CHAPTER 3

Second Order Linear Differential Equations

3.1 Introduction; Basic Terminology and Results

Any second order differential equation can be written as

$$F(x, y, y', y'') = 0$$

This chapter is concerned with special yet very important second order equations, namely linear equations.

Recall that a first order linear differential equation is an equation which can be written in the form

$$y' + p(x)y = q(x)$$

where p and q are continuous functions on some interval I. A second order, linear differential equation has an analogous form.

DEFINITION 1. A *second order linear differential equation* is an equation which can be written in the form

$$y'' + p(x)y' + q(x)y = f(x) \tag{1}$$

where p, q, and f are continuous functions on some interval I.

The functions p and q are called the *coefficients* of the equation; the function f on the right-hand side is called the *forcing function* or the *nonhomogeneous term*. The term "forcing function" comes from applications of second-order linear equations; the description "nonhomogeneous" is given below.

A second order equation which is not linear is said to be *nonlinear*.

Examples

(a) $y'' - 5y' + 6y = 3\cos 2x$. Here $p(x) = -5$, $q(x) = 6$, $f(x) = 3\cos 2x$ are continuous functions on $(-\infty, \infty)$.

(b) $x^2 y'' - 2x\,y' + 2y = 0$. This equation is linear because it can be written in the form (1) as

$$y'' - \frac{2}{x}\,y' + \frac{2}{x^2}\,y = 0$$

where $p(x) = 2/x$, $q(x) = 2/x^2$, $f(x) = 0$ are continuous on any interval that does not contain $x = 0$. For example, we could take $I = (0, \infty)$.

(c) $y'' + xy^2y' - y^3 = e^{xy}$ is a nonlinear equation; this equation cannot be written in the form (1). ■

Remarks on "Linear." Intuitively, a second order differential equation is linear if y'' appears in the equation with exponent 1 only, and if either or both of y and y' appear in the equation, then they do so with exponent 1 only. Also, there are no so-called "cross-product" terms, $y\,y', y\,y'', y'\,y''$. In this sense, it is easy to see that the equations in (a) and (b) are linear, and the equation in (c) is nonlinear.

Set $L[y] = y'' + p(x)y' + q(x)y$. If we view L as an "operator" that transforms a twice differentiable function $y = y(x)$ into the continuous function

$$L[y(x)] = y''(x) + p(x)y'(x) + q(x)y(x),$$

then, for any two twice differentiable functions $y_1(x)$ and $y_2(x)$,

$$\begin{aligned} L[y_1(x) + y_2(x)] &= [y_1(x) + y_2(x)]'' + p(x)[y_1(x) + y_2(x)]' + q(x)[y_1(x) + y_2(x)] \\ &= y_1''(x) + y_2''(x) + p(x)[y_1'(x) + y_2'(x)] + q(x)[y_1(x) + y_2(x)] \\ &= y_1''(x) + p(x)y_1'(x) + q(x)y_1(x) + y_2''(x) + p(x)y_2'(x) + q(x) + y_2(x) \\ &= L[y_1(x)] + L[y_2(x)] \end{aligned}$$

and, for any constant c,

$$\begin{aligned} L[cy(x)] &= [cy(x)]'' + p(x)[cy(x)]' + q(x)[cy(x)] \\ &= cy''(x) + p(x)[cy'(x)] + cq(x)y(x) \\ &= c[y''(x) + p(x)y'(x) + q(x)y(x)] \\ &= cL[y(x)]. \end{aligned}$$

Therefore, as introduced in Section 2.1, L is a *linear differential operator.* This is the real reason that equation (1) is said to be a *linear* differential equation. ■

The first thing we need to know is that an initial-value problem has a solution, and that it is unique.

THEOREM 1. (Existence and Uniqueness Theorem) Given the second order linear equation (1). Let a be any point on the interval I, and let α and β be any two real numbers. Then the initial-value problem

$$y'' + p(x)\,y' + q(x)\,y = f(x), \quad y(a) = \alpha,\ y'(a) = \beta$$

has a unique solution.

As before, a proof of this theorem is beyond the scope of this course.

Remark: We can solve any first order linear differential equation; Section 2-1 gives a method for finding the general solution of *any* first order linear equation. In contrast, *there is no general method for solving second (or higher) order linear differential equations.* There are, however, methods for solving certain special types of second order linear equations and we shall study these in this chapter. Extensions of these methods to higher order linear equations will be given later. ■

DEFINITION 2. The linear differential equation (1) is *homogeneous* [1] if the function f on the right side of the equation is 0 for all $x \in I$. In this case, equation (1) becomes

$$y'' + p(x)\, y' + q(x)\, y = 0. \tag{2}$$

Equation (1) is *nonhomogeneous* if f is not the zero function on I, i.e., (1) is nonhomogeneous if $f(x) \neq 0$ for some $x \in I$.

As you will see in the work which follows, almost all of our attention will be focused on homogeneous equations.

[1]This use of the term "homogeneous" is completely different from its use to categorize the first order equation $y' = f(x, y)$ in Exercises 2.2.

3.2 Second Order Linear Homogeneous Equations

As defined in the previous section, a second order linear homogeneous differential equation is an equation that can be written in the form

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{H}$$

where p and q are continuous functions on some interval I.

The trivial solution The first thing to note is that the zero function, $y(x) = 0$ for all $x \in I$, (also denoted by $y \equiv 0$) is a solution of (H) ($y \equiv 0$ implies $y' \equiv 0$ and $y'' \equiv 0$). The zero solution is called the *trivial solution* . Obviously our main interest is in finding *nontrivial* solutions. Unless specified otherwise, the term "solution" will mean "nontrivial solution." ■

First we establish some essential facts about homogeneous equations.

THEOREM 1. If $y = y(x)$ is a solution of (H) and if C is any real number, then $u(x) = Cy(x)$ is also a solution of (H).

Proof Let $y = y(x)$ be a solution of (H). Then

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0.$$

Let C be any real number, and set $u(x) = Cy(x)$. Then

$$\begin{aligned} u(x) &= Cy(x) \\ u'(x) &= Cy'(x) \\ u''(x) &= Cy''(x) \end{aligned}$$

Substituting u into (H), we get

$$\begin{aligned} u''(x) + p(x)\, u'(x) + q(x)\, u(x) &= Cy''(x) + p(x)[Cy'(x)] + q(x)[Cy(x)] \\ &= C[y''(x) + p(x)\, y'(x) + q(x)\, y(x)] \\ &= C[0] \\ &= 0. \end{aligned}$$

Alternate Proof Consider the linear differential operator $L[y] = y'' + p(x)y' + q(x)y$. Since $y = y(x)$ is a solution of (H), $L[y(x)] = 0$. Since L is a linear operator,

$$L[Cy(x)] = C\, L[y(x)] = C(0) = 0.$$

Thus, $u(x) = Cy(x)$ is a solution of (H). ■

In words, Theorem 1 says that *any constant multiple of a solution of (H) is also a solution of (H).*

THEOREM 2. If $y = y_1(x)$ and $y = y_2(x)$ are any two solutions of (H), then $u(x) = y_1(x) + y_2(x)$ is also a solution of (H).

Proof Let $y = y_1(x)$ and $y = y_2(x)$ be any two solutions of (H). Then

$$y_1''(x) + p(x)y_1'(x) + q(x)y_1(x) = 0 \qquad \text{and} \qquad y_2''(x) + p(x)y_2'(x) + q(x)y_2(x) = 0.$$

Now set $u(x) = y_1(x) + y_2(x)$. Then

$$\begin{aligned} u(x) &= y_1(x) + y_2(x) \\ u'(x) &= y_1'(x) + y_2'(x) \\ u''(x) &= y_1''(x) + y_2''(x) \end{aligned}$$

Substituting u into (H), we get

$$\begin{aligned} u''(x) + p(x)\,u'(x) + q(x)\,u(x) &= y_1''(x) + y_2''(x) + p(x)[y_1'(x) + y_2'(x)] + q(x)[y_1(x) + y_2(x)] \\ &= [y_1''(x) + p(x)y_1'(x) + q(x)y_1(x)] + [y_2''(x) + p(x)y_2'(x) + q(x)y_2(x)] \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

Alternate Proof Set $L[y] = y'' + p(x)y' + q(x)y$; L is a linear operator. Since $y = y_1(x)$ and $y = y_2(x)$ are solutions of (H), $L[y_1(x)] = L[y_2(x)] = 0$. Since L is linear,

$$L[y_1(x) + y_2(x)] = L[y_1(x)] + L[y_2(x)] = 0 + 0 = 0.$$

Thus, $u(x) = y_1(x) + y_2(x)$ is a solution of (H). ■

Theorem 2 says that *the sum of any two solutions of (H) is also a solution of (H).* (Some authors call this property the *superposition principle.*)

Combining Theorems 1 and 2, we get

THEOREM 3. If $y = y_1(x)$ and $y = y_2(x)$ are any two solutions of (H), and if C_1 and C_2 are any two real numbers, then

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

is also a solution of (H).

DEFINITION 1. (**Linear Combinations**) Let $f = f(x)$ and $g = g(x)$ be functions defined on some interval I, and let C_1 and C_2 be real numbers. The expression

$$C_1 f(x) + C_2 g(x)$$

is called a *linear combination* of f and g.

Theorem 3 says that *any linear combination of solutions of (H) is also a solution of (H).*

Note that the equation

$$y(x) = C_1 y_1(x) + C_2 y_2(x) \tag{1}$$

where C_1 and C_2 are arbitrary constants, has the form of a general solution of equation (H). So the question is: If y_1 and y_2 are solutions of (H), is the expression (1) the general solution of (H)? That is, can every solution of (H) be written as a linear combination of y_1 and y_2? It turns out that (1) may or not be the general solution; it depends on the relation between the solutions y_1 and y_2.

Example 1. As you can verify, $y_1(x) = e^x$ and $y_2(x) = e^{2x}$ are each solutions of

$$y'' - 3y' + 2y = 0. \tag{a}$$

We want to determine whether or not the two-parameter family

$$y = C_1 e^x + C_2 e^{2x} \tag{b}$$

is the general solution of (a).

Let $u = u(x)$ be *any* solution of (a) and let $\alpha = u(0), \beta = u'(0)$. We will try to find values for C_1 and C_2 such that $y(x) = C_1 e^x + C_2 e^{2x}$ satisfies $y(0) = \alpha$ and $y'(0) = \beta$. We have

$$y(x) = C_1 e^x + C_2 e^{2x}, \quad y'(x) = C_1 e^x + 2C_2 e^{2x}$$

Setting $x = 0$, we obtain the pair of equations

$$\begin{aligned} y(0) = C_1 + C_2 &= \alpha \\ y'(0) = C_1 + 2C_2 &= \beta. \end{aligned}$$

This pair of equations has the unique solution $C_1 = 2\alpha - \beta,\ C_2 = \beta - \alpha$. Now,

$$y = (2\alpha - \beta)e^x + (\beta - \alpha)e^{2x} \quad \text{and} \quad u(x)$$

are each solutions of (a), and $y(0) = u(0) = \alpha,\ y'(0) = u'(0) = \beta$. By the Existence and Uniqueness Theorem, $u(x) \equiv y(x)$. Thus

$$u(x) = (2\alpha - \beta)e^x + (\beta - \alpha)e^{2x}$$

is a member of the two-parameter family (b). Since u was *any* solution of (a), we can conclude that (b) is the general solution of (a); (b) represents *all* solutions of (a). ■

Example 2. The functions $y_1(x) = x^2$ and $y_2(x) = 5x^2$ are solutions of

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0. \tag{a}$$

We want to determine whether or not the two-parameter family

$$y = C_1x^2 + C_2(5x^2) \tag{b}$$

is the general solution of (a).

By the Existence and Uniqueness Theorem there exists a unique solution u of (a) such that $u(1) = 1,\ u'(1) = 0$. If we try to find values for $C_1,\ C_2$ such that $y(1) = 1,\ y'(1) = 0$ we obtain the pair of equations:

$$\begin{aligned} y(1) = C_1 + 5C_2 &= 1 \\ y'(1) = 2C_1 + 10C_2 &= 0 \end{aligned}$$

There is no solution to this pair of equations. Therefore u is not a member of the two-parameter family (b) and (b) is not the general solution of (a)

The problem here is that y_1 and y_2 are constant multiples of each other $(y_2 = 5y_1;$ or $y_1 = y_2/5)$. Notice that while (b) "appears" to be a two-parameter family, it is, in fact, a one-parameter family:

$$y = C_1x^2 + C_2(5x^2) = (C_1 + 5C_2)x^2 = Kx^2.$$

You can verify that $y_3(x) = x$ is a solution of (a) which is "different" from y_1 (i.e., not a constant multiple of y_1), and that

$$y = C_1x + C_2x^2$$

is the general solution of (a). ■

Let's consider the problem in general. Suppose that $y = y_1(x)$ and $y = y_2(x)$ are solutions of equation (H). Under what conditions is (1) the general solution of (H)?

