous at all other points.
Properties of Continuous Functions

The following properties of continuous functions follow directly from the definition of continuity and the corresponding properties of limits. They are stated without proof.

1. The constant function $f(x) = c$ is continuous everywhere.
2. The identity function $f(x) = x$ is continuous everywhere.

If $f$ and $g$ are continuous at $x = a$, then

3. $[f(x)]^n$, where $n$ is a real number, is continuous at $x = a$ whenever it is defined at that point.
4. $f + g$ is continuous at $x = a$.
5. $fg$ is continuous at $x = a$.
6. $f/g$ is continuous at $x = a$ provided $g(a) \neq 0$.

Using these properties of continuous functions, we can prove the following results. (A proof is sketched in Exercise 102, page 664.)

Continuity of Polynomial and Rational Functions

1. A polynomial function $y = P(x)$ is continuous at every point $x$.
2. A rational function $R(x) = p(x)/q(x)$ is continuous at every point $x$ where $q(x) \neq 0$.

Find the values of $x$ for which each of the following functions is continuous.

- **a.** $f(x) = 3x^3 + 2x^2 - x + 10$  
  **b.** $g(x) = \frac{8x^{10} - 4x + 1}{x^2 + 1}$  
  **c.** $h(x) = \frac{4x^3 - 3x^2 + 1}{x^2 - 3x + 2}$

**SOLUTION**

- **a.** The function $f$ is a polynomial function of degree 3, so $f(x)$ is continuous for all values of $x$.
- **b.** The function $g$ is a rational function. Observe that the denominator of $g$—namely, $x^2 + 1$—is never equal to zero. Therefore, we conclude that $g$ is continuous for all values of $x$.
- **c.** The function $h$ is a rational function. In this case, however, the denominator of $h$ is equal to zero at $x = 1$ and $x = 2$, which can be seen by factoring it. Thus,
  $$x^2 - 3x + 2 = (x - 2)(x - 1)$$
  We therefore conclude that $h$ is continuous everywhere except at $x = 1$ and $x = 2$, where it is discontinuous.
APPLICATIONS

Up to this point, most of the applications we have discussed involved functions that are continuous everywhere. In Example 4 we consider an application from the field of educational psychology that involves a discontinuous function.

Figure 10.36 depicts the learning curve associated with a certain individual. Beginning with no knowledge of the subject being taught, the individual makes steady progress toward understanding it over the time interval $0 \leq t < t_1$. In this instance, the individual’s progress slows as we approach time $t_1$ because he fails to grasp a particularly difficult concept. All of a sudden, a breakthrough occurs at time $t_1$, propelling his knowledge of the subject to a higher level. The curve is discontinuous at $t_1$.

**Intermediate Value Theorem**

Let’s look again at our model of the motion of the maglev on a straight stretch of track. We know that the train cannot vanish at any instant of time and it cannot skip portions of the track and reappear someplace else. To put it another way, the train cannot occupy the positions $s_1$ and $s_2$ without at least, at some time, occupying an intermediate position (Figure 10.37).

To state this fact mathematically, recall that the position of the maglev as a function of time is described by

$$f(t) = 4t^2 \quad (0 \leq t \leq 10)$$

Suppose the position of the maglev is $s_1$ at some time $t_1$ and its position is $s_2$ at some time $t_2$ (Figure 10.38).
Then, if $s_3$ is any number between $s_1$ and $s_2$ giving an intermediate position of the maglev, there must be at least one $t_3$ between $t_1$ and $t_2$ giving the time at which the train is at $s_3$—that is, $f(t_3) = s_3$.

This discussion carries the gist of the intermediate value theorem. The proof of this theorem can be found in most advanced calculus texts.

**THEOREM 4**

The Intermediate Value Theorem

If $f$ is a continuous function on a closed interval $[a, b]$ and $M$ is any number between $f(a)$ and $f(b)$, then there is at least one number $c$ in $[a, b]$ such that $f(c) = M$ (Figure 10.39).

![Figure 10.39](image)

(a) $f(c) = M$

(b) $f(c_1) = f(c_2) = f(c_3) = M$

To illustrate the intermediate value theorem, let’s look at the example involving the motion of the maglev again (see Figure 10.21, page 621). Notice that the initial position of the train is $f(0) = 0$ and the position at the end of its test run is $f(10) = 400$. Furthermore, the function $f$ is continuous on $[0, 10]$. So, the intermediate value theorem guarantees that if we arbitrarily pick a number between 0 and 400—say, 100—giving the position of the maglev, there must be a $t$ (read “t bar”) between 0 and 10 at which time the train is at the position $s = 100$. To find the value of $t$, we solve the equation $f(t) = s$, or

$$4t^2 = 100$$

giving $t = 5$ ($t$ must lie between 0 and 10).

It is important to remember when we use Theorem 4 that the function $f$ must be continuous. The conclusion of the intermediate value theorem may not hold if $f$ is not continuous (see Exercise 103).

The next theorem is an immediate consequence of the intermediate value theorem. It not only tells us when a zero of a function $f$ [root of the equation $f(x) = 0$] exists but also provides the basis for a method of approximating it.
Theorem 5

Existence of Zeros of a Continuous Function
If $f$ is a continuous function on a closed interval $[a, b]$, and if $f(a)$ and $f(b)$ have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval $(a, b)$ (Figure 10.40).

**Figure 10.40**
If $f(a)$ and $f(b)$ have opposite signs, there must be at least one number $c$ ($a < c < b$) such that $f(c) = 0$.

Geometrically, this property states that if the graph of a continuous function goes from above the $x$-axis to below the $x$-axis, or vice versa, it must cross the $x$-axis. This is not necessarily true if the function is discontinuous (Figure 10.41).

**Figure 10.41**
$f(a) < 0$ and $f(b) > 0$, but the graph of $f$ does not cross the $x$-axis between $a$ and $b$ because $f$ is discontinuous.

Example 5

Let $f(x) = x^3 + x + 1$.

a. Show that $f$ is continuous for all values of $x$.
b. Compute $f(-1)$ and $f(1)$ and use the results to deduce that there must be at least one point $x = c$, where $c$ lies in the interval $(-1, 1)$ and $f(c) = 0$.

**Solution**

a. The function $f$ is a polynomial function of degree 3 and is therefore continuous everywhere.
b. $f(-1) = (-1)^3 + (-1) + 1 = -1$
   $f(1) = 1^3 + 1 + 1 = 3$
Since \( f(-1) \) and \( f(1) \) have opposite signs, Theorem 5 tells us that there must be at least one point \( x = c \) with \(-1 < c < 1\) such that \( f(c) = 0 \).

The next example shows how the intermediate value theorem can be used to help us find the zero of a function.

**EXAMPLE 6**

Let \( f(x) = x^3 + x - 1 \). Since \( f \) is a polynomial function, it is continuous everywhere. Observe that \( f(0) = -1 \) and \( f(1) = 1 \) so that Theorem 5 guarantees the existence of at least one root of the equation \( f(x) = 0 \) in \((0, 1)\).*

We can locate the root more precisely by using Theorem 5 once again as follows: Evaluate \( f(x) \) at the midpoint of \([0, 1]\), obtaining

\[
f(0.5) = -0.375
\]

Because \( f(0.5) < 0 \) and \( f(1) > 0 \), Theorem 5 now tells us that the root must lie in \((0.5, 1)\).

Repeat the process: Evaluate \( f(x) \) at the midpoint of \([0.5, 1]\), which is

\[
\frac{0.5 + 1}{2} = 0.75
\]

Thus,

\[
f(0.75) = 0.1719
\]

Because \( f(0.5) < 0 \) and \( f(0.75) > 0 \), Theorem 5 tells us that the root is in \((0.5, 0.75)\). This process can be continued. Table 10.3 summarizes the results of our computations through nine steps.

From Table 10.3 we see that the root is approximately 0.68, accurate to two decimal places. By continuing the process through a sufficient number of steps, we can obtain as accurate an approximation to the root as we please.

**REMARK** The process of finding the root of \( f(x) = 0 \) used in Example 6 is called the method of bisection. It is crude but effective.

**SELF-CHECK EXERCISES 10.5**

1. Evaluate \( \lim_{x \to -1^-} f(x) \) and \( \lim_{x \to -1^+} f(x) \), where

\[
f(x) = \begin{cases} 
1 & \text{if } x < -1 \\
1 + \sqrt{x + 1} & \text{if } x \geq -1
\end{cases}
\]

Does \( \lim_{x \to -1} f(x) \) exist?

*It can be shown that \( f \) has precisely one zero in \((0, 1)\) (see Exercise 97, Section 12.1.).
2. Determine the values of $x$ for which the given function is discontinuous. At each point of discontinuity, indicate which condition(s) for continuity are violated. Sketch the graph of the function.

a. $f(x) = \begin{cases} -x^2 + 1 & \text{if } x \leq 1 \\ x - 1 & \text{if } x > 1 \end{cases}$

b. $g(x) = \begin{cases} 2 & \text{if } -1 < x \leq 1 \\ -x + 3 & \text{if } x > 1 \end{cases}$

Solutions to Self-Check Exercises 10.5 can be found on page 664.
In Exercises 15–20, refer to the graph of the function \( f \) and determine whether each statement is true or false.

15. \( \lim_{x \to -3^-} f(x) = 2 \)
16. \( \lim_{x \to 0} f(x) = 2 \)
17. \( \lim_{x \to 1^-} f(x) = 1 \)
18. \( \lim_{x \to 3^-} f(x) = 3 \)
19. \( \lim_{x \to 4^-} f(x) \) does not exist.  
20. \( \lim_{x \to 5^-} f(x) = 2 \)

In Exercises 21–42, find the indicated one-sided limit, if it exists.

