The Heat Equation for a Rectangular Plate whose Edges are Held at Temperature Zero. The problem we consider is

\[
\frac{\partial u}{\partial t}(x,y,t) = \kappa \left( \frac{\partial^2 u}{\partial x^2}(x,y,t) + \frac{\partial^2 u}{\partial y^2}(x,y,t) \right),
\]

for \(0 \leq x \leq L, 0 \leq y \leq H\) and \(t \geq 0\).

(2) \quad u(0,y,t) = 0 \quad \text{for } 0 \leq y \leq H \text{ and } t \geq 0,

(3) \quad u(L,y,t) = 0 \quad \text{for } 0 \leq y \leq H \text{ and } t \geq 0,

(4) \quad u(x,0,t) = 0 \quad \text{for } 0 \leq x \leq L \text{ and } t \geq 0,

(5) \quad u(x,H,t) = 0 \quad \text{for } 0 \leq x \leq L \text{ and } t \geq 0,

(6) \quad u(x,y,0) = \varphi(x,y) \quad \text{for } 0 \leq x \leq L \text{ and } 0 \leq y \leq H

Solution: Suppose that

\[ u(x,y,t) = \varphi(x,y) \cdot \Phi(t) \]

From (1) we have

\[ \varphi(x,y) \cdot \Phi'(t) = \kappa \nabla^2 \varphi(x,y) \cdot \Phi(t) \]

Assuming for now that \(\varphi(x,y) \cdot \Phi(t) \neq 0\) and dividing each side of the last equation by this quantity, we have...
\[
\frac{\nabla^2 \phi(x,y)}{\phi(x,y)} = \frac{\kappa(t)}{\kappa_H(t)}
\]
for \(0 \leq x \leq L, 0 \leq y \leq H, \) and \(t \geq 0\)

Setting \(-\lambda\) be the common constant value we have

(7) \[ -\nabla^2 \phi(x,y) = \lambda \phi(x,y) \quad \text{for} \ 0 \leq x \leq L \quad \text{and} \quad 0 \leq y \leq H \quad \text{and} \]

(8) \[ \kappa(t) + \kappa_H(t) = 0 \quad \text{for} \ t \geq 0 \]

Note that if (7) and (8) hold and

(2) \[ \phi(x,y,0) = \phi(x,y) \quad \text{and} \]

then (1) will hold and we no longer need to assume that \(\phi(x,y,0) \neq 0\).

From (3) through (5) it follows that

(9) \[ \phi(x,y) = 0 \quad \text{for (x,y) on the boundary of the rectangle}. \]

A proper listing of eigenvalues and eigenfunctions for (7) and (9) is

\(\{k_{ij}, \lambda_{ij}\}_{i=1}^{\infty}, \) and \(\{\Phi_{ij}, \lambda_{ij}\}_{i=1}^{\infty}, \)

where

\[ \lambda_{ij} = \left( \frac{k_i \pi}{L} \right)^2 + \left( \frac{l_j \pi}{H} \right)^2 \quad \text{and} \]

\[ \Phi_{ij} = \sin \left( \frac{k_i \pi x}{L} \right) \sin \left( \frac{l_j \pi y}{H} \right) \]
\[ \phi_{kj}(x, y) = \sin \frac{k \pi x}{L} \sin \frac{j \pi y}{H} \text{ for } 0 \leq x \leq L \text{ and } 0 \leq y \leq H. \]

See "A two-dimensional rectangular eigenvalue problem."

When \( z = 2 \alpha_j \), the solutions to (8) are multiples of \( \phi_{kj} \) where

\[ \phi_{kj}(t) = e^{-\alpha_j z kj t} \]

Thus for a solution to (1) - (6) we expect

\[ u(x, y, t) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} A_{kj} \phi_{kj}(x, y) \phi_{kj}(t) \]

Noting that \( \phi_{kj}(0) = 1 \) for each \((k, j)\),

we see that (6) will hold if and only if

\[ \alpha = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} A_{kj} \phi_{kj} \]

and this will hold if

\[ A_{kj} = \frac{\langle \alpha, \phi_{kj} \rangle}{\langle \phi_{kj}, \phi_{kj} \rangle} \]

for \( k = 1, 2, \ldots \) and \( j = 1, 2, \ldots \).
Thus

\[ u(x,y,t) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} [A_{nj} \sin \frac{k \pi x}{L} \sin \frac{j \pi y}{H} \exp \left[-\kappa t \left( \frac{(k \pi)^2}{L^2} + \frac{j \pi (j \pi)}{H^2} \right) \right] \]

where \( A_{nj} = \frac{4}{LH} \int_0^L \int_0^H \alpha(x,y) \sin \frac{k \pi x}{L} \sin \frac{j \pi y}{H} \, dy \, dx \)