Let $u = u(x)$ be *any* solution of (H) and choose *any* point $a \in I$. Suppose that $\alpha = u(a),\ \beta = u'(a)$. Then u is a member of the two-parameter family (1) if and only if there are values for C_1 and C_2 such that

$$\begin{aligned} C_1y_1(a) + C_2y_2(a) &= \alpha \\ C_1y_1'(a) + C_2y_2'(a) &= \beta \end{aligned}$$

If we multiply the first equation by $y_2'(a)$, the second equation by $-y_2(a)$, and add, we get

$$[y_1(a)y_2'(a) - y_2(a)y_1'(a)]C_1 = \alpha y_2'(a) - \beta y_2(a).$$

Similarly, if we multiply the first equation by $-y_1'(a)$, the second equation by $y_1(a)$, and add, we get

$$[y_1(a)y_2'(a) - y_2(a)y_1'(a)]C_2 = -\alpha y_1'(a) + \beta y_1(a).$$

We are guaranteed that this pair of equations has solutions $C_1,\ C_2$ if and only if

$$y_1(a)y_2'(a) - y_2(a)y_1'(a) \neq 0$$

in which case

$$C_1 = \frac{\alpha y_2'(a) - \beta y_2(a)}{y_1(a)y_2'(a) - y_2(a)y_1'(a)} \quad \text{and} \quad C_2 = \frac{-\alpha y_1'(a) + \beta y_1(a)}{y_1(a)y_2'(a) - y_2(a)y_1'(a)}.$$

Since a was chosen to be any point on I, we conclude that (2) is the general solution of (H) if

$$y_1(x)y_2'(x) - y_2(x)y_1'(x) \neq 0 \quad \text{for all} \quad x \in I.$$

DEFINITION 2. (Wronskian) Let $y = y_1(x)$ and $y = y_2(x)$ be solutions of (H). The function W defined by

$$W[y_1, y_2](x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

is called the *Wronskian* of $y_1,\ y_2$.

We use the notation $W[y_1, y_2](x)$ to emphasize that the Wronskian is a function of x that is determined by two solutions y_1, y_2 of equation (H). When there is no danger of confusion, we'll shorten the notation to $W(x)$.

Remark There is a short-hand way to represent the Wronskian of two solutions of equation (H) using a 2×2 determinant. Determinants will be defined and discussed in general in Chapter 5. For now

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x). \qquad \blacksquare$$

Example 3. From Example 1, the functions $y_1(x) = e^x$ and $y_2(x) = e^{2x}$ are each solutions of

$$y'' - 3y' + 2y = 0.$$

Their Wronskian is:

$$W(x) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^x(2e^{2x}) - e^{2x}(e^x) = e^{3x} \neq 0 \text{ for all } x \in (-\infty, \infty).$$

From Example 2, the functions $y_1(x) = x^2$ and $y_2(x) = 5x^2$ are solutions of

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0.$$

Their Wronskian is:

$$W(x) = \begin{vmatrix} x^2 & 5x^2 \\ 2x & 10x \end{vmatrix} = x^2(10x) - 2x(5x^2) = 10x^2 - 10x^2 \equiv 0.$$

Also from Example 2, the functions $y_1(x) = x^2$ and $y_3(x) = x$ are solutions of

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0.$$

Their Wronskian is:

$$W(x) = \begin{vmatrix} x^2 & x \\ 2x & 1 \end{vmatrix} = x^2(1) - 2x(x) = -x^2 \neq 0 \text{ for all } x \in (0, \infty). \qquad \blacksquare$$

Here is the general result.

THEOREM 4. Let $y = y_1(x)$ and $y = y_2(x)$ be solutions of equation (H), and let $W(x)$ be their Wronskian. Exactly one of the following holds:

(i) $W(x) = 0$ for all $x \in I$; y_1 is a constant multiple of y_2 and vice versa.

(ii) $W(x) \neq 0$ for all $x \in I$ and $y = C_1y_1(x) + C_2y_2(x)$ is the general solution of (H)

Proof Let $y = y_1(x)$ and $y = y_2(x)$ be solutions of equation (H). Then

$$y_1'' + py_1' + qy_1 = 0 \quad \text{which implies} \quad y_1'' = -py_1' - qy_1$$

$$y_2'' + py_2' + qy_2 = 0 \quad \text{which implies} \quad y_2'' = -py_2' - qy_2$$

Set $W = y_1y_2' - y_2y_1'$. Then

$$\begin{aligned} W' = y_1y_2'' + y_2'y_1' - y_2y_1'' - y_1'y_2' &= y_1y_2'' - y_2y_1'' \\ &= y_1\left[-py_2' - qy_2\right] - y_2\left[-py_1' - qy_1\right] \\ &= -p\left[y_1y_2' - y_2y_1'\right] = -pW \end{aligned}$$

which implies that

$$W' + p(x)W = 0.$$

Therefore W is a solution of the first order linear equation

$$y' + p(x)y = 0.$$

Now, as we showed in Section 2.1,

$$W(x) = C\,e^{-\int p(x)\,dx}, \quad \text{for some constant } C.$$

If $C = 0$, then $W(x) = 0$ for all $x \in I$; if $C \neq 0$, then $W(x) \neq 0$ for all $x \in I$.

We have already shown that if $W(x) \neq 0$ for all $x \in I$, then (1) is the general solution of (H). We leave it as an exercise (Exercise 25) to show that if $W \equiv 0$ on I then y_1 is a constant multiple of y_2 (and vice versa). $\blacksquare$

Example 4. Finishing Example 3, $y = C_1\, e^x + C_2\, e^{2x}$ is the general solution of

$$y'' - 3y' + 2y = 0;$$

$y = C_1\, x^2 + C_2\, x$ is the general solution of

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0. \qquad \blacksquare$$

DEFINITION 3. (Fundamental Set) A pair of solutions $y = y_1(x),\ y = y_2(x)$ of equation (H) forms a *fundamental set of solutions* if

$$W[y_1, y_2](x) \neq 0 \quad \text{for all } x \in I.$$

Linear Dependence; Linear Independence

By Theorem 4, if y_1 and y_2 are solutions of equation (H) such that $W[y_1, y_2] \equiv 0$, then y_1 is a constant multiple of y_2. The question as to whether or not one function is a multiple of another function and the consequences of this are of fundamental importance in differential equations and in linear algebra. We introduce the concept here; we will deal with it in more generality later.

In this sub-section we are dealing with functions in general, not just solutions of the differential equation (H)

DEFINITION 4. (Linear Dependence; Linear Independence) Given two functions $f = f(x),\ g = g(x)$ defined on an interval I. The functions f and g are *linearly dependent on* I if one of the functions is a constant multiple of the other. That is, f and g are linearly dependent on I if there exists a number λ such that $g(x) = \lambda f(x)$ for all $x \in I$, or if there is a number γ such that $f(x) = \gamma\, g(x)$ for all $x \in I$. The functions f and g are *linearly independent on* I if they are not linearly dependent.

Remark The case where one of the functions is 0 is special: If either f or g is the zero function, then f and g are linearly dependent. For example, suppose $g \equiv 0$, then $g = 0 \cdot f$ is a multiple of f. $\blacksquare$

The term Wronskian defined above for two solutions of equation (H) can be extended to any two differentiable functions f and g. Let $f = f(x)$ and $g = g(x)$ be differentiable functions on an interval I. The function $W[f, g]$ defined by

$$W[f, g](x) = f(x)g'(x) - g(x)f'(x)$$

is called the *Wronskian* of f, g.

There is a connection between linear dependence/independence and Wronskian.

THEOREM 5. Let $f = f(x)$ and $g = g(x)$ be differentiable functions on an interval I. If f and g are linearly dependent on I, then $W(x) = 0$ for all $x \in I$ ($W \equiv 0$ on I).

Proof If f and g are linearly dependent on I, then there exists a number λ such that $g(x) = \lambda f(x)$ on I. Since $g'(x) = \lambda f'(x)$, we have

$$\begin{aligned} W(x) = f(x)g'(x) - g(x)f'(x) &= f(x)[\lambda f'(x)] - [\lambda f(x)]f'(x) \\ &= \lambda f(x)f'(x) - \lambda f(x)f'(x) = 0 \quad \text{for all } x \in I. \qquad \blacksquare \end{aligned}$$

This theorem can be stated equivalently as: Let $f = f(x)$ and $g = g(x)$ be differentiable functions on an interval I. If $W(x) \neq 0$ for at least one $x \in I$, then f and g are linearly independent on I.

Going back to differential equations, Theorem 4 can be restated as

Theorem 4′ Let $y = y_1(x)$ and $y = y_2(x)$ be solutions of equation (H). Exactly one of the following holds:

(i) $W(x) = 0$ for all $x \in I$; y_1 and y_2 are linear dependent.

(ii) $W(x) \neq 0$ for all $x \in I$; y_1 and y_2 are linearly independent and $y = C_1y_1(x) + C_2y_2(x)$ is the general solution of (H).

The statements "$y_1(x)$, $y_2(x)$ form a fundamental set of solutions of (H)" and "$y_1(x)$, $y_2(x)$ are linearly independent solutions of (H)" are synonymous.

Exercises 3.2

Verify that the functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions of the equation?

1. $y'' - y' - 6y = 0$; $y_1(x) = e^{3x}$, $y_2(x) = e^{-2x}$.
2. $y'' - 9y = 0$; $y_1(x) = e^{3x}$, $y_2(x) = e^{-3x}$.
3. $y'' + 9y = 0$; $y_1(x) = \cos 3x$, $y_2(x) = \sin 3x$.
4. $y'' - 4y' + 4y = 0$; $y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$.
5. $x^2y'' - x(x+2)y' + (x+2)y = 0$; $y_1(x) = x$, $y_2(x) = xe^{x}$.
6. Given the differential equation $y'' - 3y' - 4y = 0$.

   (a) Find two values of r such that $y = e^{rx}$ is a solution of the equation.

(b) Determine a fundamental set of solutions and give the general solution of the equation.

(c) Find the solution of the equation satisfying the initial conditions $y(0) = 1,\ y'(0) = 0$.

7. Given the differential equation $y'' - \left(\dfrac{2}{x}\right) y' - \left(\dfrac{4}{x^2}\right) y = 0$.

(a) Find two values of r such that $y = x^r$ is a solution of the equation.

(b) Determine a fundamental set of solutions and give the general solution of the equation.

(c) Find the solution of the equation satisfying the initial conditions $y(1) = 2,\ y'(1) = -1$.

(d) Find the solution of the equation satisfying the initial conditions $y(2) = y'(2) = 0$.

8. Given the differential equation $(x^2 + 2x - 1)y'' - 2(x+1)y' + 2y = 0$.

(a) Show that the equation has a linear polynomial and a quadratic polynomial as solutions.

b Find two linearly independent solutions of the equation and give the general solution.

Show that the given functions are linearly independent on the interval I and find a second-order linear homogeneous equation having the pair as a fundamental set of solutions.

9. $y_1(x) = e^{3x},\ y_2(x) = e^{-x};\quad I = (-\infty, \infty)$.

10. $y_1(x) = e^{-x},\ y_2(x) = xe^{-x};\quad I = (-\infty, \infty)$.

11. $y_1(x) = 1,\ y_2(x) = x;\quad I = (0, \infty)$.

12. $y_1(x) = \cos 2x,\ y_2(x) = \sin 2x;\quad I = (-\infty, \infty)$.

13. $y_1(x) = x,\ y_2(x) = x^2;\quad I = (0, \infty)$.

14. $y_1(x) = x,\ y_2(x) = x \ln x;\quad I = (0, \infty)$.

15. Let $y = y_1(x)$ be a solution of (H): $y'' + p(x)y' + q(x)y = 0$ where p and q are continuous function on an interval I. Let $a \in I$ and assume that $y_1(x) \neq 0$ on I. Set

$$y_2(x) = y_1(x) \int_a^x \frac{e^{-\int_a^t p(u)\,du}}{y_1^2(t)}\,dt.$$

Show that y_2 is a solution of (H) and that y_1 and y_2 are linearly independent.

Use Exercise 15 to find a fundamental set of solutions of the given equation starting from the given solution y_1.

16. $y'' - 6y' + 9y = 0; \quad y_1(x) = e^{3x}$.

17. $y'' - \dfrac{2}{x}y' + \dfrac{2}{x^2}y = 0; \quad y_1(x) = x$.

18. $y'' - \dfrac{1}{x}y' + \dfrac{1}{x^2}y = 0; \quad y_1(x) = x$.

19. $y'' - \dfrac{1}{x}y' - 4x^2 y = 0; \quad y_1(x) = e^{x^2}$.

20. $y'' - \dfrac{2x-1}{x}y' + \dfrac{x-1}{x}y = 0; \quad y_1(x) = e^{x}$.

21. Let $y = y_1(x)$ and $y = y_2(x)$ be solutions of equation (H) on an interval I. Let $a \in I$ and suppose that

$$y_1(a) = \alpha,\ y_1'(a) = \beta \qquad \text{and} \qquad y_2(a) = \gamma,\ y_2'(a) = \delta.$$

Under what conditions on $\alpha,\ \beta,\ \gamma, \delta$ will the functions y_1 and y_2 be linearly independent on I?

22. Suppose that the functions y_1 and y_2 are linearly independent solutions of (H). Does it follow that c_1y_1 and c_2y_2 are also linearly independent solutions of (H)? If not, why not.

23. Suppose that the functions y_1 and y_2 are linearly independent solutions of (H). Prove that $y_3 = y_1 + y_2$ and $y_4 = y_1 - y_2$ are also linearly independent solutions of (H). Conversely, prove that if y_3 and y_4 are linearly independent solutions of (H), then y_1 and y_2 are linearly independent solutions of (H).