21. \( \lim_{x \to 1^-} (2x + 4) \)
22. \( \lim_{x \to 1^-} (3x - 4) \)
23. \( \lim_{x \to 2^-} \frac{x - 3}{x + 2} \)
24. \( \lim_{x \to 1^-} \frac{x + 2}{x + 1} \)
25. \( \lim_{x \to 0^-} \frac{1}{x} \)
26. \( \lim_{x \to 0^-} \frac{1}{x} \)
27. \( \lim_{x \to 0^-} \frac{x - 1}{x^2 + 1} \)
28. \( \lim_{x \to 2^-} \frac{x + 1}{x^2 - 2x + 3} \)
29. \( \lim_{x \to 0^-} \sqrt{x} \)
30. \( \lim_{x \to 0^-} 2\sqrt{x - 2} \)
31. \( \lim_{x \to 2^-} (2x + \sqrt{2 + x}) \)
32. \( \lim_{x \to 5^-} x(1 + \sqrt{5 + x}) \)
33. \( \lim_{x \to 1^-} \frac{1 + x}{1 - x} \)
34. \( \lim_{x \to 1^-} \frac{1 + x}{1 - x} \)
35. \( \lim_{x \to 2^-} \frac{x^2 - 4}{x - 2} \)
36. \( \lim_{x \to -3^-} \frac{\sqrt{x} + 3}{x^2 + 1} \)
37. \( \lim_{x \to -3^-} \frac{x^2 - 9}{x + 3} \)
38. \( \lim_{x \to -2^-} \frac{\sqrt{x} + 10}{2x^2 + 1} \)
39. \( \lim_{x \to 0^-} f(x) \) and \( \lim_{x \to 0^-} f(x) \), where

\[
\begin{align*}
f(x) &= \begin{cases} 
2x & \text{if } x < 0 \\
x^2 & \text{if } x \geq 0 
\end{cases}
\end{align*}
\]

40. \( \lim_{x \to 0^-} f(x) \) and \( \lim_{x \to 0^-} f(x) \), where

\[
\begin{align*}
f(x) &= \begin{cases} 
-x + 1 & \text{if } x \leq 0 \\
2x + 3 & \text{if } x > 0 
\end{cases}
\end{align*}
\]

41. \( \lim_{x \to 1^-} f(x) \) and \( \lim_{x \to 1^-} f(x) \), where

\[
\begin{align*}
f(x) &= \begin{cases} 
\sqrt{x + 3} & \text{if } x \geq 1 \\
2 + \sqrt{x} & \text{if } x < 1 
\end{cases}
\end{align*}
\]

42. \( \lim_{x \to 1^-} f(x) \) and \( \lim_{x \to 1^-} f(x) \), where

\[
\begin{align*}
f(x) &= \begin{cases} 
x + 2\sqrt{x - 1} & \text{if } x \geq 1 \\
1 - \sqrt{1 - x} & \text{if } x < 1 
\end{cases}
\end{align*}
\]

In Exercises 43–50, determine the values of \( x \), if any, at which each function is discontinuous. At each point of discontinuity, state the condition(s) for continuity that are violated.

43. \( f(x) = \begin{cases} 
2x - 4 & \text{if } x \leq 0 \\
1 & \text{if } x > 0 
\end{cases} \)
44. \( f(x) = \begin{cases} 
x^2 + 1 & \text{if } x \neq 0 \\
0 & \text{if } x = 0 
\end{cases} \)
45. \( f(x) = \begin{cases} 
x + 5 & \text{if } x \leq 0 \\
-x^2 + 5 & \text{if } x > 0 
\end{cases} \)
In Exercises 51–66, find the values of $x$ for which each function is continuous.

**51.** $f(x) = 2x^2 + x - 1$

**52.** $f(x) = x^3 - 2x^2 + x - 1$

**53.** $f(x) = \frac{2}{x^2 + 1}$

**54.** $f(x) = \frac{x}{2x^2 + 1}$

**55.** $f(x) = \frac{2}{2x - 1}$

**56.** $f(x) = \frac{x + 1}{x - 1}$

**57.** $f(x) = \frac{2x^2 + 1}{x^2 + x - 2}$

**58.** $f(x) = \frac{x - 1}{x^2 + 2x - 3}$

**59.** $f(x) = \begin{cases} x & \text{if } x \leq 1 \\ 2x - 1 & \text{if } x > 1 \end{cases}$

**60.** $f(x) = \begin{cases} -x + 1 & \text{if } x \leq -1 \\ x + 1 & \text{if } x > -1 \end{cases}$

**61.** $f(x) = \begin{cases} -2x + 1 & \text{if } x < 0 \\ x^2 + 1 & \text{if } x \geq 0 \end{cases}$

**62.** $f(x) = \begin{cases} x + 1 & \text{if } x \leq 1 \\ -x^2 + 1 & \text{if } x > 1 \end{cases}$

**63.** $f(x) = \begin{cases} x^2 - 1 & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

**64.** $f(x) = \begin{cases} x^2 - 4 & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$

**65.** $f(x) = |x + 1|$

**66.** $f(x) = \frac{|x - 1|}{x - 1}$

In Exercises 67–70, determine all values of $x$ at which the function is discontinuous.

**67.** $f(x) = \frac{2x}{x^2 - 1}$

**68.** $f(x) = \frac{1}{(x - 1)(x - 2)}$

**69.** $f(x) = \frac{x^2 - 2x}{x^2 - 3x + 2}$

**70.** $f(x) = \frac{x^2 - 3x + 2}{x^2 - 2x}$

**71. The Postage Function** The graph of the “postage function”

$$f(x) = \begin{cases} 34 & \text{if } 0 < x \leq 1 \\ 55 & \text{if } 1 < x \leq 2 \\ \vdots & \text{if } \vdots \\ 265 & \text{if } 11 < x \leq 12 \end{cases}$$

where $x$ denotes the weight of a parcel in ounces and $f(x)$ the postage in cents, is shown in the figure on page 658. Determine the values of $x$ for which $f$ is discontinuous.
72. **Inventory Control** As part of an optimal inventory policy, the manager of an office supply company orders 500 reams of photocopy paper every 20 days. The accompanying graph shows the actual inventory level of paper in an office supply store during the first 60 business days of 2001. Determine the values of $t$ for which the “inventory function” is discontinuous and give an interpretation of the graph.

73. **Learning Curves** The following graph describes the progress Michael made in solving a problem correctly during a mathematics quiz. Here, $y$ denotes the percentage of work completed, and $x$ is measured in minutes. Give an interpretation of the graph.

74. **Ailing Financial Institutions** The Franklin Savings and Loan Company acquired two ailing financial institutions in 1992. One of them was acquired at time $t = T_1$, and the other was acquired at time $t = T_2$ ($t = 0$ corresponds to the beginning of 1992). The following graph shows the total amount of money on deposit with Franklin. Explain the significance of the discontinuities of the function at $T_1$ and $T_2$.

75. **Energy Consumption** The following graph shows the amount of home heating oil remaining in a 200-gallon tank over a 120-day period ($t = 0$ corresponds to October 1). Explain why the function is discontinuous at $t = 40$, $t = 70$, $t = 95$, and $t = 110$.

76. **Prime Interest Rate** The function $P$, whose graph follows, gives the prime rate (the interest rate banks charge their best corporate customers) as a function of time for the first 32 wk in 1989. Determine the values of $t$ for which $P$ is discontinuous and interpret your results.
77. Administration of an Intravenous Solution A dextrose solution is being administered to a patient intravenously. The 1-liter (L) bottle holding the solution is removed and replaced by another as soon as the contents drop to approximately 5% of the initial (1-L) amount. The rate of discharge is constant, and it takes 6 hr to discharge 95% of the contents of a full bottle. Draw a graph showing the amount of dextrose solution in a bottle in the IV system over a 24-hr period, assuming that we started with a full bottle.

78. Commissions The base salary of a salesman working on commission is $12,000. For each $50,000 of sales beyond $100,000, he is paid a $1000 commission. Sketch a graph showing his earnings as a function of the level of his sales x. Determine the values of x for which the function f is discontinuous.

79. Parking Fees The fee charged per car in a downtown parking lot is $1 for the first half hour and $.50 for each additional half hour or part thereof, subject to a maximum of $5. Derive a function f relating the parking fee to the length of time a car is left in the lot. Sketch the graph of f and determine the values of x for which the function f is discontinuous.

80. Commodity Prices The function that gives the cost of a certain commodity is defined by

\[ C(x) = \begin{cases} 
5x & \text{if } 0 < x < 10 \\
4x & \text{if } 10 \leq x < 30 \\
3.5x & \text{if } 30 \leq x < 60 \\
3.25x & \text{if } x \geq 60 
\end{cases} \]

where x is the number of pounds of a certain commodity sold and C(x) is measured in dollars. Sketch the graph of the function C and determine the values of x for which the function C is discontinuous.

81. Energy Expended by a Fish Suppose that a fish swimming a distance of L ft at a speed of v ft/sec relative to the water and against a current flowing at the rate of u ft/sec (u < v) expends a total energy given by

\[ E(v) = \frac{aLv^3}{v - u} \]

where E is measured in foot-pounds (ft-lb) and a is a constant.

a. Evaluate \( \lim_{v \to u^+} E(v) \) and interpret your result.

b. Evaluate \( \lim_{v \to u^-} E(v) \) and interpret your result.

82. Let

\[ f(x) = \begin{cases} 
x + 2 & \text{if } x \leq 1 \\
kx^2 & \text{if } x > 1 
\end{cases} \]

Find the value of k that will make f continuous on \(( -\infty, \infty )\).

83. Let

\[ f(x) = \begin{cases} 
x^3 - 4 & \text{if } x \neq -2 \\
k & \text{if } x = -2 
\end{cases} \]

For what value of k will f be continuous on \(( -\infty, \infty )\)?

84. a. Suppose f is continuous at a and g is discontinuous at a. Is the sum \( f + g \) discontinuous at a? Explain.

b. Suppose f and g are both discontinuous at a. Is the sum \( f + g \) necessarily discontinuous at a? Explain.

85. a. Suppose f is continuous at a and g is discontinuous at a. Is the product \( fg \) necessarily discontinuous at a? Explain.

b. Suppose f and g are both discontinuous at a. Is the product \( fg \) necessarily discontinuous at a? Explain.

In Exercises 86–89, (a) show that the function f is continuous for all values of x in the interval \([a, b]\) and (b) prove that f must have at least one zero in the interval \((a, b)\) by showing that \( f(a) \) and \( f(b) \) have opposite signs.

86. \( f(x) = x^3 - 6x + 8; a = 1, b = 3 \)

87. \( f(x) = x^3 - 3x^2 + 2x; a = -1, b = 1 \)

88. \( f(x) = 2x^3 - 3x^2 - 36x + 14; a = 0, b = 1 \)

89. \( f(x) = 2x^{3/3} - 5x^{4/3}; a = 14, b = 16 \)
Using Technology

Finding the Points of Discontinuity of a Function

You can very often recognize the points of discontinuity of a function \( f \) by examining its graph. For example, Figure T1 shows the graph of \( f(x) = \frac{x}{x^2 - 1} \) obtained using a graphing utility. It is evident that \( f \) is discontinuous at \( x = -1 \) and \( x = 1 \). This observation is also borne out by the fact that both these points are not in the domain of \( f \).

Consider the function

\[
g(x) = \frac{2x^3 + x^2 - 7x - 6}{x^2 - x - 2}
\]

Using a graphing utility we obtain the graph of \( g \) shown in Figure T2. An examination of this graph does not reveal any points of discontinuity. However, if we factor both the numerator and the denominator of the rational expression, we see that

\[
g(x) = \frac{(x + 1)(x - 2)(2x + 3)}{(x + 1)(x - 2)}
= 2x + 3
\]

provided \( x \neq -1 \) and \( x \neq 2 \), so that its graph in fact looks like that shown in Figure T3.
This example shows the limitation of the graphing utility and reminds us of the importance of studying functions analytically!

**Graphing Functions Defined Piecewise**

The following example illustrates how to plot the graphs of functions defined in a piecewise manner on a graphing utility.

