Example 1

Problem: Find the solution $u$ to Laplace’s equation in the rectangle $[0, \pi] \times [0, \pi]$ such that $u(x, y) = F(x, y)$ on the boundary of $[0, \pi] \times [0, \pi]$ when $F(x, y) = x^2$. Stating the problem in more detail, it becomes that of finding $u$ such that

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0 \text{ for } (x, y) \text{ in } [0, \pi] \times [0, \pi],$$

$$u(0, y) = 0 \text{ when } 0 \leq y \leq \pi,$$
$$u(\pi, y) = \pi^2 \text{ when } 0 \leq y \leq \pi,$$
$$u(x, 0) = x^2 \text{ when } 0 \leq x \leq \pi,$$
$$u(x, \pi) = x^2 \text{ when } 0 \leq x \leq \pi.$$

Solution. First find the function $v$ such that

$$v(x, y) = ax + by + cxy + d$$

and $v(x, y) = F(x, y)$ at each of the four corners of $[0, \pi] \times [0, \pi]$. Thus $v$ is to satisfy

\begin{align*}
    v(0, 0) & = 0 \\
    v(\pi, 0) & = \pi^2 \\
    v(0, \pi) & = 0 \\
    v(\pi, \pi) & = \pi^2
\end{align*}

so

\begin{align*}
    d & = 0 \\
    a\pi & = \pi^2 \text{ implying } a = \pi \\
    b\pi & = 0 \text{ amplifying } b = 0 \\
    \pi^2 + c\pi^2 & = \pi^2 \text{ implying } c = 0.
\end{align*}

Thus

$$v(x, y) = \pi x$$

Next let

$$w(x, y) = u(x, y) - v(x, y)$$
and state the problem for $w$. Using the definition of $v$ and the fact that $v$ is a solution to Laplace’s equation we have

$$\frac{\partial^2 w}{\partial x^2}(x, y) + \frac{\partial^2 w}{\partial y^2}(x, y) = 0 \text{ for } (x, y) \text{ in } [0, \pi] \times [0, \pi],$$

$$w(0, y) = u(0, y) - v(0, y) = 0 - 0 = 0,$$
$$w(\pi, y) = u(\pi, y) - v(\pi, y) = \pi^2 - \pi^2 = 0,$$
$$w(x, 0) = u(x, 0) - v(x, 0) = x^2 - \pi x,$$
$$w(x, \pi) = u(x, \pi) - v(x, \pi) = x^2 - \pi x.$$

To find $w$ we follow the procedure given in Laplace Equation Problem I to find $w_1$ ($w_1$ is the same as $u_{\text{lower}}$) such that

$$\frac{\partial^2 w_1}{\partial x^2}(x, y) + \frac{\partial^2 w_1}{\partial y^2}(x, y) = 0 \text{ for } 0 \leq x \leq \pi \text{ and } 0 \leq y \leq \pi,$$
$$w_1(0, y) = 0 \text{ for } 0 \leq y \leq \pi,$$
$$w_1(\pi, y) = 0 \text{ for } 0 \leq y \leq \pi,$$
$$w_1(x, \pi) = 0 \text{ for } 0 \leq x \leq \pi,$$
$$w_1(x, 0) = x^2 - \pi x \text{ for } 0 \leq x \leq \pi,$$

the procedure given in Laplace Equation Problem II to find $w_2$ ($w_2$ is the same as $u_{\text{upper}}$) such that

$$\frac{\partial^2 w_2}{\partial x^2}(x, y) + \frac{\partial^2 w_2}{\partial y^2}(x, y) = 0 \text{ for } 0 \leq x \leq \pi \text{ and } 0 \leq y \leq \pi,$$
$$w_2(0, y) = 0 \text{ for } 0 \leq y \leq \pi,$$
$$w_2(\pi, y) = 0 \text{ for } 0 \leq y \leq \pi,$$
$$w_2(x, \pi) = x^2 - \pi x \text{ for } 0 \leq x \leq \pi,$$
$$w_2(x, 0) = 0 \text{ for } 0 \leq x \leq \pi,$$

the procedure given in Laplace Equation Problem III to find $w_3$ ($w_3$ is the same as $u_{\text{left}}$) such that

$$\frac{\partial^2 w_3}{\partial x^2}(x, y) + \frac{\partial^2 w_3}{\partial y^2}(x, y) = 0 \text{ for } 0 \leq x \leq \pi \text{ and } 0 \leq y \leq \pi,$$
$$w_3(0, y) = 0 \text{ for } 0 \leq y \leq \pi,$$
$$w_3(\pi, y) = 0 \text{ for } 0 \leq y \leq \pi,$$
$$w_3(x, \pi) = 0 \text{ for } 0 \leq x \leq \pi,$$
$$w_3(x, 0) = 0 \text{ for } 0 \leq x \leq \pi,$$

and the procedure given in Laplace Equation Problem IV to find $w_4$ ($w_4$ is the same as $u_{\text{right}}$)
such that
\[
\frac{\partial^2 w_4}{\partial x^2}(x, y) + \frac{\partial^2 w_4}{\partial y^2}(x, y) = 0 \text{ for } 0 \leq x \leq \pi \text{ and } 0 \leq y \leq \pi,
\]
\[
w_4(0, y) = 0 \text{ for } 0 \leq y \leq \pi,
\]
\[
w_4(\pi, y) = 0 \text{ for } 0 \leq y \leq \pi,
\]
\[
w_4(x, \pi) = 0 \text{ for } 0 \leq x \leq \pi, \text{ and }
\]
\[
w_4(x, 0) = 0 \text{ for } 0 \leq x \leq \pi.
\]

Then let

\[
w = w_1 + w_2 + w_3 + w_4.
\]

Using the formulas that have been worked out for each case and using the fact (integrate by parts) that

\[
\frac{2}{\pi \sinh \pi} \int_0^\pi (x^2 - \pi x) \sin kx \, dx = \frac{4((-1)^k - 1)}{k^3 \pi \sinh k\pi}
\]

we find that

\[
w_1(x, y) = \sum_{k=1}^{\infty} \frac{4((-1)^k - 1)}{k^3 \pi \sinh k\pi} \sin kx \sinh k(\pi - y),
\]

\[
w_2(x, y) = \sum_{k=1}^{\infty} \frac{4((-1)^k - 1)}{k^3 \pi \sinh k\pi} \sin kx \sinh ky,
\]

and

\[
w_3(x, y) = w_4(x, y) = 0
\]

for all $(x, y)$ in $[0, \pi] \times [0, \pi]$. Thus since $u = v + w,$

\[
u(x, y) = \pi x + \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{((-1)^k - 1)}{k^3 \sinh k\pi} (\sin kx)(\sinh k(\pi - y) + \sinh ky)
\]

for all $(x, y)$ in $[0, \pi] \times [0, \pi].$