24. Suppose that the functions y_1 and y_2 are linearly independent solutions of (H). Under what conditions will the functions $y_3 = \alpha y_1 + \beta y_2$ and $y_4 = \gamma y_1 + \delta y_2$ be linearly independent solutions of (H)?

25. Suppose that $y = y_1(x)$ and $y = y_2(x)$ are solutions of (H). Show that if $y_1(x) \neq 0$ on I and $W[y_1, y_2](x) \equiv 0$ on I, then $y_2(x) = \lambda y_1(x)$ on I.

3.3 Homogeneous Equations with Constant Coefficients

We emphasized in Sections 3.1 and 3.2 that there are no general methods for solving second (or higher) order linear differential equations. However, there are some special cases for which solution methods do exist. In this and the following sections we consider such a case, linear equations with constant coefficients. In this section we treat homogeneous equations; nonhomogeneous equations will be treated in the next two sections.

A *second order linear homogeneous differential equation with constant coefficients* is an equation which can be written in the form

$$y'' + ay' + by = 0 \tag{1}$$

where a and b are real numbers.

You have seen that the function $y = e^{-ax}$ is a solution of the first-order linear equation

$$y' + ay = 0. \quad \text{(the model for exponential growth and decay)}$$

This suggests the possibility that equation (1) may also have an exponential function $y = e^{rx}$ as a solution.

If $y = e^{rx}$, then $y' = r\,e^{rx}$ and $y'' = r^2\,e^{rx}$. Substitution into (1) gives

$$r^2\,e^{rx} + a\,(r\,e^{rx}) + b\,(e^{rx}) = e^{rx}\left(r^2 + ar + b\right) = 0.$$

Since $e^{rx} \neq 0$ for all x, we conclude that $y = e^{rx}$ is a solution of (1) if and only if

$$r^2 + ar + b = 0. \tag{2}$$

Thus, if r is a root of the quadratic equation (2), then $y = e^{rx}$ is a solution of equation (1); we can find solutions of (1) by finding the roots of the quadratic equation (2).

DEFINITION 1. Given the differential equation (1). The corresponding quadratic equation

$$r^2 + ar + b = 0$$

is called the *characteristic equation of* (1); the quadratic polynomial $r^2 + ar + b$ is called the *characteristic polynomial.* The roots of the characteristic equation are called the *characteristic roots* .

The nature of the solutions of the differential equation (1) depends on the nature of the roots of its characteristic equation (2). There are three cases to consider:

(1) Equation (2) has two, distinct real roots, $r_1 = \alpha,\ r_2 = \beta$.

(2) Equation (2) has only one real root, $r = \alpha$.

(3) Equation (2) has complex conjugate roots, $r_1 = \alpha + i\,\beta,\ r_2 = \alpha - i\,\beta,\ \beta \neq 0.$

Case I: The characteristic equation has two, distinct real roots, $r_1 = \alpha,\ r_2 = \beta.$ In this case,

$$y_1(x) = e^{\alpha x} \qquad \text{and} \qquad y_2(x) = e^{\beta x}$$

are solutions of (1). Since $\alpha \neq \beta$, y_1 and y_2 are not constant multiples of each other, the pair y_1, y_2 forms a fundamental set of solutions of equation (1) and

$$y = C_1\, e^{\alpha x} + C_2\, e^{\beta x}$$

is the general solution.

Note: We can use the Wronskian to verify the independence of y_1 and y_2:

$$W(x) = \begin{vmatrix} e^{\alpha x} & e^{\beta x} \\ \alpha\, e^{\alpha x} & \beta\, e^{\beta x} \end{vmatrix} = e^{\alpha x}\left(\beta\, e^{\beta x}\right) - e^{\beta x}\,(\alpha\, e^{\alpha x}) = (\alpha - \beta)\, e^{(\alpha+\beta)x} \neq 0. \qquad \blacksquare$$

Example 1. Find the general solution of the differential equation

$$y'' + 2y' - 8y = 0.$$

SOLUTION The characteristic equation is

$$\begin{aligned} r^2 + 2r - 8 &= 0 \\ (r+4)(r-2) &= 0 \end{aligned}$$

The characteristic roots are: $r_1 = -4,\ r_2 = 2.$ The functions $y_1(x) = e^{-4x},\ y_2(x) = e^{2x}$ form a fundamental set of solutions of the differential equation and

$$y = C_1\, e^{-4x} + C_2\, e^{2x}$$

is the general solution of the equation. $\blacksquare$

Example 2. Find a linearly independent pair of solutions of

$$y'' + 3y' = 0.$$

and give the general solution of the equation.

SOLUTION The characteristic equation is $r^2 + 3r = r(r+3) = 0$, and the characteristic roots are $r_1 = 0,\ r_2 = -3.$ Therefore the functions $y_1(x) = e^{0x} \equiv 1$ and $y_2(x) = e^{-3x}$ are linearly independent solutions of the differential equation.

The general solution is

$$y = C_1(1) + C_2\, e^{-3x} = C_1 + C_2\, e^{-3x}. \qquad \blacksquare$$

Case II: The characteristic equation has only one real root, $r = \alpha$.[2] Then

$$y_1(x) = e^{\alpha x} \qquad \text{and} \qquad y_2(x) = x\, e^{\alpha x}$$

are linearly independent solutions of equation (1) and

$$y = C_1\, e^{\alpha x} + C_2\, x\, e^{\alpha x}$$

is the general solution.

Proof: We know that $y_1(x) = e^{\alpha x}$ is one solution of the differential equation; we need to find another solution which is independent of y_1. Since the characteristic equation has only one real root, α, the equation must be

$$r^2 + ar + b = (r - \alpha)^2 = r^2 - 2\alpha r + \alpha^2 = 0$$

and the differential equation (1) must have the form

$$y'' - 2\alpha\, y' + \alpha^2 y = 0. \qquad (*)$$

Now, $z = C\, e^{\alpha x}$, C any constant, is also a solution of $(*)$, but z is not independent of y_1 since it is simply a multiple of y_1. We replace C by a function u which is to be determined (if possible) so that $y = ue^{\alpha x}$ is a solution of $(*)$.[3] Calculating the derivatives of y, we have

$$\begin{aligned} y &= u\, e^{\alpha x} \\ y' &= \alpha\, u\, e^{\alpha x} + u'\, e^{\alpha x} \\ y'' &= \alpha^2 u\, e^{\alpha x} + 2\alpha\, u'\, e^{\alpha x} + u''\, e^{\alpha x} \end{aligned}$$

Substitution into $(*)$ gives

$$\alpha^2 u\, e^{\alpha x} + 2\alpha\, u'\, e^{\alpha x} + u''\, e^{\alpha x} - 2\alpha \left[\alpha\, u\, e^{\alpha x} + u'\, e^{\alpha x}\right] + \alpha^2\, u\, e^{\alpha x} = 0.$$

This reduces to

$$u''\, e^{\alpha x} = 0 \qquad \text{which implies} \qquad u'' = 0 \qquad \text{since } e^{\alpha x} \neq 0.$$

Now, $u'' = 0$ is the simplest second order, linear differential equation with constant coefficients; the general solution is $u = C_1 + C_2 x = C_1 \cdot 1 + C_2 \cdot x$, and $u_1(x) = 1$ and $u_2(x) = x$ form a fundamental set of solutions.

Since $y = u\, e^{\alpha x}$, we conclude that

$$y_1(x) = 1 \cdot e^{\alpha x} = e^{\alpha x} \qquad \text{and} \qquad y_2(x) = x\, e^{\alpha x}$$

[2] In this case, α is said to be a *double root* of the characteristic equation.

[3] This is an application of a general method called *variation of parameters.* We will use the method several times in the work that follows.

are solutions of $(*)$. In particular, $y_2 = x\,e^{\alpha x}$ is a solution of $(*)$ which is independent of $y_1 = e^{\alpha x}$. That is, y_1 and y_2 form a fundamental set of solutions of $(*)$. This can also be checked by using the Wronskian:

$$W(x) = \begin{vmatrix} e^{\alpha x} & xe^{\alpha x} \\ \alpha\, e^{\alpha x} & e^{\alpha x} + \alpha\, xe^{\alpha x} \end{vmatrix} = e^{\alpha x}\,[e^{\alpha x} + \alpha\, x\, e^{\alpha x}] - \alpha\, x\, e^{\alpha x} = e^{2\alpha x} \neq 0.$$

Finally, the general solution of $(*)$ is

$$y = C_1\, e^{\alpha x} + C_2\, x\, e^{\alpha x}.$$

Note: The solution $y_2(x) = xe^{\alpha x}$ can also be obtained by using Problem 15 in Exercises 3.2. ■

Example 3. Find the general solution of the differential equation

$$y'' - 6y' + 9y = 0.$$

SOLUTION The characteristic equation is

$$\begin{aligned} r^2 - 6r + 9 &= 0 \\ (r-3)^2 &= 0 \end{aligned}$$

There is only one characteristic root: $r_1 = r_2 = 3$. The functions $y_1(x) = e^{3x},\ y_2(x) = x\,e^{3x}$ are linearly independent solutions of the differential equation and

$$y = C_1\, e^{3x} + C_2\, x\, e^{3x}$$

is the general solution. ■

Case III: The characteristic equation has complex conjugate roots:

$$r_1 = \alpha + i\,\beta, \quad r_2 = \alpha - i\,\beta, \quad \beta \neq 0$$

In this case

$$y_1(x) = e^{\alpha x}\cos\beta x \qquad \text{and} \qquad y_2(x) = e^{\alpha x}\sin\beta x$$

are linearly independent solutions of equation (1) and

$$y = C_1\, e^{\alpha x}\cos\beta x + C_2\, e^{\alpha x}\sin\beta x = e^{\alpha x}\,[C_1\cos\beta x + C_2\sin\beta x]$$

is the general solution.

Proof: It is true that the functions $z_1(x) = e^{(\alpha+i\beta)x}$ and $z_2(x) = e^{(\alpha-i\beta)x}$ are linearly independent solutions of (1), but these are complex-valued functions and we are not equipped to handle such functions in this course. We want real-valued solutions of (1). The characteristic equation in this case is

$$r^2 + ar + b = (r - [\alpha + i\,\beta])(r - [\alpha - i\,\beta]) = r^2 - 2\alpha\, r + \alpha^2 + \beta^2 = 0$$

and the differential equation (1) has the form

$$y'' - 2\alpha\, y' + \left(\alpha^2 + \beta^2\right) y = 0. \qquad (*)$$

We'll proceed in a manner similar to Case II. Set $y = u\,e^{\alpha x}$ where u is to be determined (if possible) so that y is a solution of (*). Calculating the derivatives of y, we have

$$\begin{aligned} y &= u\,e^{\alpha x} \\ y' &= \alpha\, u\, e^{\alpha x} + u'\, e^{\alpha x} \\ y'' &= \alpha^2 u\, e^{\alpha x} + 2\alpha\, u'\, e^{\alpha x} + u''\, e^{\alpha x} \end{aligned}$$

Substitution into (*) gives

$$\alpha^2 u\, e^{\alpha x} + 2\alpha\, u'\, e^{\alpha x} + u''\, e^{\alpha x} - 2\alpha\left[\alpha\, u\, e^{\alpha x} + u'\, e^{\alpha x}\right] + \left(\alpha^2 + \beta^2\right)\, u\, e^{\alpha x} = 0.$$

This reduces to

$$u''\, e^{\alpha x} + \beta^2\, u\, e^{\alpha x} = 0 \qquad \text{which implies} \qquad u'' + \beta^2\, u = 0 \quad \text{since} \quad e^{\alpha x} \neq 0.$$

Now,

$$u'' + \beta^2\, u = 0$$

is the equation of *simple harmonic motion* (for example, it models the oscillatory motion of a weight suspended on a spring). The functions $u_1(x) = \cos\,\beta x$ and $u_2(x) = \sin\,\beta x$ form a fundamental set of solutions. (Verify this.)

Since $y = u\,e^{\alpha x}$, we conclude that

$$y_1(x) = e^{\alpha x} \cos\,\beta x \qquad \text{and} \qquad y_2(x) = e^{\alpha x} \sin\,\beta x$$

are solutions of (*). It's easy to see that y_1 and y_2 form a fundamental set of solutions. This can also be checked by using the Wronskian

Finally, we conclude that the general solution of equation (1) is:

$$y = C_1\, e^{\alpha x}\, \cos\,\beta x + C_2 e^{\alpha x}\, \sin\,\beta x = e^{\alpha x}\left[C_1\, \cos\,\beta x + C_2\, \sin\,\beta x\right]. \qquad \blacksquare$$

Example 4. Find the general solution of the differential equation

$$y'' - 4y' + 13y = 0.$$

SOLUTION The characteristic equation is: $r^2 - 4r + 13$. By the quadratic formula, the roots are

$$r_1,\ r_2 = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2} = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6\,i}{2} = 2 \pm 3\,i.$$

The characteristic roots are the complex numbers: $r_1 = 2 + 3\,i,\ r_2 = 2 - 3\,i$. The functions $y_1(x) = e^{2x} \cos 3x,\ y_2(x) = e^{2x} \sin 3x$ are linearly independent solutions of the differential equation and

$$y = C_1\, e^{2x} \cos 3x + C_2\, e^{2x} \sin 3x = e^{2x}\,[C_1 \cos 3x + C_2 \sin 3x]$$

is the general solution. ■

Example 5. Find two linearly independent solutions of $y'' + 16y = 0$.