**Example 1**

Plot the graph of

\[ f(x) = \begin{cases} 
  x + 1 & \text{if } x \leq 1 \\
  \frac{2}{x} & \text{if } x > 1 
\end{cases} \]

**Solution**

We enter the function

\[ y1 = (x + 1)(x \leq 1) + (2/x)(x > 1) \]

The graph of the function in the viewing rectangle \([-5, 5] \times [-2, 4]\) is shown in Figure T4.
The percentage of U.S. households, \( P(t) \), watching television during weekdays between the hours of 4 P.M. and 4 A.M. is given by

\[
P(t) = \begin{cases} 
0.01354t^4 - 0.49375t^3 + 2.58333t^2 + 3.8t + 31.60704 & \text{if } 0 \leq t \leq 8 \\
1.35t^2 - 33.05t + 208 & \text{if } 8 < t \leq 12 
\end{cases}
\]

where \( t \) is measured in hours, with \( t = 0 \) corresponding to 4 P.M. Plot the graph of \( P \) in the viewing rectangle \([0, 12] \times [0, 80]\).

Source: A. C. Nielsen Co.

**SOLUTION**

We enter the function

\[
y2 = (0.01354x^4 - 0.49375x^3 + 2.58333x^2 + 3.8x + 31.60704)(x \geq 0)(x \leq 8) + (1.35x^2 - 33.05x + 208)(x > 8)(x \leq 12)
\]

The graph of \( P \) is shown in Figure T5.

**Exercises**

In Exercises 1–10, use a graphing utility to plot the graph of \( f \) and to spot the points of discontinuity of \( f \). Then use analytical means to verify your observation and find all points of discontinuity.

1. \( f(x) = \frac{2}{x^2 - x} \)

2. \( f(x) = \frac{2x + 1}{x^2 + x - 2} \)

3. \( f(x) = \frac{\sqrt{x}}{x^2 - x - 2} \)

4. \( f(x) = \frac{3}{\sqrt{x(x + 1)}} \)

5. \( f(x) = \frac{6x^4 + x^2 - 2x}{2x^2 - x} \)

6. \( f(x) = \frac{2x^4 - x^2 - 13x - 6}{2x^2 - 5x - 3} \)

7. \( f(x) = \frac{2x^4 - 3x^3 - 2x^2}{2x^3 - 3x - 2} \)

8. \( f(x) = \frac{6x^4 - x^3 + 5x^2 - 1}{6x^2 - x - 1} \)

9. \( f(x) = \frac{x^3 + x^2 - 2x}{x^4 + 2x^3 - x - 2} \)

Hint: \( x^4 + 2x^3 - x - 2 = (x^2 - 1)(x + 2) \)

10. \( f(x) = \frac{x^3 - x}{x^{4/3} - x + x^{1/3} - 1} \)

Hint: \( x^{4/3} - x + x^{1/3} - 1 = (x^{1/3} - 1)(x + 1) \)

Can you explain why part of the graph is missing?
In Exercises 11–14, use a graphing utility to plot the graph of \( f \) in the indicated viewing rectangle.

11. \( f(x) = \begin{cases} 
-1 & \text{if } x \leq 1 \\
-1 + 1 & \text{if } x > 1; [-5, 5] \times [-2, 8]
\end{cases} \)

12. \( f(x) = \begin{cases} 
\frac{1}{3}x^2 - 2x & \text{if } x \leq 3 \\
-x + 6 & \text{if } x > 3; [0, 7] \times [-5, 5]
\end{cases} \)

13. \( f(x) = \begin{cases} 
2 & \text{if } x \leq 0 \\
\sqrt{4 - x^2} & \text{if } x > 0; [-2, 2] \times [-4, 4]
\end{cases} \)

14. \( f(x) = \begin{cases} 
-x^2 + x + 2 & \text{if } x \leq 1 \\
2x^3 - x^2 - 4 & \text{if } x > 1; [-4, 4] \times [-5, 5]
\end{cases} \)

15. **Flight Path of a Plane** The function

\[
f(x) = \begin{cases} 
0 & \text{if } 0 \leq x < 1 \\
-0.00411523x^3 + 0.0679012x^2 - 0.123457x + 0.0596708 & \text{if } 1 \leq x < 10 \\
1.5 & \text{if } 10 \leq x \leq 100
\end{cases} \]

where both \( x \) and \( f(x) \) are measured in units of 1000 ft, describes the flight path of a plane taking off from the origin and climbing to an altitude of 15,000 ft. Plot the graph of \( f \) to visualize the trajectory of the plane.

16. **Home Shopping Industry** According to industry sources, revenue from the home shopping industry for the years since its inception may be approximated by the function

\[
R(t) = \begin{cases} 
-0.03t^3 + 0.25t^2 - 0.12 & \text{if } 0 \leq t \leq 3 \\
0.57t - 0.63 & \text{if } 3 < t \leq 11
\end{cases}
\]

where \( R(t) \) measures the revenue in billions of dollars and \( t \) is measured in years, with \( t = 0 \) corresponding to the beginning of 1984. Plot the graph of \( R \).

*Source: Paul Kagan Associates*
In Exercises 90–91, use the intermediate value theorem to find the value of \( c \) such that \( f(c) = M \).

90. \( f(x) = x^2 - x + 1 \) on \([-1, 4]; \quad M = 7\)
91. \( f(x) = x^3 - 4x + 6 \) on \([0, 3]; \quad M = 2\)

92. Use the method of bisection (see Example 6) to find the root of the equation \( x^3 - x + 1 = 0 \) accurate to two decimal places.

93. Use the method of bisection to find the root of the equation \( x^4 + 2x - 7 = 0 \) accurate to two decimal places.

94. Joan is looking straight out a window of an apartment building at a height of 32 ft from the ground. A boy throws a tennis ball straight up by the side of the building where the window is located. Suppose the height \( h(t) \) of the ball (measured in feet) from the ground at time \( t \) is \( h(t) = 4 + 64t - 16t^2 \).
   a. Show that \( h(0) = 4 \) and \( h(2) = 68 \).
   b. Use the intermediate value theorem to conclude that the ball must cross Joan’s line of sight at least once.
   c. At what time(s) does the ball cross Joan’s line of sight? Interpret your results.

95. In Exercises 95–99, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

95. Suppose the function \( f \) is defined on the interval \([a, b]\). If \( f(a) \) and \( f(b) \) have the same sign, then \( f \) has no zero in \([a, b]\).

96. If \( \lim_{x \to a} f(x) = L \), then \( \lim_{x \to a} f(x) - \lim_{x \to a} f(x) \neq 0 \).

97. If \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} f(x) = L \), then \( f(a) = L \).

98. If \( \lim f(x) = L \) and \( g(a) = M \), then \( \lim f(x)g(x) = LM \).

99. If \( f \) is continuous on \([-2, 3] \), \( f(-2) = 3 \) and \( f(3) = 1 \), then there exists at least one number \( c \) in \([-2, 3] \) such that \( f(c) = 2 \).

100. Let \( f(x) = x - \sqrt{1 - x^2} \).
   a. Show that \( f \) is continuous for all values of \( x \) in the interval \([-1, 1]\).
   b. Show that \( f \) has at least one zero in \([-1, 1]\).
   c. Find the zeros of \( f \) in \([-1, 1]\) by solving the equation \( f(x) = 0 \).

101. Let \( f(x) = \frac{x^2}{x^2 + 1} \).
   a. Show that \( f \) is continuous for all values of \( x \).
   b. Show that \( f(x) \) is nonnegative for all values of \( x \).
   c. Show that \( f \) has a zero at \( x = 0 \). Does this contradict Theorem 5?

102. a. Prove that a polynomial function \( y = P(x) \) is continuous at every point \( x \). Follow these steps:
   (1) Use Properties 2 and 3 of continuous functions to establish that the function \( g(x) = x^n \), where \( n \) is a positive integer, is continuous everywhere.
   (2) Use Properties 1 and 5 to show that \( f(x) = cx^n \), where \( c \) is a constant and \( n \) is a positive integer, is continuous everywhere.
   (3) Use Property 4 to complete the proof of the result.
   b. Prove that a rational function \( R(x) = \frac{p(x)}{q(x)} \) is continuous at every point \( x \), where \( q(x) \neq 0 \).
   Hint: Use the result of part (a) and Property 6.

103. Show that the conclusion of the intermediate value theorem does not hold if \( f \) is discontinuous on \([a, b]\).

Solutions to Self-Check Exercises 10.5

1. For \( x < -1, f(x) = 1 \), and so
   \[ \lim_{x \to -1} f(x) = \lim_{x \to -1} 1 = 1 \]
   For \( x \geq -1, f(x) = 1 + \sqrt{x + 1} \), and so
   \[ \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (1 + \sqrt{x + 1}) = 1 \]
Since the left-hand and right-hand limits of \( f \) exist as \( x \) approaches \( x = -1 \) and both are equal to 1, we conclude that
\[
\lim_{x \to -1} f(x) = 1
\]

2. a. The graph of \( f \) is as follows:

We see that \( f \) is continuous everywhere.

b. The graph of \( g \) is as follows:

Since \( g \) is not defined at \( x = -1 \), it is discontinuous there. It is continuous everywhere else.

### An Intuitive Example

We mentioned in Section 10.4 that the problem of finding the *rate of change* of one quantity with respect to another is mathematically equivalent to the problem of finding the *slope of the tangent line* to a curve at a given point on the curve. Before going on to establish this relationship, let’s show its plausibility by looking at it from an intuitive point of view.

Consider the motion of the maglev discussed in Section 10.4. Recall that the position of the maglev at any time \( t \) is given by
\[
s = f(t) = 4t^2 \quad (0 \leq t \leq 30)
\]
where \( s \) is measured in feet and \( t \) in seconds. The graph of the function \( f \) is sketched in Figure 10.42.
Observe that the graph of \( f \) rises slowly at first but more rapidly as \( t \) increases, reflecting the fact that the speed of the maglev is increasing with time. This observation suggests a relationship between the speed of the maglev at any time \( t \) and the steepness of the curve at the point corresponding to this value of \( t \). Thus, it would appear that we can solve the problem of finding the speed of the maglev at any time if we can find a way to measure the steepness of the curve at any point on the curve.

To discover a yardstick that will measure the steepness of a curve, consider the graph of a function \( f \) such as the one shown in Figure 10.43a. Think of the curve as representing a stretch of roller coaster track (Figure 10.43b). When the car is at the point \( P \) on the curve, a passenger sitting erect in the car and looking straight ahead will have a line of sight that is parallel to the line \( T \), the tangent to the curve at \( P \).

As Figure 10.43a suggests, the steepness of the curve—that is, the rate at which \( y \) is increasing or decreasing with respect to \( x \)—is given by the slope of the tangent line to the graph of \( f \) at the point \( P(x, f(x)) \). But for now we will show how this relationship can be used to estimate the rate of change of a function from its graph.

**EXAMPLE 1**

The graph of the function \( y = N(t) \), shown in Figure 10.44, gives the number of Social Security beneficiaries from the beginning of 1990 \((t = 0)\) through the year 2045 \((t = 55)\).
Use the graph of \( y = N(t) \) to estimate the rate at which the number of Social Security beneficiaries was growing at the beginning of the year 2000 \((t = 10)\). How fast will the number be growing at the beginning of 2025 \((t = 35)\)? [Assume that the rate of change of the function \( N \) at any value of \( t \) is given by the slope of the tangent line at the point \( P(t, N(t)) \).]

Source: Social Security Administration

From the figure, we see that the slope of the tangent line \( T_1 \) to the graph of \( y = N(t) \) at \( P_1(10, 44.7) \) is approximately 0.5. This tells us that the quantity \( y \) is increasing at the rate of 1/2 unit per unit increase in \( t \), when \( t = 10 \). In other words, at the beginning of the year 2000, the number of Social Security beneficiaries was increasing at the rate of approximately 0.5 million, or 500,000, per year.

The slope of the tangent line \( T_2 \) at \( P_2(35, 71.9) \) is approximately 1.15. This tells us that at the beginning of 2025 the number of Social Security beneficiaries will be growing at the rate of approximately 1.15 million, or 1,150,000, per year.

Slope of a Tangent Line

In Example 1 we answered the questions raised by drawing the graph of the function \( N \) and estimating the position of the tangent lines. Ideally, however, we would like to solve a problem analytically whenever possible. To do this we need a precise definition of the slope of a tangent line to a curve.

To define the tangent line to a curve \( C \) at a point \( P \) on the curve, fix \( P \) and let \( Q \) be any point on \( C \) distinct from \( P \) (Figure 10.45). The straight line passing through \( P \) and \( Q \) is called a secant line.

Now, as the point \( Q \) is allowed to move toward \( P \) along the curve, the secant line through \( P \) and \( Q \) rotates about the fixed point \( P \) and approaches a fixed line through \( P \). This fixed line, which is the limiting position of the secant lines through \( P \) and \( Q \) as \( Q \) approaches \( P \), is the tangent line to the graph of \( f \) at the point \( P \).
We can describe the process more precisely as follows. Suppose the curve $C$ is the graph of a function $f$ defined by $y = f(x)$. Then the point $P$ is described by $P(x, f(x))$ and the point $Q$ by $Q(x + h, f(x + h))$, where $h$ is some appropriate nonzero number (Figure 10.46a). Observe that we can make $Q$ approach $P$ along the curve $C$ by letting $h$ approach zero (Figure 10.46b).

Next, using the formula for the slope of a line, we can write the slope of the secant line passing through $P(x, f(x))$ and $Q(x + h, f(x + h))$ as

$$
\frac{f(x + h) - f(x)}{(x + h) - x} = \frac{f(x + h) - f(x)}{h}
$$

As observed earlier, $Q$ approaches $P$, and therefore the secant line through $P$ and $Q$ approaches the tangent line $T$ as $h$ approaches zero. Consequently, we might expect that the slope of the secant line would approach the slope of the tangent line $T$ as $h$ approaches zero. This leads to the following definition.

**Slope of a Tangent Line**

The slope of the tangent line to the graph $f$ at the point $P(x, f(x))$ is given by

$$
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
$$

if it exists.
Rates of Change

We now show that the problem of finding the slope of the tangent line to the graph of a function \( f \) at the point \( P(x, f(x)) \) is mathematically equivalent to the problem of finding the rate of change of \( f \) at \( x \). To see this, suppose we are given a function \( f \) that describes the relationship between the two quantities \( x \) and \( y \):

\[
y = f(x)
\]

The number \( f(x + h) - f(x) \) measures the change in \( y \) that corresponds to a change \( h \) in \( x \) (Figure 10.47).

Then, the difference quotient

\[
\frac{f(x + h) - f(x)}{h} \quad (5)
\]

measures the average rate of change of \( y \) with respect to \( x \) over the interval \([x, x + h]\). For example, if \( y \) measures the position of a car at time \( x \), then quotient (5) gives the average velocity of the car over the time interval \([x, x + h]\).

Observe that the difference quotient (5) is the same as (3). We conclude that the difference quotient (5) also measures the slope of the secant line that passes through the two points \( P(x, f(x)) \) and \( Q(x + h, f(x + h)) \) lying on the graph of \( y = f(x) \). Next, by taking the limit of the difference quotient (5) as \( h \) goes to zero—that is, by evaluating

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \quad (6)
\]

we obtain the rate of change of \( f \) at \( x \). For example, if \( y \) measures the position of a car at time \( x \), then the limit (6) gives the velocity of the car at time \( x \). For emphasis, the rate of change of a function \( f \) at \( x \) is often called the instantaneous rate of change of \( f \) at \( x \). This distinguishes it from the average
rate of change of \( f \), which is computed over an interval \([x, x + h]\) rather than at a point \( x \).

Observe that the limit (6) is the same as (4). Therefore, the limit of the difference quotient also measures the slope of the tangent line to the graph of \( y = f(x) \) at the point \((x, f(x))\).

The following summarizes this discussion.

### Average and Instantaneous Rates of Change

The average rate of change of \( f \) over the interval \([x, x + h]\) or slope of the secant line to the graph of \( f \) through the points \((x, f(x))\) and \((x + h, f(x + h))\) is

\[
\frac{f(x + h) - f(x)}{h}
\]

(7)

The instantaneous rate of change of \( f \) at \( x \) or slope of the tangent line to the graph of \( f \) at \((x, f(x))\) is

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

(8)

### The Derivative

The limit (4), or (8), which measures both the slope of the tangent line to the graph of \( y = f(x) \) at the point \( P(x, f(x)) \) and the (instantaneous) rate of change of \( f \) at \( x \) is given a special name: the derivative of \( f \) at \( x \).

The derivative of a function \( f \) with respect to \( x \) is the function \( f' \) (read “\( f \) prime”), defined by

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

(9)

The domain of \( f' \) is the set of all \( x \) where the limit exists.

Thus, the derivative of a function \( f \) is a function \( f' \) that gives the slope of the tangent line to the graph of \( f \) at any point \((x, f(x))\) and also the rate of change of \( f \) at \( x \) (Figure 10.48).
Other notations for the derivative of \( f \) include:

\[
D_x f(x) \quad \text{(Read “d sub x of f of x”)}
\]

\[
\frac{dy}{dx} \quad \text{(Read “d y d x”)}
\]

\[
y’ \quad \text{(Read “y prime”)}
\]

The last two are used when the rule for \( f \) is written in the form \( y = f(x) \).

The calculation of the derivative of \( f \) is facilitated using the following four-step process.

**Four-Step Process for Finding \( f'(x) \)**

1. Compute \( f(x + h) \).
2. Form the difference \( f(x + h) - f(x) \).
3. Form the quotient \( \frac{f(x + h) - f(x)}{h} \).
4. Compute \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \).

**EXAMPLE 2**

Find the slope of the tangent line to the graph of \( f(x) = 3x + 5 \) at any point \( (x, f(x)) \).

**SOLUTION ✔**

The slope of the tangent line at any point on the graph of \( f \) is given by the derivative of \( f \) at \( x \). To find the derivative, we use the four-step process:

**Step 1** \( f(x + h) = 3(x + h) + 5 = 3x + 3h + 5 \)

**Step 2** \( f(x + h) - f(x) = (3x + 3h + 5) - (3x + 5) = 3h \)

**Step 3** \( \frac{f(x + h) - f(x)}{h} = \frac{3h}{h} = 3 \)

**Step 4** \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} 3 = 3 \)

We expect this result since the tangent line to any point on a straight line must coincide with the line itself and therefore must have the same slope as the line. In this case the graph of \( f \) is a straight line with slope 3.

**EXAMPLE 3**

Let \( f(x) = x^2 \).

**a.** Compute \( f'(x) \).

**b.** Compute \( f'(2) \) and interpret your result.
To find \( f'(x) \), we use the four-step process:

**Step 1**

\[
\frac{f(x + h) - f(x)}{h} = \frac{(x + h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h}
\]

**Step 2**

\[
\frac{f(x + h) - f(x)}{h} = \frac{2xh + h^2}{h} = 2x + h
\]

**Step 3**

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} (2x + h) = 2x
\]

**Step 4**

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} (2x + h) = 2x
\]

b. \( f'(2) = 2(2) = 4 \). This result tells us that the slope of the tangent line to the graph of \( f \) at the point \((2, 4)\) is 4. It also tells us that the function \( f \) is changing at the rate of 4 units per unit change in \( x \) at \( x = 2 \). The graph of \( f \) and the tangent line at \((2, 4)\) are shown in Figure 10.49.

---

**Exploring with Technology**

1. Consider the function \( f(x) = x^2 \) of Example 3. Suppose we want to compute \( f'(2) \) using Equation (9). Thus,

\[
f'(2) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h}
\]

Use a graphing utility to plot the graph of

\[
g(x) = \frac{(x + h)^2 - x^2}{h}
\]

in the viewing rectangle \([-3, 3] \times [-2, 6]\).

2. Use **ZOOM** and **TRACE** to find \( \lim_{x \to 0} g(x) \).

3. Explain why the limit found in part 2 is \( f'(2) \).

---

**Example 4**

Let \( f(x) = x^2 - 4x \).

a. Compute \( f'(x) \).

b. Find the point on the graph of \( f \) where the tangent line to the curve is horizontal.

c. Sketch the graph of \( f \) and the tangent line to the curve at the point found in part (b).

d. What is the rate of change of \( f \) at this point?

**Solution**

a. To find \( f'(x) \), we use the four-step process:

**Step 1**

\[
f(x + h) = (x + h)^2 - 4(x + h) = x^2 + 2xh + h^2 - 4x - 4h
\]

**Step 2**

\[
f(x + h) - f(x) = x^2 + 2xh + h^2 - 4x - 4h - (x^2 - 4x)
\]

\[
= 2xh + h^2 - 4h = h(2x + h - 4)
\]
FIGURE 10.50
The tangent line to the graph of \( y = x^2 - 4x \) at \((2, -4)\) is \( y = -4 \).

Step 3 \( \frac{f(x + h) - f(x)}{h} = \frac{h(2x + h - 4)}{h} = 2x + h - 4 \)

Step 4 \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} (2x + h - 4) = 2x - 4 \)

b. At a point on the graph of \( f \) where the tangent line to the curve is horizontal and hence has slope zero, the derivative \( f' \) of \( f \) is zero. Accordingly, to find such point(s) we set \( f'(x) = 0 \), which gives \( 2x - 4 = 0 \), or \( x = 2 \). The corresponding value of \( y \) is given by \( y = f(2) = -4 \), and the required point is \((2, -4)\).

c. The graph of \( f \) and the tangent line are shown in Figure 10.50.

d. The rate of change of \( f \) at \( x = 2 \) is zero.