SOLUTION The characteristic equation is $r^2 + 16 = 0$ and the complex numbers $r_1 = 0 + 4i = 4i,\ r_2 = 0 - 4i = -4i$ are the characteristic roots. The functions

$$y_1(x) = e^{0x} \cos 4x = \cos 4x,\ y_2(x) = e^{0x} \sin 4x = \sin 4x$$

are linearly independent solutions of the differential equation. ■

In our next example we find the solution of an initial-value problem.

Example 6. Find the solution of the initial-value problem:

$$y'' + 2y' - 15y = 0, \quad y(0) = 2, \quad y'(0) = -6.$$

SOLUTION The characteristic equation is

$$\begin{aligned} r^2 + 2r - 15 &= 0 \\ (r+5)(r-3) &= 0 \end{aligned}$$

The characteristic roots are: $r_1 = -5,\ r_2 = 3$. The functions $y_1(x) = e^{-5x},\ y_2(x) = e^{3x}$ are linearly independent solutions of the differential equation and

$$y = C_1\, e^{-5x} + C_2\, e^{3x}$$

is the general solution.

Before applying the initial conditions we need to calculate y':

$$y'(x) = -5C_1\, e^{-5x} + 3C_2\, e^{3x}$$

Now, the conditions $y(0) = 2,\ y'(0) = -6$ are satisfied if and only if

$$\begin{aligned} C_1 + C_2 &= 2 \\ -5C_1 + 3C_2 &= -6. \end{aligned}$$

The solution of this pair of equations is: $C_1 = \frac{3}{2}$, $C_2 = \frac{1}{2}$ and the solution of the initial-value problem is

$$y = \tfrac{3}{2}\, e^{-5x} + \tfrac{1}{2}\, e^{3x}. \qquad \blacksquare$$

Recovering a Differential Equation from Solutions

You can also work backwards using the results above. That is, we can determine a second order, linear, homogeneous differential equation with constant coefficients that has given functions u and v as solutions. Here are some examples.

Example 7. Find a second order, linear, homogeneous differential equation with constant coefficients that has the functions $u(x) = e^{2x}$, $v(x) = e^{-3x}$ as solutions.

SOLUTION Since e^{2x} is a solution, 2 must be a root of the characteristic equation and $r-2$ must be a factor of the characteristic polynomial. Similarly, e^{-3x} a solution means that -3 is a root and $r-(-3) = r+3$ is a factor of the characteristic polynomial. Thus the characteristic equation must be

$$(r-2)(r+3) = 0 \qquad \text{which expands to} \qquad r^2 + r - 6 = 0.$$

Therefore, the differential equation is

$$y'' + y' - 6y = 0. \qquad \blacksquare$$

Example 8. Find a second order, linear, homogeneous differential equation with constant coefficients that has $y = C_1\, e^{-4x} + C_2\, x\, e^{-4x}$ as its general solution.

SOLUTION Since e^{-4x} and xe^{-4x} are solutions, -4 must be a double root of the characteristic equation. Therefore, the characteristic equation is

$$(r - [-4])^2 = (r+4)^2 = 0 \qquad \text{which expands to} \qquad r^2 + 8r + 16 = 0$$

and the differential equation is

$$y'' + 8y' + 16y = 0. \qquad \blacksquare$$

Example 9. Find a second order, linear, homogeneous differential equation with constant coefficients that has $y(x) = e^x \cos 2x$ as a solution.

SOLUTION Since $e^x \cos 2x$ is a solution, the characteristic equation must have the complex numbers $1+2i$ and $1-2i$ as roots. (Although we didn't state it explicitly, $e^x \sin 2x$ must also be a solution.) The characteristic equation must be

$$(r - [1+2i])(r - [1-2i]) = 0 \qquad \text{which expands to} \qquad r^2 - 2r + 5 = 0$$

and the differential equation is

$$y'' - 2y' + 5y = 0. \qquad \blacksquare$$

Exercises 3.3

Find the general solution of the given differential equation.

1. $y'' + 2y' - 8y = 0.$
2. $y'' - 13y' + 42y = 0.$
3. $y'' - 10y' + 25y = 0.$
4. $y'' + 2y' + 5y = 0.$
5. $y'' + 4y' + 13y = 0.$
6. $y'' = 0.$
7. $y'' + 2y' = 0.$
8. $2y'' + 5y' - 3y = 0.$
9. $y'' - 12y = 0.$
10. $y'' + 12y = 0.$
11. $y'' - 2y' + 2y = 0.$
12. $y'' - 3y' + \frac{9}{4}y = 0.$
13. $y'' - y' - 30y = 0.$
14. $2y'' + 3y' = 0.$
15. $2y'' + 2y' + y = 0.$
16. $y'' + 2y' + 3y = 0.$
17. $y'' - 8y' + 16y = 0.$
18. $5y'' + \frac{11}{4}y' - \frac{3}{4}y = 0.$

Find the solution of the initial-value problem.

19. $y'' - 5y' + 6y = 0; \quad y(0) = 1,\ y'(0) = 1.$
20. $y'' + 4y' + 3y = 0; \quad y(0) = 2,\ y'(0) = -1.$
21. $y'' + 2y' + y = 0; \quad y(0) = -3,\ y'(0) = 1.$
22. $y'' + \frac{1}{4}y = 0; \quad y(\pi) = 1,\ y'(\pi) = -1.$
23. $y'' - 2y' + 2y = 0; \quad y(0) = -1,\ y'(0) = -1.$

24. $y'' + 4y' + 4y = 0; \quad y(-1) = 2,\ y'(-1) = 1.$

Find a differential equation $y'' + ay' + by = 0$ that is satisfied by the given functions.

25. $y_1(x) = e^{2x},\ y_2(x) = e^{-5x}.$

26. $y_1(x) = 3e^{3x},\ y_2(x) = 2xe^{3x}.$

27. $y_1(x) = \cos 2x,\ y_2(x) = 2\sin 2x.$

28. $y_1(x) = e^{-2x}\cos 4x,\ y_2(x) = e^{-2x}\sin 4x.$

Find a differential equation $y'' + ay' + by = 0$ whose general solution is the given expression.

29. $y = C_1e^{x/2} + C_2e^{2x}.$

30. $y = C_1e^{3x} + C_2e^{-4x}.$

31. $y = C_1e^{-x}\cos 3x + C_2e^{-x}\sin 3x.$

32. $y = C_1e^{x/2} + C_2xe^{x/2}.$

33. $y = C_1\cos 4x + C_2\sin 4x.$

34. Find the solution $y = y(x)$ of the initial-value problem $y'' - y' - 2y = 0;\ y(0) = \alpha,\ y'(0) = 2$. Then find α such that $y(x) \to 0$ as $x \to \infty$.

35. Find the solution $y = y(x)$ of the initial-value problem $4y'' - y = 0;\ y(0) = 2,\ y'(0) = \beta$. Then find β such that $y(x) \to 0$ as $x \to \infty$.

36. Given the differential equation $y'' - (2a - 1)y' + a(a - 1)y = 0$.

    (a) Determine the values of a (if any) for which all solutions have limit 0 as $x \to \infty$.

    (b) Determine the values of a (if any) for which all solutions are unbounded as $x \to \infty$.

Exercises 37 - 39 are concerned with the differential equation (1): $y'' + ay' + by = 0$ where a and b are constants.

37. Give a condition on a and b which will imply that:

    (a) (1) has solutions of the form $y_1 = e^{\alpha x},\ y_2 = e^{\beta x},\ \alpha,\ \beta$ distinct real numbers.

    (b) (1) has solutions of the form $y_1 = e^{\alpha x},\ y_2 = xe^{\alpha x},\ \alpha$ a real number.

    (c) (1) has solutions of the form $y_1 = e^{\alpha x}\cos\beta x,\ y_2 = e^{\alpha x}\sin\beta x,\ \alpha,\ \beta$ real numbers.

38. Prove that if a and b are both positive, then all solutions have limit 0 as $x \to \infty$.

39. Prove:

(a) If $a = 0$ and $b > 0$, then all solutions of the equation are bounded.

(b) If $a > 0$ and $b = 0$, and $y = y(x)$ is a solution, then

$$\lim_{x\to\infty} y(x) = k \quad \text{for some constant } k.$$

Determine k for the solution that satisfies the initial conditions $y(0) = \alpha,\ y'(0) = \beta$.

40. Show that the general solution of the differential equation

$$y'' - \omega^2 y = 0, \quad \omega \text{ a positive constant},$$

can be written

$$y = C_1 \cosh \omega x + C_2 \sinh \omega x.$$

41. Suppose that the roots $r_1,\ r_2$ of the characteristic equation (2) are real and distinct. Then they can be written as $r_1 = \alpha + \beta,\ r_2 = \alpha - \beta$ where α and β are real. Show that the general solution of equation (1) in this case can be expressed in the form

$$y = e^{\alpha x}\left(C_1 \cosh \beta x + C_2 \sinh \beta x\right).$$

Euler Equations A second order linear homogeneous equation of the form

$$x^2 \frac{d^2y}{dx^2} + \alpha x \frac{dy}{dx} + \beta y = 0 \tag{E}$$

where α and β are constants, is called an *Euler equation* .

42. Prove that the Euler equation (E) can be transformed into the second order equation with constant coefficients

$$\frac{d^2y}{dz^2} + a\frac{dy}{dz} + by = 0$$

where a and b are constants, by means of the change of independent variable $z = \ln x$.

Find the general solution of the Euler equations.

43. $x^2y'' - xy' - 8y = 0$.

44. $x^2y'' - 2xy' + 2y = 0$.

45. $x^2y'' - 3xy' + 4y = 0$.

46. $x^2y'' - xy' + 5y = 0$.

3.4 Second Order Linear Nonhomogeneous Equations

In this section we consider the general second order linear nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{N}$$

where p, q, f are continuous functions on an interval I.

The objectives of this section are to determine the "structure" of the set of solutions of (N) and to develop a method for constructing a solution of (N) using two linearly independent solutions of the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0. \tag{H}$$

As we shall see, there is a close connection between equations (N) and (H). In this context, equation (H) is called the *reduced equation* of equation (N).

General Results

THEOREM 1. If $z = z_1(x)$ and $z = z_2(x)$ are solutions of equation (N), then

$$y(x) = z_1(x) - z_2(x)$$

is a solution of equation (H).

Proof: Since z_1 and z_2 are solutions of (N),

$$z_1''(x) + p(x)z_1'(x) + q(x)z_1(x) = f(x) \quad \text{and} \quad z_2''(x) + p(x)z_2'(x) + q(x)z_2(x) = f(x).$$

Let $y(x) = z_1(x) - z_2(x)$. Then

$$\begin{aligned} y'' - py' + qy &= (z_1'' - z_2'') + p(z_1' - z_2') + q(z_1 - z_2) \\ &= \left(z_1'' + pz_1' + qz_1\right) - \left(z_2'' + pz_2' + qz_2\right) \\ &= f(x) - f(x) = 0. \end{aligned}$$

Thus, $y = z_1 - z_2$ is a solution of (H).

Alternate Proof Set $L[y] = y'' + p(x)y' + q(x)y$; L is a linear operator. Since z_1 and z_2 are solutions of (N), $L[z_1(x)] = L[z_2(x)] = f(x)$. Since L is a linear operator,

$$L[z_1(x) - z_2(x)] = L[z_1(x)] - L[z_2(x)] = f(x) - f(x) = 0.$$

Thus, $y = z_1 - z_2$ is a solution of (H). ■

In words, Theorem 1 says that *the difference of any two solutions of the nonhomogeneous equation (N) is a solution of its reduced equation (H).*

Our next theorem gives the "structure" of the set of solutions of (N).

THEOREM 2. Let $y = y_1(x)$ and $y = y_2(x)$ be linearly independent solutions of the reduced equation (H) and let $z = z(x)$ be a particular solution of (N). If $u = u(x)$ is *any* solution of (N), then there exist constants C_1 and C_2 such that

$$u(x) = C_1 y_1(x) + C_2 y_2(x) + z(x)$$

Proof: Let $z = z(x)$ be a particular of (N) and let $u = u(x)$ be *any other* solution of (N). By Theorem 1, $u(x) - z(x)$ is a solution of the reduced equation (H). Since $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (H), there exist constants C_1 and C_2 such that

$$u(x) - z(x) = C_1 y_1(x) + C_2 y_2(x).$$

Thus,

$$u(x) = C_1 y_1(x) + C_2 y_2(x) + z(x). \qquad \blacksquare$$

According to Theorem 2, if $y = y_1(x)$ and $y = y_2(x)$ are linearly independent solutions of the reduced equation (H) and $z = z(x)$ is a particular solution of (N), then

$$y = C_1 y_1(x) + C_2 y_2(x) + z(x) \tag{1}$$

represents the set of all solutions of (N). That is, (1) is the general solution of (N). Another way to look at (1) is: The general solution of (N) consists of the general solution of the reduced equation (H) *plus* a particular solution of (N):

$$\underbrace{y}_{\text{general solution of (N)}} = \underbrace{C_1 y_1(x) + C_2 y_2(x)}_{\text{general solution of (H)}} + \underbrace{z(x).}_{\text{particular solution of (N)}}$$

The following result is sometimes useful in finding particular solutions of nonhomogeneous equations. It is known as the *superposition principle.*

THEOREM 3. Given the second order linear nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x) + g(x). \tag{$*$}$$

If $z = z_f(x)$ and $z = z_g(x)$ are particular solutions of

$$y'' + p(x)y' + q(x)y = f(x) \quad \text{and} \quad y'' + p(x)y' + q(x)y = g(x),$$

respectively, then $z(x) = z_f(x) + z_g(x)$ is a particular solution of $(*)$.

The proof is left as an exercise. $\blacksquare$

This result can be extended to nonhomogeneous equations whose right-hand side is the sum of an arbitrary number of functions.

COROLLARY If

$z = z_1(x)$ is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x),$$

$z = z_2(x)$ is a particular solution of

$$y'' + p(x)y' + q(x)y = f_2(x),$$

$\vdots$

$z = z_n(x)$ is a particular solution of

$$y'' + p(x)y' + q(x)y = f_n(x),$$

then $z(x) = z_1(x) + z_2(x) + \cdots + z_n(x)$ is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x) + \cdots + f_n(x). \qquad \blacksquare$$

The importance of Theorem 3 and its Corollary is that we need only consider non-homogeneous equations in which the function on the right-hand side consists of one term only.

Variation of Parameters

By our work above, to find the general solution of (N) we need to find:

(i) a linearly independent pair of solutions y_1, y_2 of the reduced equation (H), and

(ii) a particular solution z of (N).

The *method of variation of parameters* uses a pair of linearly independent solutions of the reduced equation to construct a particular solution of (N).

Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the reduced equation

$$y'' + p(x)y' + q(x)y = 0. \tag{H}$$

Then

$$y = C_1y_1(x) + C_2y_2(x)$$

is the general solution of (H). We replace the arbitrary constants C_1 and C_2 by functions $u = u(x)$ and $v = v(x)$, which are to be determined so that

$$z(x) = u(x)y_1(x) + v(x)y_2(x)$$

is a particular solution of the nonhomogeneous equation (N). The replacement of the parameters C_1 and C_2 by the "variables" u and v is the basis for the term "variation of parameters." Since there are two unknowns u and v to be determined, we shall impose two conditions on these unknowns. One condition is that z should solve the differential equation (N). The second condition is at our disposal and we shall choose it in a manner that will simplify our calculations.

Differentiating z we get

$$z' = u\,y_1' + y_1\,u' + v\,y_2' + y_2\,v'.$$

For our second condition on u and v, we set

$$y_1\,u' + y_2\,v' = 0. \tag{a}$$

This condition is chosen because it simplifies the first derivative z' and because it will lead to a simple pair of equations in the unknowns u and v. With this condition the equation for z' becomes

$$z' = u\,y_1' + v\,y_2' \tag{b}$$

and

$$z'' = u\,y_1'' + y_1'\,u' + v\,y_2'' + y_2'\,v'.$$

Now substitute z, z' (given by (b)), and z'' into the left side of equation (N). This gives

$$\begin{aligned} z'' + pz' + qz &= (u\,y_1'' + y_1'\,u' + v\,y_2'' + y_2'\,v') + p(u\,y_1' + v\,y_2') + q(u\,y_1 + v\,y_2) \\ &= u(y_1'' + py_1' + qy_1) + v(y_2'' + py_2' + qy_2) + y_1'\,u' + y_2'\,v'. \end{aligned}$$

Since y_1 and y_2 are solutions of (H),

$$y_1'' + py_1' + qy_1 = 0 \quad \text{and} \quad y_2'' + py_2' + qy_2 = 0$$

and so

$$z'' + pz' + qz = y_1'\,u' + y_2'\,v'.$$

The condition that z should satisfy (N) is

$$y_1'\,u' + y_2'\,v' = f(x). \tag{c}$$

Equations (a) and (c) constitute a system of two equations in the two unknowns u and v:

$$\begin{aligned} y_1\,u' + y_2\,v' &= 0 \\ y_1'\,u' + y_2'\,v' &= f(x) \end{aligned}$$

Obviously this system involves u' and v' not u and v, but if we can solve for u' and v', then we can integrate to find u and v. Solving for u' and v', we find that

$$u' = \frac{-y_2\, f}{y_1\, y_2' - y_2\, y_1'} \quad \text{and} \quad v' = \frac{y_1\, f}{y_1\, y_2' - y_2\, y_1'}$$

We know that the denominators here are non-zero because the expression

$$y_1(x)y_2'(x) - y_2(x)y_1'(x) = W(x)$$

is the Wronskian of y_1 and y_2, and y_1, y_2 are linearly independent solutions of the reduced equation.

We can now get u and v by integrating:

$$u = \int \frac{-y_2(x)f(x)}{W(x)}\, dx \quad \text{and} \quad v = \int \frac{y_1(x)f(x)}{W(x)}\, dx.$$

Finally

$$z(x) = y_1(x)\int \frac{-y_2(x)f(x)}{W(x)}\, dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}\, dx \tag{2}$$

is a particular solution of the nonhomogeneous equation (N).

Remark This result illustrates why the emphasis is on linear homogeneous equations. To find the general solution of the nonhomogeneous equation (N) we need a fundamental set of solutions of the reduced equation (H) and one particular solution of (N). But, as we have just shown, if we have a fundamental set of solutions of (H), then we can use them to construct a particular solution of (N). Thus, all we really need to solve (N) is a fundamental set of solutions of its reduced equation (H). ■

Example 1. Find a particular solution of the nonhomogeneous equation

$$y'' - \frac{2}{x}\, y' + \frac{2}{x^2}\, y = 2x^3 \tag{$*$}$$

given that $y_1(x) = x$ and $y_2(x) = x^2$ are linearly independent solutions of the corresponding reduced equation. Also give the general solution of the nonhomogeneous equation.

SOLUTION The Wronskian of y_1, y_2 is $W(x) = y_1\, y_2' - y_2\, y_1' = x(2x) - x^2(1) = x^2$. By the method of variation of parameters, a particular solution of the nonhomogeneous equation is

$$z(x) = u(x)\, x + v(x)\, x^2$$

where, from (2),

$$u(x) = \int \frac{-y_2(x)\, f(x)}{W(x)}\, dx = \int \frac{-x^2(2x^3)}{x^2}\, dx = \int -2x^3\, dx = -\tfrac{1}{2}\, x^4$$

and

$$v(x) = \int \frac{y_1(x)\, f(x)}{W(x)}\, dx = \int \frac{x(2x^3)}{x^2}\, dx = \int 2x^2\, dx = \tfrac{2}{3}\, x^3$$

(NOTE: Since we are seeking only one function u and one function v we have not included arbitrary constants in the integration steps.)

Now

$$z(x) = -\tfrac{1}{2}\, x^4 \cdot x + \tfrac{2}{3}\, x^3 \cdot x^2 = \tfrac{1}{6}\, x^5.$$

is a particular solution of the nonhomogeneous equation (∗) and

$$y = C_1 x + C_2 x^2 + \tfrac{1}{6}\, x^5.$$

is the general solution. ■

Remark Rather than simply memorizing the formula (4) for the particular solution z of (N), some people prefer to use the variation of parameters method to construct the solution z. We'll use this approach in the next example. ■

Example 2. Find the general solution of

$$y'' - 5y' + 6y = 4e^{2x}. \qquad (*)$$

SOLUTION The reduced equation $y'' - 5y' + 6y = 0$ has characteristic equation

$$r^2 - 5r - 6 = (r-2)(r-3) = 0,$$

and $y_1(x) = e^{2x}$, $y_2(x) = e^{3x}$ are linearly independent solutions. The general solution of the reduced equation is

$$y = C_1 e^{2x} + C_2 e^{3x}.$$

We replace the constants C_1 and C_2 by functions u and v which are to be determined such that

$$z = ue^{2x} + ve^{3x}$$

is a solution of (∗).

Now,

$$z' = 2ue^{2x} + e^{2x}u' + 3ve^{3x} + e^{3x}v'.$$

Imposing the condition

$$e^{2x}u' + e^{3x}v' = 0, \qquad \text{(a)}$$

the first derivative simplifies to

$$z' = 2ue^{2x} + 3ve^{3x} \qquad \text{(b)}$$

and

$$z'' = 4ue^{2x} + 2e^{2x}u' + 9ve^{3x} + 3e^{3x}v'.$$

Substituting z, z' [given by (b)] and z'' into the left side of $(*)$ gives

$$4ue^{2x} + 2e^{2x}u' + 9ve^{3x} + 3e^{3x}v' - 5(2ue^{2x} + 3ve^{3x}) + 6(u^{2x} + ve^{3x}) = 2e^{2x}u' + 3e^{3x}v'.$$

Setting this equal to $4e^{2x}$ we have our second equation

$$2e^{2x}u' + 3e^{3x}v' = 4e^{2x}. \tag{c}$$

Taking (a) and (c) together, we get the system

$$\begin{aligned} e^{2x}u' + e^{3x}v' &= 0 \\ 2e^{2x}u' + 3e^{3x}v' &= 4e^{2x} \end{aligned}$$

Multiplying the first equation by -3 and adding gives

$$-e^{2x}u' = 4e^{2x} \quad \text{which implies} \quad u' = -4 \quad \text{and} \quad u = -4x;$$

multiplying the first equation by -2 and adding gives

$$e^{3x}v' = 4e^{2x} \quad \text{which implies} \quad v' = 4e^{-x} \quad \text{and} \quad v = -4e^{-x}.$$

Therefore,

$$z = (-4x)e^{2x} + (-4e^{-x})e^{3x} = -4xe^{2x} - 4e^{2x}$$

is a particular solution of (*).

Finally, the general solution of (*) is

$$y = C_1\, e^{2x} + C_2\, e^{3x} - 4x\, e^{2x} - 4e^{2x}$$

which can be written equivalently as

$$y = C_1\, e^{2x} + C_2\, e^{3x} - 4x\, e^{2x}$$

by combining $-4e^{2x}$ with $C_1\, e^{2x}$. ■

Exercises 3.4

Verify that the given functions y_1 and y_2 form a fundamental set of solutions of the reduced equation of the given nonhomogeneous equation; then find a particular solution of the nonhomogeneous equation and give the general solution of the equation.

1. $y'' - \dfrac{2}{x^2}\, y = 3 - x^{-2}$; $y_1(x) = x^2,\ y_2(x) = x^{-1}$.
2. $y'' - \dfrac{1}{x}y' + \dfrac{1}{x^2}y = \dfrac{2}{x}$; $y_1(x) = x,\ y_2(x) = x\,\ln\, x$.

3. $x^2y'' - 2xy' + 2y = x^2 \ln x; \quad y_1(x) = x,\ y_2(x) = x^2.$

4. $y'' - \dfrac{1+x}{x}y' + \dfrac{1}{x}y = xe^{2x}; \quad y_1(x) = 1 + x,\ y_2(x) = e^x.$

5. $(x-1)y'' - xy' + y = (x-1)^2; \quad y_1(x) = x,\ y_2(x) = e^x.$

6. $x^2y'' - xy' + y = 4x \ln x.$

Find the general solution of the given nonhomogeneous differential equation.

7. $y'' - y' - 2y = 2e^{-x}.$

8. $y'' + y = \tan x.$

9. $y'' + 4y = \sec 2x.$

10. $y'' - 2y' + y = xe^x.$

11. $y'' - 2y' + y = e^x \cos x.$

12. $y'' - 4y' + 4y = \frac{1}{3}\,x^{-1}e^{2x}.$

13. $y'' + 4y' + 4y = \dfrac{e^{-2x}}{x^2}.$

14. $y'' + 2y' + y = e^{-x} \ln x.$

15. $y'' + 9y = 9\sec^2 3x.$

16. $y'' - 2y' + 2y = e^x \sec x.$

17. The function $y_1(x) = x$ is a solution of $x^2y'' + xy' - y = 0$. Find the general solution of the differential equation

$$x^2y'' + xy' - y = 2x.$$

HINT: See Exercise 15, Section 3.2.

18. The function $y_1(x) = x$ is a solution of $(x^2+1)y'' - 2xy' + 2y = 0$. Find the general solution of the differential equation

$$(x^2+1)y'' - 2xy' + 2y = (x^2+1)^2.$$

19. The functions $y_1(x) = x^2 + x \ln x$, $y_2(x) = x + x^2$ and $y_3(x) = x^2$ are solutions of a second order linear nonhomogeneous equation. What is the general solution of the equation?

20. The functions $y_1(x) = x - 2x^3$ $y_2(x) = xe^x + x - 2x^3$ and $y_3(x) = -2x^3$ are solutions of a second order linear nonhomogeneous equation. What is the general solution of the equation?