EXAMPLE 5
Let \( f(x) = \frac{1}{x} \).

a. Compute \( f'(x) \).

b. Find the slope of the tangent line \( T \) to the graph of \( f \) at the point where \( x = 1 \).

c. Find an equation of the tangent line \( T \) in part (b).

a. To find \( f'(x) \), we use the four-step process:

Step 1 \( f(x + h) = \frac{1}{x + h} \)

Step 2 \( f(x + h) - f(x) = \frac{1}{x + h} - \frac{1}{x} = \frac{x - (x + h)}{x(x + h)} = -\frac{h}{x(x + h)} \)

Step 3 \( \frac{f(x + h) - f(x)}{h} = -\frac{h}{x(x + h)} \cdot \frac{1}{h} = -\frac{1}{x(x + h)} \)

Step 4 \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} -\frac{1}{x(x + h)} = -\frac{1}{x^2} \)

b. The slope of the tangent line \( T \) to the graph of \( f \) where \( x = 1 \) is given by \( f'(1) = -1 \).

c. When \( x = 1 \), \( y = f(1) = 1 \) and \( T \) is tangent to the graph of \( f \) at the point \((1, 1)\). From part (b), we know that the slope of \( T \) is \(-1 \). Thus, an equation of \( T \) is

\[
y - 1 = -1(x - 1)
\]

\[
y = -x + 2
\]

(Figure 10.51).
Exploring with Technology

1. Use the results of Example 5 to draw the graph of \( f(x) = \frac{1}{x} \) and its tangent line at the point \((1, 1)\) by plotting the graphs of \( y_1 = 1/x \) and \( y_2 = -x + 2 \) in the viewing rectangle \([-4, 4] \times [-4, 4]\).

2. Some graphing utilities draw the tangent line to the graph of a function at a given point automatically—you need only specify the function and give the x-coordinate of the point of tangency. If your graphing utility has this feature, verify the result of part 1 without finding an equation of the tangent line.

Group Discussion

Consider the following alternative approach to the definition of the derivative of a function: Let \( h \) be a positive number and suppose \( P(x-h, f(x-h)) \) and \( Q(x+h, f(x+h)) \) are two points on the graph of \( f \).

1. Give a geometric and a physical interpretation of the quotient

\[
\frac{f(x+h) - f(x-h)}{2h}
\]

Make a sketch to illustrate your answer.

2. Give a geometric and a physical interpretation of the limit

\[
\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}
\]

Make a sketch to illustrate your answer.

3. Explain why it makes sense to define

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h} \]

4. Using the definition given in part (c), formulate a four-step process for finding \( f'(x) \) similar to that given on page 671 and use it to find the derivative of \( f(x) = x^2 \). Compare your answer with that obtained in Example 3 on page 672.

Applications

Example 6

Suppose the distance (in feet) covered by a car moving along a straight road \( t \) seconds after starting from rest is given by the function \( f(t) = 2t^2 (0 \leq t \leq 30) \).

a. Calculate the average velocity of the car over the time intervals \([22, 23]\), \([22, 22.1]\), and \([22, 22.01]\).

b. Calculate the (instantaneous) velocity of the car when \( t = 22 \).

c. Compare the results obtained in part (a) with that obtained in part (b).
a. We first compute the average velocity (average rate of change of \( f \)) over the interval \([t, t + h]\) using Formula (7). We find

\[
\frac{f(t + h) - f(t)}{h} = \frac{2(t + h)^2 - 2t^2}{h} = \frac{2t^2 + 4th + 2h^2 - 2t^2}{h} = 4t + 2h
\]

Next, using \( t = 22 \) and \( h = 1 \), we find that the average velocity of the car over the time interval \([22, 23]\) is

\[
4(22) + 2(1) = 90
\]

or 90 feet per second. Similarly, using \( t = 22, h = 0.1, \) and \( h = 0.01 \), we find that its average velocities over the time intervals \([22, 22.1]\) and \([22, 22.01]\) are 88.2 and 88.02 feet per second, respectively.

b. Using the limit (8), we see that the instantaneous velocity of the car at any time \( t \) is given by

\[
\lim_{h \to 0} \frac{f(t + h) - f(t)}{h} = \lim_{h \to 0} (4t + 2h) \quad \text{[Using the results from part (a)]}
\]

\[
= 4t
\]

In particular, the velocity of the car 22 seconds from rest \((t = 22)\) is given by

\[
v = 4(22)
\]

or 88 feet per second.

c. The computations in part (a) show that, as the time intervals over which the average velocity of the car are computed become smaller and smaller, the average velocities over these intervals do approach 88 feet per second, the instantaneous velocity of the car at \( t = 22 \).

---

**EXAMPLE 7**

The management of the Titan Tire Company has determined that the weekly demand function for their Super Titan tires is given by

\[
p = f(x) = 144 - x^2
\]

where \( p \) is measured in dollars and \( x \) is measured in units of a thousand (Figure 10.52).

a. Find the average rate of change in the unit price of a tire if the quantity demanded is between 5000 and 6000 tires, between 5000 and 5100 tires, and between 5000 and 5010 tires.

b. What is the instantaneous rate of change of the unit price when the quantity demanded is 5000 units?
SOLUTION

a. The average rate of change of the unit price of a tire if the quantity demanded is between $x$ and $x + h$ is

$$\frac{f(x + h) - f(x)}{h} = \frac{[144 - (x + h)^2] - (144 - x^2)}{h} = \frac{144 - x^2 - 2x(h) - h^2 - 144 + x^2}{h} = -2x - h$$

To find the average rate of change of the unit price of a tire when the quantity demanded is between 5000 and 6000 tires (that is, over the interval $[5, 6]$), we take $x = 5$ and $h = 1$, obtaining

$$-2(5) - 1 = -11$$

or $-11$ per 1000 tires. (Remember, $x$ is measured in units of a thousand.) Similarly, taking $h = 0.1$ and $h = 0.01$ with $x = 5$, we find that the average rates of change of the unit price when the quantities demanded are between 5000 and 5100 and between 5000 and 5010 are $-10.10$ and $-10.01$ per 1000 tires, respectively.

b. The instantaneous rate of change of the unit price of a tire when the quantity demanded is $x$ units is given by

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} (-2x - h) \quad \text{[Using the results from part (a)]}$$

$$= -2x$$

In particular, the instantaneous rate of change of the unit price per tire when the quantity demanded is 5000 is given by $-2(5)$, or $-10$ per 1000 tires.

The derivative of a function provides us with a tool for measuring the rate of change of one quantity with respect to another. Table 10.4 lists several other applications involving this limit.

<table>
<thead>
<tr>
<th>$x$ stands for</th>
<th>$y$ stands for</th>
<th>$\frac{f(a + h) - f(a)}{h}$ measures the</th>
<th>$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$ measures the</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Time</strong></td>
<td><strong>Concentration of a drug</strong> in the bloodstream at time $x$</td>
<td>Average rate of change in the concentration of the drug over the time interval $[a, a + h]$</td>
<td>Instantaneous rate of change in the concentration of the drug in the bloodstream at time $x = a$</td>
</tr>
<tr>
<td><strong>Number of items sold</strong></td>
<td><strong>Revenue</strong> at a sales level of $x$ units</td>
<td>Average rate of change in the revenue when the sales level is between $x = a$ and $x = a + h$</td>
<td>Instantaneous rate of change in the revenue when the sales level is $a$ units</td>
</tr>
</tbody>
</table>

(continued)
10.6  THE DERIVATIVE  677

**DIFFERENTIABILITY AND CONTINUITY**

In practical applications, one encounters functions that fail to be differentiable—that is, do not have a derivative at certain values in the domain of the function $f$. It can be shown that a continuous function $f$ fails to be differentiable at a point $x = a$ when the graph of $f$ makes an abrupt change of direction at that point. We call such a point a "corner." A function also fails to be differentiable at a point where the tangent line is vertical since the slope of a vertical line is undefined. These cases are illustrated in Figure 10.53.

<table>
<thead>
<tr>
<th>$x$ stands for</th>
<th>$y$ stands for</th>
<th>$\frac{f(a + h) - f(a)}{h}$ measures the</th>
<th>$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$ measures the</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td><strong>Volume of sales</strong> at time $x$</td>
<td>Average rate of change in the volume of sales over the time interval $[a, a + h]$</td>
<td>Instantaneous rate of change in the volume of sales at time $x = a$</td>
</tr>
<tr>
<td>Time</td>
<td><strong>Population of Drosophila</strong> (fruit flies) at time $x$</td>
<td>Average rate of growth of the fruit fly population over the time interval $[a, a + h]$</td>
<td>Instantaneous rate of change of the fruit fly population at time $x = a$</td>
</tr>
<tr>
<td>Temperature in a chemical reaction</td>
<td><strong>Amount of product formed in the chemical reaction</strong> when the temperature is $x$ degrees</td>
<td>Average rate of formation of chemical product over the temperature range $[a, a + h]$</td>
<td>Instantaneous rate of formation of chemical product when the temperature is $a$ degrees</td>
</tr>
</tbody>
</table>

The next example illustrates a function that is not differentiable at a point.

**Example 8**

Mary works at the B&O department store, where, on a weekday, she is paid $6 per hour for the first 8 hours and $9 per hour for overtime. The function

$$f(x) = \begin{cases} 6x & \text{if } 0 \leq x \leq 8 \\ 9x - 24 & \text{if } 8 < x \end{cases}$$

gives Mary’s earnings on a weekday in which she worked $x$ hours. Sketch the graph of the function $f$ and explain why it is not differentiable at $x = 8$.  

---

**Table 10.4 (continued)**

<table>
<thead>
<tr>
<th>$x$ stands for</th>
<th>$y$ stands for</th>
<th>$\frac{f(a + h) - f(a)}{h}$ measures the</th>
<th>$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$ measures the</th>
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<tr>
<td>Time</td>
<td><strong>Volume of sales</strong> at time $x$</td>
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<td>Temperature in a chemical reaction</td>
<td><strong>Amount of product formed in the chemical reaction</strong> when the temperature is $x$ degrees</td>
<td>Average rate of formation of chemical product over the temperature range $[a, a + h]$</td>
<td>Instantaneous rate of formation of chemical product when the temperature is $a$ degrees</td>
</tr>
</tbody>
</table>
The graph of \( f \) is shown in Figure 10.54. Observe that the graph of \( f \) has a corner at \( x = 8 \) and consequently is not differentiable at \( x = 8 \).

We close this section by mentioning the connection between the continuity and the differentiability of a function at a given value \( x = a \) in the domain of \( f \). By reexamining the function of Example 8, it becomes clear that \( f \) is continuous everywhere and, in particular, when \( x = 8 \). This shows that in general the continuity of a function at a point \( x = a \) does not necessarily imply the differentiability of the function at that point. The converse, however, is true: If a function \( f \) is differentiable at a point \( x = a \), then it is continuous there.

**Exploring with Technology**

1. Use a graphing utility to plot the graph of \( f(x) = x^{1/3} \) in the viewing rectangle \([-2, 2] \times [-2, 2] \).
2. Use the graphing utility to draw the tangent line to the graph of \( f \) at the point \((0, 0)\). Can you explain why the process breaks down?

**Differentiability and Continuity**

If a function is differentiable at \( x = a \), then it is continuous at \( x = a \).

For a proof of this result, see Exercise 59, page 684.

**Example 9**

Figure 10.55 depicts a portion of the graph of a function. Explain why the function fails to be differentiable at each of the points \( x = a, b, c, d, e, f, \) and \( g \).
The function fails to be differentiable at the points \( x = a, b, \) and \( c \) because it is discontinuous at each of these points. The derivative of the function does not exist at \( x = d, e, \) and \( f \) because it has a kink at each of these points. Finally, the function is not differentiable at \( x = g \) because the tangent line is vertical at that point.