3.5 Nonhomogeneous Equations with Constant Coefficients; Undetermined Coefficients

Solving a linear nonhomogeneous equation depends, in part, on finding a particular solution of the equation. We have seen one method for finding a particular solution, the method of variation of parameters. In this section we present another method, the method of *undetermined coefficients.*

Remark: Limitations of the method. In contrast to variation of parameters, which can be applied to any nonhomogeneous equation, the method of undetermined coefficients can be applied only to nonhomogeneous equations of the form

$$y'' + ay' + by = f(x) \tag{1}$$

where a and b are constants and the nonhomogeneous term f is a polynomial, an exponential function, a sine, a cosine, or a combination of such functions. ■

To motivate the method of undetermined coefficients, consider the linear operator L on the left side of (1):

$$L[y] = y'' + ay' + by. \tag{2}$$

If we calculate $L[y]$ for an exponential function $z = Ae^{rx}$, A a constant, we have

$$z = Ae^{rx}, \qquad z' = Are^{rx}, \qquad z'' = Ar^2e^{rx}$$

and

$$\begin{aligned} L[y] = y'' + ay' + by &= Ar^2e^{rx} + a(Are^{rx}) + b(Ae^{rx} = \left(Ar^2 + aAr + bA\right)e^{rx} \\ &= K\,e^{rx} \qquad \text{where } K = Ar^2 + aAr + bA. \end{aligned}$$

That is, the operator L "transforms" Ae^{rx} into a constant multiple of e^{rx}. We can use this result to determine a particular solution of a nonhomogeneous equation of the form

$$y'' + ay' + by = ce^{rx}.$$

Here is a specific example.

Example 1. Find a particular solution of the nonhomogeneous equation

$$y'' - 2y' + 5y = 6e^{3x}.$$

SOLUTION As we saw above, if we "apply" $y'' - 2y' + 5y$ to $z(x) = Ae^{3x}$ we will get an expression of the form Ke^{3x}. We want to determine A so that $K = 6$. The constant A is called an *undetermined coefficient.* We have

$$z = Ae^{3x}, \quad z' = 3Ae^{3x}, \quad z'' = 9Ae^{3x}.$$

Substituting z and its derivatives into the left side of the differential equation, we get

$$9Ae^{3x} - 2\left(3Ae^{3x}\right) + 5\left(Ae^{3x}\right) = (9A - 6A + 5A)e^{3x} = 8Ae^{3x}.$$

We want

$$z'' - 2z' + 5z = 6e^{3x},$$

so we set

$$8Ae^{3x} = 6e^{3x} \quad \text{which gives} \quad 8A = 6 \quad \text{and} \quad A = \tfrac{3}{4}.$$

Thus, $z(x) = \frac{3}{4}\,e^{3x}$ is a particular solution of $y'' - 2y' + 5y = 6e^{3x}$. (Verify this.)

You can also verify that

$$y = C_1 e^x \cos 2x + C_2 e^x \sin 2x + \tfrac{3}{4}\,e^{3x}$$

is the general solution of the equation. ■

If we set $z(x) = A\cos\beta x$ and calculate z' and z'', we get

$$z = A\cos\beta x, \quad z' = -\beta A\sin\beta x, \quad z'' = -\beta^2 A\cos\beta x.$$

Therefore, $L[y] = y'' + ay' + by$ applied to z gives

$$\begin{aligned} L[z] = z'' + az' + bz &= -\beta^2 A\cos\beta x + a\left(-\beta A\sin\beta x\right) + b(A\,cos\,\beta x) \\ &= (-\beta^2 A + bA)\cos\beta x + (-a\beta A)\sin\beta x. \end{aligned}$$

That is, L "transforms" $z = A\cos\beta x$ into an expression of the form

$$K\cos\beta x + M\sin\beta x$$

where K and M are constants which depend on a, b, β and A. We will get exactly the same type of result if we apply L to $z = B\sin\beta x$. Combining these two results, it follows that $L[y] = y'' + ay' + by$ applied to

$$z = A\cos\beta x + B\sin\beta x$$

will produce the expression

$$K\cos\beta x + M\sin\beta x$$

where K and M are constants which depend on $a, b, \beta, A,$ and B.

Now suppose we have a nonhomogeneous equation of the form

$$y'' + ay' + by = c\cos\beta x \quad \text{or} \quad y'' + ay' + by = d\sin\beta x,$$

or the general form

$$y'' + ay' + by = c\cos\beta x + d\sin\beta x.$$

(Note: The first equation can be written $y'' + ay' + by = c\cos\beta x + 0\sin\beta x$ and the second equation can be written $y'' + ay' + by = 0\cos\beta x + d\sin\beta x$ so each is a special case of the general form.)

We will look for a solution of the form $z(x) = A\cos\beta x + B\sin\beta x$.

Example 2. Find a particular solution of the nonhomogeneous equation

$$y'' + 2y' + y = 10\cos 3x.$$

SOLUTION Set $z = A\cos 3x + B\sin 3x$ where A and B are constants which are to be determined so that $z'' + 2z' + z = 10\cos 3x$.

Calculating the derivatives of z, we have

$$z = A\cos 3x + B\sin 3x, \quad z' = -3Aa\sin 3x + 3B\cos 3x, \quad z'' = -9A\cos 3x - 9B\sin 3x.$$

Substituting z and its derivatives into the left side of the differential equation gives

$$\begin{aligned} -9A\cos 3x - 9B\sin 3x + 2(-3A\sin 3x + 3B\cos 3x) + A\cos 3x + B\sin 3x \\ = (-8A + 6B)\cos 3x + (-6A - 8B)\sin 3x. \end{aligned}$$

Since we want $z'' + 2z' + z = 10\cos 3x = 10\cos 3x + 0\sin 3x$, we set

$$(-8A + 6B)\cos 3x + (-6A - 8B)\sin 3x = 10\cos 3x + 0\sin 3x$$

which implies

$$\begin{aligned} -8A + 6B &= 10 \\ -6A - 8B &= 0 \end{aligned}$$

The solution of this pair of equations is: $A = -\frac{4}{5}$, $B = \frac{3}{5}$. Therefore a particular solution of the differential equation is

$$z(x) = -\tfrac{4}{5}\cos 3x + \tfrac{3}{5}\sin 3x.$$

You can verify that the general solution of the differential equation is

$$y = C_1 e^{-x} + C_2 x e^{-x} - \tfrac{4}{5}\cos 3x + \tfrac{3}{5}\sin 3x. \qquad \blacksquare$$

Example 3. Find the general solution of the differential equation

$$y'' - 6y' + 8y = 2\cos 2x - 4\sin 2x. \tag{$*$}$$

SOLUTION First we consider the reduced equation:

$$y'' - 6y' + 8y = 0.$$

The characteristic equation is: $r^2 - 6r + 8 = (r-2)(r-4) = 0$ and the roots are $r = 2, 4$. Therefore the functions $y_1(x) = e^{2x}$, $y_2(x) = e^{4x}$ form a fundamental set of solutions of the reduced equation.

Next we find a particular solution of $(*)$. Since $f(x) = 2\cos 2x - 4\sin 2x$, we seek a solution of the form $z = A\cos 2x + B\sin 2x$. We calculate the derivatives of z and substitute into the left side of $(*)$:

$$z = A\cos 2x + B\sin 2x, \quad z' = -2A\sin 2x + 2B\cos 2x, \quad z'' = -4A\cos 2x - 4B\sin 2x;$$

$$\begin{aligned} z'' - 6z' + 8z &= -4A\cos 2x - 4B\sin 2x - 6(-2A\sin 2x + 2B\cos 2x) + 8(A\cos 2x + B\sin 2x) \\ &= (4A - 12B)\cos 2x + (12A + 4B)\sin 2x; \end{aligned}$$

Now z is a solution of $(*)$ if

$$(4A - 12B)\cos 2x + (12A + 4B)\sin 2x = 2\cos 2x - 4\sin 2x$$

which implies

$$\begin{aligned} 4A - 12B &= 2 \\ 12A + 4B &= -4 \end{aligned}$$

The solution to this pair of equations is: $A = -\frac{1}{4}$, $B = -\frac{1}{4}$. Therefore, $z(x) = -\frac{1}{4}\cos 2x - \frac{1}{4}\sin 2x$ is a particular solution of $(*)$.

Finally, the general solution of $(*)$ is:

$$y = C_1 e^{2x} + C_2 e^{4x} - \tfrac{1}{4}\cos 2x - \tfrac{1}{4}\sin 2x. \qquad \blacksquare$$

Continuing with these ideas, if $y'' + ay' + by$ is applied to $z = Ae^{\alpha x}\cos \beta x + Be^{\alpha x}\sin \beta x$, then the result will have the form

$$Ke^{\alpha x}\cos \beta x + Me^{\alpha x}\sin \beta x$$

where K and M are constants which depend on $a, b, \alpha, \beta, A, B$. Therefore, we expect that a nonhomogeneous equation of the form

$$y'' + ay' + by = ce^{\alpha x}\cos \beta x + de^{\alpha x}\sin \beta x$$

will have a particular solution of the form $z = Ae^{\alpha x}\cos \beta x + Be^{\alpha x}\sin \beta x$.

The following table summarizes our discussion to this point.

A particular solution of $y'' + ay' + by = f(x)$

If $f(x) =$	try $z(x) =$
ce^{rx}	Ae^{rx}
$c\cos \beta x + d\sin \beta x$	$z(x) = A\cos \beta x + B\sin \beta x$
$ce^{\alpha x}\cos \beta x + de^{\alpha x}\sin \beta x$	$z(x) = Ae^{\alpha x}\cos \beta x + Be^{\alpha x}\sin \beta x$

Note: The first line includes the case $r = 0$;
if $f(x) = ce^{0x} = c$, then $z = Ae^{0x} = A$.

Unfortunately, the situation is not quite as simple as it appears; there is a difficulty.

Example 4. Find a particular solution of the nonhomogeneous equation

$$y'' - 5y' + 6y = 4e^{2x}. \tag{$*$}$$

SOLUTION According to the table, we should set $z(x) = Ae^{2x}$. Calculating the derivatives of z, we have

$$z = Ae^{2x}, \quad z' = 2Ae^{2x}, \quad z'' = 4Ae^{2x}.$$

Substituting z and its derivatives into the left side of $(*)$, we get

$$z'' - 5z' + 6z = 4Ae^{2x} - 5(2Ae^{2x}) + 6(Ae^{2x}) = 0Ae^{2x}.$$

Clearly the equation

$$0Ae^{2x} = 4e^{2x} \quad \text{which is equivalent to} \quad 0A = 4$$

does not have a solution. Therefore equation $(*)$ does not have a solution of the form $z = Ae^{2x}$.

The problem here is $z = Ae^{2x}$ is a solution of the reduced equation

$$y'' - 5y' + 6y = 0.$$

(The characteristic equation is $r^2 - 5r + 6 = 0$; the roots are $r = 2,\ 3$; and $y_1 = e^{2x}$, $y_2 = e^{3x}$ are linearly independent solutions.)

In Example 2 of the preceding section we saw that $z(x) = -4xe^{2x}$ is a particular solution of $(*)$. So, in the context here, since our trial solution $z = Ae^{2x}$ solves the reduced equation, we'll try $z = Axe^{2x}$. The derivatives of this z are:

$$z = Axe^{2x}, \quad z' = 2Axe^{2x} + Ae^{2x}, \quad z'' = 4Axe^{2x} + 4Ae^{2x}.$$

Substituting into the left side of $(*)$, we get

$$\begin{aligned} z'' - 5z' + 6z &= 4Axe^{2x} + 4Ae^{2x} - 5(2Axe^{2x} + Ae^{2x}) + 6(Axe^{2x}) \\ &= -Ae^{2x}. \end{aligned}$$

Setting $z'' - 5z' + 6z = 4e^{2x}$ gives

$$-Ae^{2x} = 4e^{2x} \quad \text{which implies} \quad A = -4.$$

Thus, $z(x) = -4xe^{2x}$ is a particular solution of $(*)$ (as we already know). ■

We learn from this example that we have to make an adjustment if our trial solution z (from the table) satisfies the reduced equation. Here's another example.

Example 5. Find a particular solution of

$$y'' + 6y' + 9y = 5e^{-3x}. \qquad (**)$$

SOLUTION The reduced equation, $y'' + 6y' + 9y = 0$ has characteristic equation

$$r^2 + 6r + 9 = (r+3)^2 = 0.$$

Thus, $r = -3$ is a double root and $y_1(x) = e^{-3x},\ y_2(x) = xe^{-3x}$ form a fundamental set of solutions.

According to our table, to find a particular solution of $(**)$ we should try $z = Ae^{-3x}$. But this won't work, z is a solution of the reduced equation. Based on the result of the preceding example, we should try $z = Axe^{-3x}$, but this won't work either; $z = Axe^{-3x}$ is also a solution of the reduced equation. So we'll try $z = Ax^2e^{-3x}$. The derivatives of this z are:

$$z = Ax^2e^{-3x}, \quad z' = -3Ax^2e^{-3x} + 2Axe^{-3x}, \quad z'' = 9Ax^2e^{-3x} - 12Axe^{-3x} + 2Ae^{-3x}.$$

Substituting into the left side of $(**)$, we get

$$\begin{aligned} z'' + 6z' + 9z &= 9Ax^2e^{-3x} - 12Axe^{-3x} + 2Ae^{-3x} + 6(-3Ax^2e^{-3x} + 2Axe^{-3x}) + 9(Ax^2e^{-3x}) \\ &= 2Ae^{-3x}. \end{aligned}$$

Setting $z'' + 6z' + 9z = 5e^{-3x}$ gives

$$2Ae^{-3x} = 5e^{-3x} \quad \text{which implies} \quad A = \tfrac{5}{2}.$$

Thus, $z(x) = \frac{5}{2}\,x^2e^{-3x}$ is a particular solution of $(**)$.