**Group Discussion**

Suppose a function \( f \) is differentiable at \( x = a \). Can there be two tangent lines to the graphs of \( f \) at the point \((a, f(a))\)? Explain your answer.

**Self-Check Exercises 10.6**

1. Let \( f(x) = -x^2 - 2x + 3 \).
   a. Find the derivative \( f' \) of \( f \), using the definition of the derivative.
   b. Find the slope of the tangent line to the graph of \( f \) at the point \((0, 3)\).
   c. Find the rate of change of \( f \) when \( x = 0 \).
   d. Find an equation of the tangent line to the graph of \( f \) at the point \((0, 3)\).
   e. Sketch the graph of \( f \) and the tangent line to the curve at the point \((0, 3)\).

2. The losses (in millions of dollars) due to bad loans extended chiefly in agriculture, real estate, shipping, and energy by the Franklin Bank are estimated to be
   \[ A = f(t) = -t^2 + 10t + 30 \quad (0 \leq t \leq 10) \]
   where \( t \) is the time in years (\( t = 0 \) corresponds to the beginning of 1994). How fast were the losses mounting at the beginning of 1997? At the beginning of 1999? How fast will the losses be mounting at the beginning of 2001? Interpret your results.

Solutions to Self-Check Exercises 10.6 can be found on page 684.

**10.6 Exercises**

1. **Average Weight of an Infant** The following graph shows the weight measurements of the average infant from the time of birth (\( t = 0 \)) through age 2 (\( t = 24 \)). By computing the slopes of the respective tangent lines, estimate the rate of change of the average infant’s weight when \( t = 3 \) and when \( t = 18 \). What is the average rate of change in the average infant’s weight over the first year of life?

2. **Forestry** The following graph shows the volume of wood produced in a single-species forest. Here \( f(t) \) is
measured in cubic meters per hectare and \( t \) is measured in years. By computing the slopes of the respective tangent lines, estimate the rate at which the wood grown is changing at the beginning of year 10 and at the beginning of year 30.

Source: The Random House Encyclopedia

3. TV-Viewing Patterns The following graph shows the percentage of U.S. households watching television during a 24-hr period on a weekday (\( t = 0 \) corresponds to 6 A.M.). By computing the slopes of the respective tangent lines, estimate the rate of change of the percentage of households watching television at 4 P.M. and 11 P.M.

Source: A. C. Nielsen Company

4. Crop Yield Productivity and yield of cultivated crops are often reduced by insect pests. The following graph shows the relationship between the yield of a certain crop, \( f(x) \), as a function of the density of aphids \( x \). (Aphids are small insects that suck plant juices.) Here, \( f(x) \) is measured in kilograms/4000 square meters, and \( x \) is measured in hundreds of aphids per bean stem. By computing the slopes of the respective tangent lines, estimate the rate of change of the crop yield with respect to the density of aphids when that density is 200 aphids/bean stem and when it is 800 aphids/bean stem.

Source: The Random House Encyclopedia

5. The position of car \( A \) and car \( B \), starting out side by side and traveling along a straight road, is given by \( s = f(t) \) and \( s = g(t) \), respectively, where \( s \) is measured in feet and \( t \) is measured in seconds (see the accompanying figure).

a. Which car is traveling faster at \( t_1 \)?
b. What can you say about the speed of the cars at \( t_2 \)?

Hint: Compare tangent lines.
c. Which car is traveling faster at \( t_3 \)?
d. What can you say about the positions of the cars at \( t_2 \)?

6. The velocity of car \( A \) and car \( B \), starting out side by side and traveling along a straight road, is given by \( v = f(t) \) and \( v = g(t) \), respectively, where \( v \) is measured in feet/second and \( t \) is measured in seconds (see the accompanying figure).

a. What can you say about the velocity and acceleration of the two cars at \( t_1 \)? (Acceleration is the rate of change of velocity.)
b. What can you say about the velocity and acceleration of the two cars at \( t_2 \)?

7. Effect of a Bactericide on Bacteria In the following figure, \( f(t) \) gives the population \( P_1 \) of a certain bacteria culture at time \( t \) after a portion of bactericide \( A \) was introduced into the population at \( t = 0 \). The graph of \( g \) gives the population \( P_2 \) of a similar bacteria culture at time \( t \) after a portion of bactericide \( B \) was introduced into the population at \( t = 0 \).

a. Which population is decreasing faster at \( t_1 \)?
b. Which population is decreasing faster at \( t_2 \)?
In Exercises 9–16, use the four-step process to find the slope of the tangent line to the graph of each function at the given point and determine an equation of the tangent line.

9. \( f(x) = 13 \)   
10. \( f(x) = -6 \)
11. \( f(x) = 2x + 7 \)   
12. \( f(x) = 8 - 4x \)
13. \( f(x) = 3x^2 \)   
14. \( f(x) = -\frac{1}{2}x^2 \)
15. \( f(x) = -x^2 + 3x \)   
16. \( f(x) = 2x^2 + 5x \)

In Exercises 17–22, find the slope of the tangent line to the graph of each function at the given point and determine an equation of the tangent line.

17. \( f(x) = 2x + 7 \) at \((2, 11)\)
18. \( f(x) = -3x + 4 \) at \((-1, 7)\)
19. \( f(x) = 3x^2 \) at \((1, 3)\)
20. \( f(x) = 3x - x^2 \) at \((-2, -10)\)
21. \( f(x) = -\frac{1}{x} \) at \((3, -\frac{1}{3})\)
22. \( f(x) = \frac{3}{2x} \) at \((1, \frac{3}{2})\)

23. Let \( f(x) = 2x^2 + 1 \).
   a. Find the derivative \( f' \) of \( f \).
   b. Find an equation of the tangent line to the curve at the point \((1, 3)\).
   c. Sketch the graph of \( f \).

24. Let \( f(x) = x^2 + 6x \).
   a. Find the derivative \( f' \) of \( f \).
   b. Find the point on the graph of \( f \) where the tangent line to the curve is horizontal.
   Hint: Find the value of \( x \) for which \( f'(x) = 0 \).
   c. Sketch the graph of \( f \) and the tangent line to the curve at the point found in part (b).

25. Let \( f(x) = x^2 - 2x + 1 \).
   a. Find the derivative \( f' \) of \( f \).
   b. Find the point on the graph of \( f \) where the tangent line to the curve is horizontal.
   c. Sketch the graph of \( f \) and the tangent line to the curve at the point found in part (b).
   d. What is the rate of change of \( f \) at this point?

26. Let \( f(x) = \frac{1}{x - 1} \).
   a. Find the derivative \( f' \) of \( f \).
   b. Find an equation of the tangent line to the curve at the point \((-1, -\frac{1}{2})\).
   c. Sketch the graph of \( f \).

27. Let \( y = f(x) = x^2 + x \).
   a. Find the average rate of change of \( y \) with respect to \( x \) in the interval from \( x = 2 \) to \( x = 3 \), from \( x = 2 \) to \( x = 2.5 \), and from \( x = 2 \) to \( x = 2.1 \).
   b. Find the (instantaneous) rate of change of \( y \) at \( x = 2 \).
   c. Compare the results obtained in part (a) with that of part (b).
28. Let \( y = f(x) = x^2 - 4x \).
   a. Find the average rate of change of \( y \) with respect to \( x \) in the interval from \( x = 3 \) to \( x = 4 \), from \( x = 3 \) to \( x = 3.5 \), and from \( x = 3 \) to \( x = 3.1 \).
   b. Find the (instantaneous) rate of change of \( y \) at \( x = 3 \).
   c. Compare the results obtained in part (a) with that of part (b).

29. **Velocity of a Car** Suppose the distance \( s \) (in feet) covered by a car moving along a straight road \( t \) sec after starting from rest is given by the function \( f(t) = 2t^2 + 48t \).
   a. Calculate the average velocity of the car over the time intervals \([20, 21]\), \([20, 20.1]\), and \([20, 20.01]\).
   b. Calculate the (instantaneous) velocity of the car when \( t = 20 \).
   c. Compare the results of part (a) with that of part (b).

30. **Velocity of a Ball Thrown into the Air** A ball is thrown straight up with an initial velocity of 128 ft/sec, so that its height (in feet) after \( t \) sec is given by \( s(t) = 128t - 16t^2 \).
   a. What is the average velocity of the ball over the time intervals \([2, 3]\), \([2, 2.5]\), and \([2, 2.1]\)?
   b. What is the instantaneous velocity at time \( t = 2 \)?
   c. What is the instantaneous velocity at time \( t = 5 \)? Is the ball rising or falling at this time?
   d. When will the ball hit the ground?

31. During the construction of a high-rise building, a worker accidentally dropped his portable electric screwdriver from a height of 400 ft. After \( t \) sec, the screwdriver had fallen a distance of \( s = 16t^2 \) ft.
   a. How long did it take the screwdriver to reach the ground?
   b. What was the average velocity of the screwdriver between the time it was dropped and the time it hit the ground?
   c. What was the velocity of the screwdriver at the time it hit the ground?

32. A hot air balloon rises vertically from the ground so that its height after \( t \) sec is \( h = \frac{1}{2}t^2 + \frac{1}{3}t \) ft (0 ≤ \( t \) ≤ 60).
   a. What is the height of the balloon at the end of 40 sec?
   b. What is the average velocity of the balloon between \( t = 0 \) and \( t = 40 \)?
   c. What is the velocity of the balloon at the end of 40 sec?

33. At a temperature of 20°C, the volume \( V \) (in liters) of 1.33 g of O\(_2\) is related to its pressure \( p \) (in atmospheres) by the formula \( V = \frac{1}{p} \).
   a. What is the average rate of change of \( V \) with respect to \( p \) as \( p \) increases from \( p = 2 \) to \( p = 3 \)?
   b. What is the rate of change of \( V \) with respect to \( p \) when \( p = 2 \)?

34. **Cost of Producing Surfboards** The total cost \( C(x) \) (in dollars) incurred by the Aloha Company in manufacturing \( x \) surfboards a day is given by \( C(x) = -10x^2 + 300x + 130 \) (0 ≤ \( x \) ≤ 15).
   a. Find \( C'(x) \).
   b. What is the rate of change of the total cost when the level of production is ten surfboards a day?
   c. What is the average cost Aloha incurs in manufacturing ten surfboards a day?

35. **Effect of Advertising on Profit** The quarterly profit (in thousands of dollars) of Cunningham Realty is given by \( P(x) = \frac{1}{3}x^2 + 7x + 30 \) (0 ≤ \( x \) ≤ 50) where \( x \) (in thousands of dollars) is the amount of money Cunningham spends on advertising per quarter.
   a. Find \( P'(x) \).
   b. What is the rate of change of Cunningham’s quarterly profit if the amount it spends on advertising is $10,000/quarter (\( x = 10 \)) and $30,000/quarter (\( x = 30 \))?