The general solution of $(**)$ is: $y = C_1e^{-3x} + C_2xe^{-3x} + \frac{5}{2}\,x^2e^{-3x}$. ■

Based on these examples we amend our table to read:

Table 1

A particular solution of $y'' + ay' + by = f(x)$

If $f(x) =$	try $z(x) =$*
ce^{rx}	Ae^{rx}
$c\cos\beta x + d\sin\beta x$	$z(x) = A\cos\beta x + B\sin\beta x$
$ce^{\alpha x}\cos\beta x + de^{\alpha x}\sin\beta x$	$z(x) = Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x$

*Note: If z satisfies the reduced equation, try xz; if xz also satisfies the reduced equation, then x^2z will give a particular solution

Remark In practice it is a good idea to solve the reduced equation before selecting the trial solution z of the nonhomogeneous equation. That way you will not waste your time selecting a z that satisfies the reduced equation. ■

Example 6. Without carrying out the calculations, give the form of the general solution of the nonhomogeneous differential equation

$$y'' - y' - 6y = 2\cos x - 2e^{3x} + 10.$$

SOLUTION The first step is to solve the reduced equation $y'' - y' - 6y = 0$. The characteristic equation is $r^2 - r - 6 = (r+2)(r-3) = 0$. Thus, $y_1(x) = e^{-2x}$, $y_2(x) = e^{3x}$ forms a fundamental set of solutions of the reduced equation and

$$y = C_1 e^{-2x} + C_2 e^{3x}$$

is the general solution of the reduced equation.

Now we need a particular solution z of the given equation. To find z we make use of the Corollary to Theorem 3 in the preceding section. That is, we'll find a particular solution z_1 of

$$y'' - y' - 6y = 2\cos x,$$

a particular solution z_2 of

$$y'' - y' - 6y = -2e^{3x},$$

and a particular solution z_3 of

$$y'' - y' - 6y = 10.$$

Then $z = z_1 + z_2 + z_3$ will be a particular solution of the given equation.

From Table 1, z_1 has the form $z_1(x) = A\cos x + B\sin x$ and z_2 has the form $z_2(x) = Cxe^{3x}$. To find z_3 note that $10 = 10e^{0x}$ which is simply the case $r = 0$ in the first line of Table 1. Thus z_3 has the form $z_3(x) = De^{0x} = D$. A particular solution z of the nonhomogeneous equation has the form

$$z(x) = A\cos x + B\sin x + Cxe^{3x} + D.$$

where the constants A, B, C, D are to be determined.

The general solution of the equation will have the form

$$y = C_1 e^{-2x} + C_2 e^{3x} + A\cos x + B\sin x + Cxe^{3x} + D. \qquad ■$$

You can verify that $y = C_1\, e^{-2x} + C_2\, e^{3x} - \frac{7}{25}\cos x - \frac{1}{25}\sin x - \frac{2}{5}\, xe^{3x} - \frac{5}{3}$ is the general solution.

So far we have only considered the nonhomogeneous differential equation (1) in cases where the nonhomogeneous term f is a constant multiple of one of the functions e^{rx}, $\cos\beta x$, $\sin\beta x$, $e^{\alpha x}\cos\beta x$, $e^{\alpha x}\sin\beta x$, or is a sum of such functions. In general, the method of undetermined coefficients can be applied in cases where

$$\begin{aligned} f(x) &= p(x)e^{rx} \\ f(x) &= p(x)\cos\beta x, \text{ or } p(x)\sin\beta x, \\ f(x) &= p(x)e^{\alpha x}\cos\beta x, \text{ or } p(x)e^{\alpha x}\sin\beta x \end{aligned}$$

where p is a polynomial, or where f is a sum of such functions. This follows from the fact that the expression $y'' + ay' + by$ applied to

$$z = \left(A_0 + A_1x + A_2x^2 + \cdots + A_nx^n\right)e^{rx}$$

will result in an expression of the form $P(x)e^{rx}$ where P is a polynomial of degree n (or less); $y'' + ay' + by$ applied to

$$z = \left(A_0 + A_1x + A_2x^2 + \cdots + A_nx^n\right)\cos\beta x$$

will result in an expression of the form $P(x)\cos\beta x + Q(x)\sin\beta x$ where P and Q are a polynomials of degree n (or less); and so on.

The general version of the method of undetermined coefficients can be summarized as follows:

(1) If $f(x) = p(x)e^{rx}$ where p is a polynomial of degree n, then

$$z(x) = \left(A_0 + A_1x + A_2x^2 + \cdots + A_nx^n\right)e^{rx}.$$

(2) If $f(x) = p_1(x)\cos\beta x + p_2(x)\sin\beta x$ where p_1 and p_2 are polynomials of degrees k and m, respectively, then

$$z(x) = (A_0 + A_1x + \cdots + A_nx^n)\cos\beta x + (B_0 + B_1x + \cdots + B_nx^n)\sin\beta x$$

where $n = \max\{k, m\}$.

(3) If $f(x) = p_1(x)e^{\alpha x}\cos\beta x + p_2(x)e^{\alpha x}\sin\beta x$ where p_1 and p_2 are polynomials of degrees k and m, respectively, then

$$z(x) = (A_0 + A_1x + \cdots + A_nx^n)\,e^{\alpha x}\cos\beta x + (B_0 + B_1x + \cdots + B_nx^n)\,e^{\alpha x}\sin\beta x$$

where $n = \max\{k, m\}$.

Note: If any term in z satisfies the reduced equation $y'' + ay' + by = 0$, then use xz as the trial solution; if any term in xz satisfies the reduced equation, then x^2z will give a particular solution.

Here are some examples.

Example 7. Find a particular solution of

$$y'' + 4y = (3 + 2x)e^{-2x}. \qquad (*)$$

SOLUTION The functions $y_1(x) = \cos 2x$, $y_2(x) = \sin 2x$ form a fundamental set of solutions of the reduced equation $y'' + 4y = 0$.

A particular solution of $(*)$ will have the form $z = (A + Bx)e^{-2x}$ where A and B are to be determined. The derivatives of z are:

$$z = (A + Bx)e^{-x}, \quad z' = -2(A + Bx)e^{-2x} + Be^{-2x}, \quad z'' = 4(A + Bx)e^{-2x} - 4Be^{-2x}.$$

Substituting z and its derivatives into the left side of $(*)$, we get

$$z'' + 4z = 4(A + Bx)e^{-2x} - 4Be^{-2x} + 4(A + Bx)e^{-2x} = [(8A - 4B) + 8Bx]e^{-2x}.$$

Thus z is a solution of $(*)$ if

$$[(8A - 4B) + 8Bx]e^{-2x} = (3 + 2x)e^{-2x} \quad \text{which implies} \quad 8A - 4B = 3 \text{ and } 8B = 2.$$

The solution of this pair of equations is $A = \frac{1}{2}$, $B = \frac{1}{4}$, and

$$z(x) = \left(\tfrac{1}{2} + \tfrac{1}{4}\,x\right) e^{-2x}$$

is a particular solution of $(*)$. ■

Example 8. Give the form of the general solution of each of the following nonhomogeneous equations:

(a) $y'' - 3y' + 2y = (1 + 2x - 4x^2)e^{2x}$.

(b) $y'' + 4y' + 4y = (3 - 5x)e^{-2x}$.

SOLUTION (a) The reduced equation is $y'' - 3y' + 2y = 0$. The characteristic equation is

$$r^2 - 3r + 2 = (r - 1)(r - 2) = 0.$$

Thus, $y_1(x) = e^x$, $y_2(x) = e^{2x}$ forms a fundamental set of solutions of the reduced equation.

According to the summary above, a particular solution z should have the form

$$z = (A_0 + A_1x + A_2x^2)e^{2x}$$

but A_0e^{2x} satisfies the reduced equation. Therefore we need to multiply the trial solution by x and try

$$z = (A_0x + A_1x^2 + A_2x^3)e^{2x}.$$

Since none of the terms in this z satisfies the reduced equation, this is the form of a particular solution.

The general solution of the equation will have the form

$$y = C_1e^x + C_2e^{2x} + (A_0x + A_1x^2 + A_2x^3)e^{2x}$$

where A_0, A_1, A_2 are constants which are to be determined.

(b) The reduced equation is $y'' + 4y' + 4y = 0$. The characteristic equation is

$$r^2 + 4r + 4 = (r + 2)^2 = 0.$$

Thus, $y_1(x) = e^{-2x}$, $y_2(x) = xe^{-2x}$ forms a fundamental set of solutions of the reduced equation.

According to the summary above, a particular solution z should have the form

$$z = (A_0 + A_1x)e^{-2x}$$

but A_0e^{-2x} and A_1xe^{-2x} satisfy the reduced equation. Therefore we need to multiply the trial solution by x and try

$$z = (A_0x + A_1x^2)e^{-2x}.$$

But A_0xe^{-2x} also satisfies the reduced equation so we need to multiply the initial z by x^2. Since none of the terms in

$$z = (A_0x^2 + a_1x^3)e^{-2x}$$

satisfies the reduced equation, this is the form of a particular solution.

The general solution of the equation will have the form

$$y = C_1e^{-2x} + C_2xe^{-2x} + (A_0x^2 + A_1x^3)e^{-2x}$$

where A_0, A_1 are constants which are to be determined. ■

Summary of Sections 3.4 and 3.5

The method of variation of parameters can be applied to *any* linear nonhomogeneous equation but it has the limitation of requiring a fundamental set of solutions of the reduced equation.

The method of undetermined coefficients is limited to linear nonhomogeneous equations with constant coefficients and with restrictions on the nonhomogeneous term f.

In cases where both methods are applicable, the method of undetermined coefficients is usually more efficient and, hence, the preferable method. ■

Exercises 3.5

Find the general solution.

1. $y'' - 2y' - 3y = 3e^{2x}$.
2. $y'' + 2y' + 2y = 10e^{x}$.
3. $y'' + 6y' + 9y = 9e^{3x}$.
4. $y'' + 6y' + 9y = e^{-3x}$.
5. $y'' + 2y' = 4\sin 2x$.
6. $y'' + y = 3\sin 2x + x\cos 2x$.
7. $2y'' + 3y' + y = x^2 + 3\sin x$.
8. $y'' - 6y' + 9y = e^{-3x}$.
9. $y'' + 5y' + 6y = 3x + 4$.
10. $y'' + 4y' + 4y = xe^{-x}$.
11. $y'' + 6y' + 8y = 3e^{-2x}$.
12. $y'' + 2y' + y = xe^{-x}$.
13. $y'' + 9y = x^2e^{3x} + 6$.
14. $y'' + y' - 2y = x^3 + x$.
15. $y'' - 2y' + 5y = e^{-x}\sin 2x$.
16. $y'' + 2y' + 5y = e^{2x}\cos x$.

Find the solution of the given initial-value problem.

17. $y'' + y' - 2y = 2x; \quad y(0) = 0,\ y'(0) = 1$.
18. $y'' + 4y = x^2 + 3e^{x}; \quad y(0) = 0,\ y'(0) = 2$.
19. $y'' - y' - 2y = \sin 2x; \quad y(0) = 1,\ y'(0) = -1$.
20. $y'' - 2y' + y = xe^{x} + 4; \quad y(0) = 1,\ y'(0) = 1$.

Determine a suitable form for a particular solution $z = z(x)$ of the given equation.

21. $y'' - 2y' - 3y = 6 - 3xe^{-x} + 4\cos 3x$.
22. $y'' + 2y' = 2x + x^2e^{-3x} + \sin 2x$.

23. $y'' + y = x^2 - 1 + 3\cos\, x - 2\sin\, x$.

24. $y'' - 5y' + 6y = 2e^{2x}\cos\, x - 3xe^{3x} + 5$.

25. $y'' - 4y' + 4y = 2xe^{2x} + x^2 - 1 + 2x\, \cos\, 2x$.

26. $y'' + 5y' + 6y = 2e^{2x}\cos\, x - 3xe^{3x} + 5e^{-2x}$.

27. $y'' + 2y' + 2y = 4e^{-x} + 2e^{-x}\cos\, x + 9$.

28. $y'' + 2y' + 5y = 4e^{-x}\sin\, 2x + 2e^{-x}\cos\, x$.

Find the general solution of the given differential equation.

29. $y'' - 4y' + 4y = 2\sin\, x + 3x^{-1}e^{2x}$.

30. $y'' - 2y' + y = \dfrac{e^x}{x^2+1} + 2e^{2x}$.

31. $y'' + 9y = 3\cos\, x - 9\sec^2\, 3x$.

32. $y'' + 4y = 5e^{4x} + 3 - \sec^2 2x$.

Exercises 33 and 34 are concerned with the differential equation

$$y'' + ay' + by = f(x)$$

where a and b are nonnegative constants.

33. Suppose that $a,\ b > 0$. Show that if $y_1(x)$ and $y_2(x)$ are solutions of the equation, then $y_1(x) - y_2(x) \to 0$ as $x \to \infty$. What happens if $a = 0$ and $b > 0$?

34. If $f(x) = c$, c a constant, show that every solution $y(x)$ of the equation has the property $y(x) \to c/b$ as $x \to \infty$. What happens if $b = 0$? What happens if $a = b = 0$?