36. **Demand for Tents** The demand function for the Sportsman X × 7 tents is given by \( p = f(x) = -0.1x^2 - x + 40 \) where \( p \) is measured in dollars and \( x \) is measured in units of a thousand.
   a. Find the average rate of change in the unit price of a tent if the quantity demanded is between 5000 and 5050 tents; between 5000 and 5010 tents.
   b. What is the rate of change of the unit price if the quantity demanded is 5000?

37. **A Country’s GDP** The gross domestic product (GDP) of a certain country is projected to be \( N(t) = t^2 + 2t + 50 \) (0 ≤ \( t \) ≤ 5) billion dollars \( t \) yr from now. What will be the rate of change of the country’s GDP 2 yr and 4 yr from now?

38. **Growth of Bacteria** Under a set of controlled laboratory conditions, the size of the population of a certain bacteria culture at time \( t \) (in minutes) is described by the function \( P = f(t) = 3t^2 + 2t + 1 \)
   Find the rate of population growth at \( t = 10 \) min.
In Exercises 39–43, let \( x \) and \( f(x) \) represent the given quantities. Fix \( x = a \) and let \( h \) be a small positive number. Give an interpretation of the quantities 
\[
\frac{f(a + h) - f(a)}{h} \quad \text{and} \quad \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

39. \( x \) denotes time, and \( f(x) \) denotes the population of seals at time \( x \).

40. \( x \) denotes time, and \( f(x) \) denotes the prime interest rate at time \( x \).

41. \( x \) denotes time, and \( f(x) \) denotes a country’s industrial production.

42. \( x \) denotes the level of production of a certain commodity, and \( f(x) \) denotes the total cost incurred in producing \( x \) units of the commodity.

43. \( x \) denotes altitude, and \( f(x) \) denotes atmospheric pressure.

In each of Exercises 44–49, the graph of a function is shown. For each function, state whether or not (a) \( f(x) \) has a limit at \( x = a \), (b) \( f(x) \) is continuous at \( x = a \), and (c) \( f(x) \) is differentiable at \( x = a \). Justify your answers.

44. \hspace{1cm} 45.

46. \hspace{1cm} 47.

48. \hspace{1cm} 49.

50. The distance \( s \) (in feet) covered by a motorcycle traveling in a straight line and starting from rest in \( t \) sec is given by the function 
\[
s(t) = -0.1t^3 + 2t^2 + 24t
\]
Calculate the motorcycle’s average velocity over the time interval \([2, 2 + h]\) for \( h = 1, 0.1, 0.01, 0.001, 0.0001, \) and \( 0.00001 \) and use your results to guess at the motorcycle’s instantaneous velocity at \( t = 2 \).

51. The daily total cost \( C(x) \) incurred by Trappee and Sons, Inc., for producing \( x \) cases of Texa-Pep hot sauce is given by 
\[
C(x) = 0.000002x^3 + 5x + 400
\]
Calculate 
\[
\frac{C(100 + h) - C(100)}{h}
\]
for \( h = 1, 0.1, 0.01, 0.001, \) and \( 0.0001 \) and use your results to estimate the rate of change of the total cost function when the level of production is 100 cases/day.

In Exercises 52 and 53, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

52. If \( f \) is continuous at \( x = a \), then \( f \) is differentiable at \( x = a \).

53. If \( f \) is continuous at \( x = a \) and \( g \) is differentiable at \( x = a \), then \( \lim_{x \to a} f(x)g(x) = f(a)g(a) \).

54. Sketch the graph of the function \( f(x) = |x + 1| \) and show that the function does not have a derivative at \( x = -1 \).
55. Sketch the graph of the function \( f(x) = 1/(x - 1) \) and show that the function does not have a derivative at \( x = 1 \).

56. Let

\[
f(x) = \begin{cases} 
  x^2 & \text{if } x \leq 1 \\
  ax + b & \text{if } x > 1
\end{cases}
\]

Find the values of \( a \) and \( b \) so that \( f \) is continuous and has a derivative at \( x = 1 \). Sketch the graph of \( f \).

57. Sketch the graph of the function \( f(x) = x^{2/3} \). Is the function continuous at \( x = 0 \)? Does \( f'(0) \) exist? Why, or why not?

58. Prove that the derivative of the function \( f(x) = |x| \) for \( x \neq 0 \) is given by

\[
f'(x) = \begin{cases} 
  1 & \text{if } x > 0 \\
  -1 & \text{if } x < 0
\end{cases}
\]

Hint: Recall the definition of the absolute value of a number.

59. Show that if a function \( f \) is differentiable at a point \( x = a \), then \( f \) must be continuous at that point.

Hint: Write

\[
f(x) - f(a) = \frac{|f(x) - f(a)|}{x - a} (x - a)
\]

Use the product rule for limits and the definition of the derivative to show that

\[
\lim_{x \to a} |f(x) - f(a)| = 0
\]

Solutions to Self-Check Exercises 10.6

1. a. \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \)

\[
= \lim_{h \to 0} \frac{-(x + h)^2 - 2(x + h) + 3 - (-x^2 - 2x + 3)}{h}
\]

\[
= \lim_{h \to 0} \frac{-x^2 - 2xh - h^2 - 2x - 2h + 3 + x^2 + 2x - 3}{h}
\]

\[
= \lim_{h \to 0} \frac{h(-2x - h - 2)}{h}
\]

\[
= \lim_{h \to 0} (-2x - h - 2) = -2x - 2
\]

b. From the result of part (a), we see that the slope of the tangent line to the graph of \( f \) at any point \((x, f(x))\) is given by

\[
f'(x) = -2x - 2
\]

In particular, the slope of the tangent line to the graph of \( f \) at \((0, 3)\) is

\[
f'(0) = -2
\]

c. The rate of change of \( f \) when \( x = 0 \) is given by \( f'(0) = -2 \), or \(-2\) units/unit change in \( x \).

d. Using the result from part (b), we see that an equation of the required tangent line is

\[
y - 3 = -2(x - 0)
\]

\[
y = -2x + 3
\]
2. The rate of change of the losses at any time \( t \) is given by

\[
f'(t) = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h}
= \lim_{h \to 0} \frac{[-(t + h)^2 + 10(t + h) + 30] - (-t^2 + 10t + 30)}{h}
= \lim_{h \to 0} \frac{-t^2 - 2th - h^2 + 10t + 10h + 30 + t^2 - 10t - 30}{h}
= \lim_{h \to 0} \frac{h(-2t - h + 10)}{h}
= \lim_{h \to 0} (-2t - h + 10)
= -2t + 10
\]

Therefore, the rate of change of the losses suffered by the bank at the beginning of 1997 \((t = 3)\) was

\[
f'(3) = -2(3) + 10 = 4
\]

That is, the losses were increasing at the rate of $4 million/year. At the beginning of 1999 \((t = 5)\),

\[
f'(5) = -2(5) + 10 = 0
\]

and we see that the growth in losses due to bad loans was zero at this point. At the beginning of 2001 \((t = 7)\),

\[
f'(7) = -2(7) + 10 = -4
\]

and we conclude that the losses will be decreasing at the rate of $4 million/year.

Group projects for this chapter can be found at the Brooks/Cole Web site:
http://www.brookscole.com/product/0534378420
Using Technology

Graphing a Function and Its Tangent Lines

We can use a graphing utility to plot the graph of a function $f$ and the tangent line at any point on the graph.

**Example 1**

Let $f(x) = x^2 - 4x$.

**a.** Find an equation of the tangent line to the graph of $f$ at the point $(3, -3)$.

**b.** Plot both the graph of $f$ and the tangent line found in part (a) on the same set of axes.

**Solution**

**a.** The slope of the tangent line at any point on the graph of $f$ is given by $f'(x)$. But from Example 4 (page 672) we find $f'(x) = 2x - 4$. Using this result, we see that the slope of the required tangent line is

$$f'(3) = 2(3) - 4 = 2$$

Finally, using the point-slope form of the equation of a line, we find that an equation of the tangent line is

$$y - (-3) = 2(x - 3)$$

$$y + 3 = 2x - 6$$

$$y = 2x - 9.$$  

**b.** The graph of $f$ in the standard viewing rectangle and the tangent line of interest are shown in Figure T1.

**Remark**

Some graphing utilities will draw both the graph of a function $f$ and the tangent line to the graph of $f$ at a specified point when the function and the specified value of $x$ are entered.
FINDING THE DERIVATIVE OF A FUNCTION AT A GIVEN POINT

The numerical derivative operation of a graphing utility can be used to give an approximate value of the derivative of a function for a given value of \( x \).

**EXAMPLE 2**

Let \( f(x) = \sqrt{x} \).

a. Use the numerical derivative operation of a graphing utility to find the derivative of \( f \) at \( (4, 2) \).
b. Find an equation of the tangent line to the graph of \( f \) at \( (4, 2) \).
c. Plot the graph of \( f \) and the tangent line on the same set of axes.

**SOLUTION**

a. Using the numerical derivative operation of a graphing utility, we find that

\[
f'(4) = \frac{1}{4}
\]

b. An equation of the required tangent line is

\[
y - 2 = \frac{1}{4}(x - 4)
\]

\[
y = \frac{1}{4}x + 1
\]

c. The graph of \( f \) and the tangent line in the viewing rectangle \([0, 15] \times [0, 4]\) is shown in Figure T2.

**FIGURE T2**

The graph of \( f(x) = \sqrt{x} \) and the tangent line \( y = \frac{1}{4}x + 1 \) in the viewing rectangle \([0, 15] \times [0, 4]\)
Exercises

In Exercises 1–10, (a) find an equation of the tangent line to the graph of \( f \) at the indicated point and (b) use a graphing utility to plot the graph of \( f \) and the tangent line on the same set of axes. Use a suitable viewing rectangle.

1. \( f(x) = 4x - 3; \ (2, 5) \)
2. \( f(x) = -2x + 5; \ (1, 3) \)
3. \( f(x) = 2x^2 + x; \ (-2, 6) \)
4. \( f(x) = -x^2 + 2x; \ (1, 1) \)
5. \( f(x) = 2x^3 + x - 3; \ (2, 7) \)
6. \( f(x) = -3x^2 + 2x - 1; \ (1, -2) \)
7. \( f(x) = x + \frac{1}{x}; \ (1, 2) \)
8. \( f(x) = x - \frac{1}{x}; \ (1, 0) \)
9. \( f(x) = \sqrt{x}; \ (4, 2) \)
10. \( f(x) = \frac{1}{\sqrt{x}}; \ (4, \frac{1}{2}) \)

In Exercises 11–20, (a) use the numerical derivative operation of a graphing utility to find the derivative of \( f \) for the given value of \( x \) (to two desired places of accuracy), (b) find an equation of the tangent line to the graph of \( f \) at the indicated point, and (c) plot the graph of \( f \) and the tangent line on the same set of axes. Use a suitable viewing rectangle.

11. \( f(x) = x^3 + x + 1; \ x = 1; \ (1, 3) \)
12. \( f(x) = -2x^3 + 3x^2 + 2; \ x = -1; \ (-1, 7) \)
13. \( f(x) = x^4 - 3x^2 + 1; \ x = 2; \ (2, 5) \)
14. \( f(x) = -x^4 + 3x + 1; \ x = 1; \ (1, 3) \)
15. \( f(x) = x - \sqrt[4]{x}; \ x = 4; \ (4, 2) \)
16. \( f(x) = x^{\frac{3}{2}} - x; \ x = 4; \ (4, 4) \)
17. \( f(x) = \frac{1}{x + 1}; \ x = 1; \ \left(1, \frac{1}{2}\right) \)
18. \( f(x) = \frac{x}{x + 1}; \ x = 3; \ \left(3, \frac{3}{4}\right) \)
19. \( f(x) = x\sqrt{x^2 + 1}; \ x = 2; \ (2, 2\sqrt{5}) \)
20. \( f(x) = \frac{x}{\sqrt{x^2 + 1}}; \ x = 1; \ (1, \frac{\sqrt{2}}{2}) \)
CHAPTER 10  Summary of Principal Formulas and Terms

Formulas
1. Average rate of change of $f$ over $[x, x + h]$
   or
   Slope of the secant line to the graph of $f$ through $(x, f(x))$ and $(x + h, f(x + h))$
   or
   Difference quotient
   \[
   f(x + h) - f(x) \over h
   \]
2. Instantaneous rate of change of $f$ at $(x, f(x))$
   or
   Slope of tangent line to the graph of $f$ at $(x, f(x))$ at $x$
   or
   Derivative of $f$
   \[
   \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
   \]

Terms
function rational function
domain power function
range supply function
independent variable demand function
dependent variable limit of a function
ordered pairs indeterminate form
graph of a function right-hand limit of a function
graph of an equation left-hand limit of a function
vertical-line test continuity of a function at a point
composite function secant line
polynomial function tangent line to the graph of $f$
linear function differentiable function
quadratic function

cubic function

CHAPTER 10  Review Exercises

1. Find the domain of each function:
   a. $f(x) = \sqrt{9 - x}$
   b. $f(x) = \frac{x + 3}{2x^2 - x - 3}$

2. Let $f(x) = 3x^2 + 5x - 2$. Find:
   a. $f(-2)$
   b. $f(a + 2)$
   c. $f(2a)$
   d. $f(a + h)$
3. Let \( y^2 = 2x + 1 \).
   a. Sketch the graph of this equation.
   b. Is \( y \) a function of \( x \)? Why?
   c. Is \( x \) a function of \( y \)? Why?

4. Sketch the graph of the function defined by
   \[
   f(x) = \begin{cases} 
   x + 1 & \text{if } x < 1 \\
   -x^2 + 4x - 1 & \text{if } x \geq 1
   \end{cases}
   \]

5. Let \( f(x) = 1/x \) and \( g(x) = 2x + 3 \). Find:
   a. \( f(x)g(x) \)
   b. \( f(x)/g(x) \)
   c. \( f(g(x)) \)
   d. \( g(f(x)) \)

**In Exercises 6–19, find the indicated limits, if they exist.**

6. \( \lim \limits_{x \to 0} (5x - 3) \)
7. \( \lim \limits_{x \to 1} (x^2 + 1) \)
8. \( \lim \limits_{x \to -1} (3x^2 + 4)(2x - 1) \)
9. \( \lim \limits_{x \to 3} \frac{x - 3}{x + 4} \)
10. \( \lim \limits_{x \to 3} \frac{x + 3}{x^2 - 9} \)
11. \( \lim \limits_{x \to 2} \frac{x^2 - 2x - 3}{x^2 + 5x + 6} \)
12. \( \lim \limits_{x \to 3} \sqrt[3]{2x^3 - 5} \)
13. \( \lim \limits_{x \to 3} \frac{4x - 3}{\sqrt{x} + 1} \)
14. \( \lim \limits_{x \to 1} \frac{x - 1}{x(x - 1)} \)
15. \( \lim \limits_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} \)
16. \( \lim \limits_{x \to 1} \frac{x^2}{x^2 - 1} \)
17. \( \lim \limits_{x \to 0} \frac{x + 1}{x} \)
18. \( \lim \limits_{x \to 0} \frac{3x^2 + 2x + 4}{x^2 - 3x + 1} \)
19. \( \lim \limits_{x \to -2} \frac{x^2}{x + 1} \)

20. Sketch the graph of the function
   \[
   f(x) = \begin{cases} 
   2x - 3 & \text{if } x \leq 2 \\
   -x + 3 & \text{if } x > 2
   \end{cases}
   \]
   and evaluate \( \lim \limits_{x \to 2} f(x) \), \( \lim \limits_{x \to 2} f(x) \), and \( \lim \limits_{x \to -2} f(x) \) at the point \( a = 2 \), if the limits exist.

21. Sketch the graph of the function
   \[
   f(x) = \begin{cases} 
   4 - x & \text{if } x \leq 2 \\
   x + 2 & \text{if } x > 2
   \end{cases}
   \]
   and evaluate \( \lim \limits_{x \to 2} f(x) \), \( \lim \limits_{x \to 2} f(x) \), and \( \lim \limits_{x \to -2} f(x) \) at the point \( a = 2 \), if the limits exist.

**In Exercises 22–25, determine all values of \( x \) for which each function is discontinuous.**

22. \( g(x) = \begin{cases} 
   x + 3 & \text{if } x \neq 2 \\
   0 & \text{if } x = 2
   \end{cases} \)
23. \( f(x) = \frac{3x + 4}{4x^2 - 2x - 2} \)
24. \( f(x) = \begin{cases} 
   \frac{1}{(x + 1)^2} & \text{if } x \neq -1 \\
   2 & \text{if } x = -1
   \end{cases} \)
25. \( f(x) = \frac{|2x|}{x} \)

26. Let \( y = x^2 + 2 \).
   a. Find the average rate of change of \( y \) with respect to \( x \) in the intervals \([1, 2],[1, 1.5],[1, 1.1]\).
   b. Find the (instantaneous) rate of change of \( y \) at \( x = 1 \).

27. Use the definition of the derivative to find the slope of the tangent line to the graph of the function \( f(x) = 3x + 5 \) at any point \( P(x, f(x)) \) on the graph.

28. Use the definition of the derivative to find the slope of the tangent line to the graph of the function \( f(x) = -1/x \) at any point \( P(x, f(x)) \) on the graph.

29. Use the definition of the derivative to find the slope of the tangent line to the graph of the function \( f(x) = \frac{3}{2}x + 5 \) at the point \((-2, 2)\) and determine an equation of the tangent line.

30. Use the definition of the derivative to find the slope of the tangent line to the graph of the function \( f(x) = -x^2 \) at the point \((2, -4)\) and determine an equation of the tangent line.

31. The graph of the function \( f \) is shown in the accompanying figure.
   a. Is \( f \) continuous at \( x = a \)? Why?
   b. Is \( f \) differentiable at \( x = a \)? Justify your answers.
32. Sales of a certain clock radio are approximated by the relationship \( S(x) = 6000x + 30,000 \) (0 \( x \leq 5 \)), where \( S(x) \) denotes the number of clock radios sold in year \( x \) (\( x = 0 \) corresponds to the year 1996). Find the number of clock radios expected to be sold in the year 2000.

33. A company’s total sales (in millions of dollars) are approximately linear as a function of time (in years). Sales in 1996 were $2.4 million, whereas sales in 2001 amounted to $7.4 million.
   a. Find an equation that gives the company’s sales as a function of time.
   b. What were the sales in 1999?

34. A company has a fixed cost of $30,000 and a production cost of $6 for each unit it manufactures. A unit sells for $10.
   a. What is the cost function?
   b. What is the revenue function?
   c. What is the profit function?
   d. Compute the profit (loss) corresponding to producing 3000 units.

35. Find the point of intersection of the two straight lines having the equations \( y = \frac{3}{2}x + 6 \) and \( 3x - 2y + 3 = 0 \).

36. The cost and revenue functions for a certain firm are given by \( C(x) = 12x + 20,000 \) and \( R(x) = 20x \), respectively. Find the company’s break-even point.

37. Given the demand equation \( 3x + p - 40 = 0 \) and the supply equation \( 2x - p + 10 = 0 \), where \( p \) is the unit price in dollars and \( x \) represents the quantity in units of a thousand, determine the equilibrium quantity and the equilibrium price.

38. Clark’s rule is a method for calculating pediatric drug dosages based on a child’s weight. If \( a \) denotes the adult dosage (in milligrams) and \( w \) is the weight of the child (in pounds), then the child’s dosage is given by

\[
D(w) = \frac{aw}{150}
\]

If the adult dose of a substance is 500 mg, how much should a child who weighs 35 lb receive?

39. The monthly revenue \( R \) (in hundreds of dollars) realized in the sale of Royal electric shavers is related to the unit price \( p \) (in dollars) by the equation

\[
R(p) = -\frac{1}{2}p^2 + 30p
\]

Find the revenue when an electric shaver is priced at $30.

40. The membership of the newly opened Venus Health Club is approximated by the function

\[
N(x) = 200(4 + x)^{\frac{1}{2}} \quad (1 \leq x \leq 24)
\]

where \( N(x) \) denotes the number of members \( x \) months after the club’s grand opening. Find \( N(0) \) and \( N(12) \) and interpret your results.

41. Psychologist L. L. Thurstone discovered the following model for the relationship between the learning time \( T \) and the length of a list \( n \):

\[
T = f(n) = An\sqrt{n} - b
\]

where \( A \) and \( b \) are constants that depend on the person and the task. Suppose that, for a certain person and a certain task, \( A = 4 \) and \( b = 4 \). Compute \( f(4), f(5), \ldots, f(12) \) and use this information to sketch the graph of the function \( f \). Interpret your results.

42. The monthly demand and supply functions for the Luminar desk lamp are given by

\[
p = d(x) = -1.1x^2 + 1.5x + 40
\]
\[
p = s(x) = 0.1x^2 + 0.5x + 15
\]

respectively, where \( p \) is measured in dollars and \( x \) in units of a thousand. Find the equilibrium quantity and price.

43. The Photo-Mart transfers movie films to videocassettes. The fees charged for this service are shown in the following table. Find a function \( C \) relating the cost \( C(x) \) to the number of feet \( x \) of film transferred. Sketch the graph of the function \( C \) and discuss its continuity.

<table>
<thead>
<tr>
<th>Length of Film in Feet, ( x )</th>
<th>Price ($) for Conversion</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 \leq x \leq 100 )</td>
<td>5.00</td>
</tr>
<tr>
<td>( 100 &lt; x \leq 200 )</td>
<td>9.00</td>
</tr>
<tr>
<td>( 200 &lt; x \leq 300 )</td>
<td>12.50</td>
</tr>
<tr>
<td>( 300 &lt; x \leq 400 )</td>
<td>15.00</td>
</tr>
<tr>
<td>( x &gt; 400 )</td>
<td>( 7 + 0.02x )</td>
</tr>
</tbody>
</table>

44. The average cost (in dollars) of producing \( x \) units of a certain commodity is given by

\[
\bar{C}(x) = 20 + \frac{400}{x}
\]

Evaluate \( \lim_{x \to \infty} \bar{C}(x) \) and interpret your results.