3.6 Vibrating Mechanical Systems

Undamped Vibrations

A spring of length l_0 units is suspended from a support. When an object of mass m is attached to the spring, the spring stretches to a length l_1 units. If the object is then pulled down (or pushed up) an additional y_0 units at time $t = 0$ and then released, what is the resulting motion of the object? That is, what is the position $y(t)$ of the object at time $t > 0$? Assume that time is measured in seconds

We begin by analyzing the forces acting on the object at time $t > 0$. First, there is the weight of the object (gravity):

$$F_1 = mg.$$

This is a downward force. We choose our coordinate system so that the positive direction is down. Next, there is the restoring force of the spring. By Hooke's Law, this force is proportional to the total displacement $l_1 + y(t)$ and acts in the direction opposite to the displacement:

$$F_2 = -k[l_1 + y(t)] \qquad \text{with} \quad k > 0.$$

The constant of proportionality k is called the *spring constant.* If we assume that the spring is frictionless and that there is no resistance due to the surrounding medium (for example, air resistance), then these are the only forces acting on the object. Under these conditions, the total force is

$$F = F_1 + F_2 = mg - k[l_1 + y(t)] = (mg - kl_1) - ky(t).$$

Before the object was displaced, the system was in equilibrium, so the force of gravity, mg plus the force of the spring, $-kl_1$, must have been 0:

$$mg - kl_1 = 0.$$

Therefore, the total force F reduces to

$$F = -ky(t).$$

By Newton's Second Law of Motion, $F = ma$ (force = mass × acceleration), we have

$$ma = -ky(t) \quad \text{and} \quad a = -\frac{k}{m}y(t).$$

Therefore, at any time t we have

$$a = y''(t) = -\frac{k}{m}y(t) \quad \text{or} \quad y''(t) + \frac{k}{m}y(t) = 0.$$

When the acceleration is a constant negative multiple of the displacement, the object is said to be in *simple harmonic motion.*

Since $k/m > 0$, we can set $\omega = \sqrt{k/m}$ and write this equation as

$$y''(t) + \omega^2 y(t) = 0, \tag{1}$$

a second order linear homogeneous equation with constant coefficients. The characteristic equation is

$$r^2 + \omega^2 = 0$$

and the characteristic roots are $\pm\omega i$. The general solution of (1) is

$$y = C_1 \cos \omega t + C_2 \sin \omega t.$$

In Exercises 3.6 (Problem 5) you are asked to show that the general solution can be written as

$$y = A \sin(\omega t + \phi_0), \tag{2}$$

where A and ϕ_0 are constants with $A > 0$ and $\phi_0 \in [0, 2\pi)$. For our purposes here, this is the preferred form. The motion is *periodic* with *period* T given by

$$T = \frac{2\pi}{\omega},$$

a complete oscillation takes $2\pi/\omega$ seconds. The reciprocal of the period gives the number of oscillations per second. This is called the *frequency* , denoted by f:

$$f = \frac{\omega}{2\pi}.$$

Since $\sin(\omega t + \phi_0)$ oscillates between -1 and 1,

$$y(t) = A \sin(\omega t + \phi_0)$$

oscillates between $-A$ and A. The number A is called the *amplitude* of the motion. The number ϕ_0 is called the *phase constant* or the *phase shift* . The figure gives a typical graph of (2).

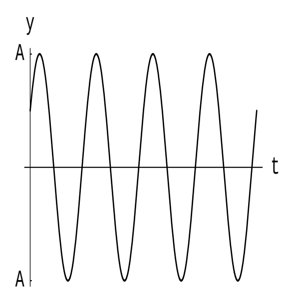

Figure 1

Example 1. Find an equation for the oscillatory motion of an object, given that the period is $2\pi/3$ and at time $t = 0$, $y = 1$, $y' = 3$.

SOLUTION In general the period is $2\pi/\omega$, so that here

$$\frac{2\pi}{\omega} = \frac{2\pi}{3} \quad \text{and therefore} \quad \omega = 3.$$

The equation of motion takes the form

$$y(t) = A \sin(3\omega + \phi_0).$$

Differentiating the equation of motion gives

$$y'(t) = 3A\cos(3t + \phi_0).$$

Applying the initial conditions, we have

$$y(0) = 1 = A\sin\,\phi_0, \qquad y'(0) = 3 = 3A\cos\,\phi_0$$

and therefore

$$A\sin\,\phi_0 = 1, \qquad A\cos\,\phi_0 = 1.$$

Adding the squares of these equations, we have

$$2 = A^2\sin^2\phi_0 + A^2\cos^2\phi_0 = A^2.$$

Since $A > 0$, $A = \sqrt{2}$.

Finally, to find ϕ_0, note that

$$\sqrt{2}\,\sin\,\phi_0 = 1 \quad \text{and} \quad \sqrt{2}\,\cos\,\phi_0 = 1.$$

These equations imply that $\phi_0 = \pi/4$. Thus, the equation of motion is

$$y(t) = \sqrt{2}\,\sin(3t + \tfrac{1}{4}\pi). \qquad \blacksquare$$

Damped Vibrations

If the spring is not frictionless or if there the surrounding medium resists the motion of the object (for example, air resistance), then the resistance tends to dampen the oscillations. Experiments show that such a resistant force R is approximately proportional to the velocity $v = y'$ and acts in a direction opposite to the motion:

$$R = -cy' \quad \text{with } c > 0.$$

Taking this force into account, the force equation reads

$$F = -ky(t) - cy'(t).$$

Newton's Second Law $F = ma = my''$ then gives

$$my''(t) = -ky(t) - cy'(t)$$

which can be written as

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0. \qquad (c,\ k,\ m \text{ all constant}) \tag{3}$$

This is the equation of motion in the presence of a *damping factor.*

The characteristic equation

$$r^2 + \frac{c}{m}r + \frac{k}{m} = 0$$

has roots

$$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}.$$

There are three cases to consider:

$$c^2 - 4km < 0, \qquad c^2 - 4km > 0, \qquad c^2 - 4km = 0.$$

Case 1: $c^2 - 4km < 0$. In this case the characteristic equation has complex roots:

$$r_1 = -\frac{c}{2m} + i\omega, \quad r_2 = -\frac{c}{2m} - i\omega \quad \text{where } \omega = \frac{\sqrt{4km - c^2}}{2m}.$$

The general solution is

$$y = e^{(-c/2m)t}\,(C_1 \cos \omega t + C_2 \sin \omega t)$$

which can also be written as

$$y(t) = A\,e^{(-c/2m)t} \sin(\omega t + \phi_0) \tag{4}$$

where, as before, A and ϕ_0 are constants, $A > 0$, $\phi_0 \in [0, 2\pi)$. This is called the *underdamped case.* The motion is similar to simple harmonic motion except that the damping factor $e^{(-c/2m)t}$ causes $y(t) \to 0$ as $t \to \infty$. The oscillations continue indefinitely with constant frequency $f = \omega/2\pi$ but diminishing amplitude $Ae^{(-c/2m)t}$. This motion is illustrated in Figure 2. ■

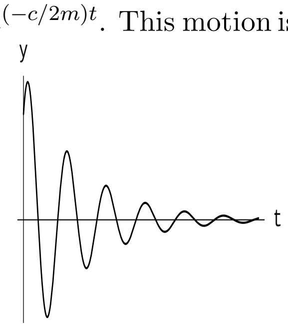

Figure 2

Case 2: $c^2 - 4km > 0$. In this case the characteristic equation has two distinct real roots:

$$r_1 = \frac{-c + \sqrt{c^2 - 4km}}{2m}, \qquad r_2 = \frac{-c - \sqrt{c^2 - 4km}}{2m}.$$

The general solution is

$$y(t) = y = C_1 e^{r_1 t} + C_2 e^{r_2 t}. \tag{5}$$

This is called the *overdamped case.* The motion is nonoscillatory. Since

$$\sqrt{c^2 - 4km} < \sqrt{c^2} = c,$$

r_1 and r_2 are both negative and $y(t) \to 0$ as $t \to \infty$. ■

Case 3: $c^2 - 4km = 0$. In this case the characteristic equation has only one real root:

$$r_1 = \frac{-c}{2m},$$

and the general solution is

$$y(t) = y = C_1 e^{-(c/2m)\,t} + C_2 t\, e^{-(c/2m)\,t}. \tag{6}$$

This is called the *critically damped case.* Once again, the motion is nonoscillatory and $y(t) \to 0$ as $t \to \infty$. ■

In both the overdamped and critically damped cases, the object moves back to the equilibrium position $(y(t) \to 0$ as $t \to \infty)$. The object may move through the equilibrium position once, but only once. Two typical examples of the motion are shown in Figure 3.

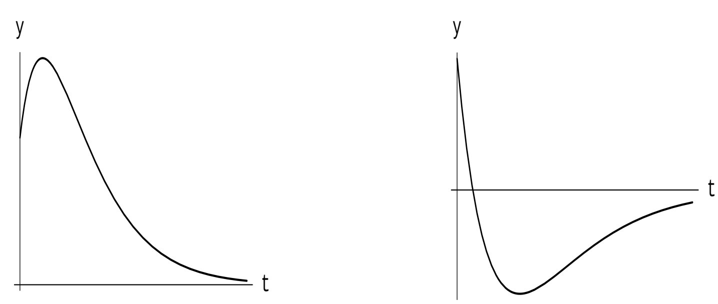

Figure 3

Forced Vibrations

The vibrations that we have considered thus far result from the interplay of three forces: gravity, the restoring force of the spring, and the retarding force of friction or the surrounding medium. Such vibrations are called *free vibrations* .

The application of an external force to a freely vibrating system modifies the vibrations and produces what are called *forced vibrations* . As an example we'll investigate the effect of a periodic external force $F_0 \cos \gamma t$ where F_0 and γ are positive constants.

In an undamped system the force equation is

$$F = -kx + F_0 \cos \gamma t$$

and the equation of motion takes the form

$$y'' + \frac{k}{m} y = \frac{F_0}{m} \cos \gamma t.$$

We set $\omega = \sqrt{k/m}$ and write the equation of motion as

$$y'' + \omega^2 y = \frac{F_0}{m} \cos \gamma t. \tag{7}$$

As we'll see, the nature of the motion depends on the relation between the *applied frequency* , $\gamma/2\pi$, and the *natural frequency* of the system, $\omega/2\pi$.

Case 1: $\gamma \neq \omega$. In this case the method of undetermined coefficients gives the particular solution

$$z(t) = \frac{F_0/m}{\omega^2 - \gamma^2} \cos \gamma t$$

and the general equation of motion is

$$y = A \sin(\omega t + \phi_0) + \frac{F_0/m}{\omega^2 - \gamma^2} \cos \gamma t. \tag{8}$$

If ω/γ is rational, the vibrations are periodic. If ω/γ is not rational, then the vibrations are not periodic and can be highly irregular. In either case, the vibrations are bounded by

$$|A| + \left| \frac{F_0/m}{\omega^2 - \gamma^2} \right|. \qquad \blacksquare$$

Case 2: $\gamma = \omega$. In this case the method of undetermined coefficients gives

$$z(t) = \frac{F_0}{2\omega m} t \sin \omega t$$

and the general solution has the form

$$y = A \sin(\omega t + \phi_0) + \frac{F_0}{2\omega m} t \sin \omega t. \tag{9}$$

The system is said to be in *resonance* . The motion is oscillatory but, because of the t factor in the second term, it is not periodic. As $t \to \infty$, the amplitude of the vibrations increases without bound.

A typical illustration of the motion is given in Figure 4. ■

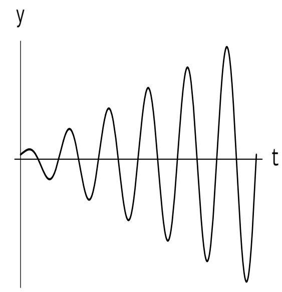

Figure 4

Exercises 3.6

1. An object is in simple harmonic motion. Find an equation for the motion given that the period is $\frac{1}{4}\pi$ and, at time $t = 0$, $y = 1$, $y' = 0$. What is the amplitude? What is the frequency?

2. An object is in simple harmonic motion. Find an equation for the motion given that the frequency is $1/\pi$ and, at time $t = 0$, $y = 0$, $y' = -2$. What is the amplitude? What is the period?

3. An object is in simple harmonic motion with period T and amplitude A. What is the velocity at the equilibrium point $y = 0$?

4. An object in simple harmonic motion passes through the equilibrium point $y = 0$ at time $t = 0$ and every three seconds thereafter. Find the equation of motion given that $y(0) = 5$.

5. Show that simple harmonic motion $y(t) = C_1 \cos \omega t + C_2 \sin \omega t$ can be written as: (a) $y(t) = A \sin(\omega t + \phi_0)$; (b) $y(t) = A \cos(\omega t + \psi_0)$.

6. What is the effect of an increase in the resistance constant c on the amplitude and frequency of the vibrations given by (4)?

7. Show that the motion given by (5) can pass through the equilibrium point at most once. How many times can the motion change directions?

8. Show that the motion given by (6) can pass through the equilibrium point at most once. How many times can the motion change directions?

9. Show that if $\gamma \neq \omega$, then the method of undetermined coefficients applied to (7) gives
$$z = \frac{F_0/m}{\omega^2 - \gamma^2} \cos \gamma t.$$

10. Show that if ω/γ is rational, then the vibrations given by (8) are periodic.

11. Show that if $\gamma = \omega$, then the method of undetermined coefficients applied to (7) gives

$$z = \frac{F_0}{2\omega m}\, t \sin \omega t.$$