The Wave Equation for a Vibrating Rectangular Membrane

We consider the wave equation for a vibrating rectangular membrane whose edges are fixed to a flat frame. The problem is as follows.

\[
\frac{\partial^2 u}{\partial t^2}(x,y,t) = c^2 \left( \frac{\partial^2 u}{\partial x^2}(x,y,t) + \frac{\partial^2 u}{\partial y^2}(x,y,t) \right) \tag{1}
\]

for \(0 \leq x \leq L, \ 0 \leq y \leq H,\) and all \(t,\)

\[
u(x,y,t) = 0 \text{ for } (x,y) \text{ on the boundary of } [0,L] \times [0,H] \text{ and all } t, \tag{2}
\]

\[
u(x,y,0) = \alpha(x,y) \text{ for } 0 \leq x \leq L, \text{ and } 0 \leq y \leq H, \text{ and } \tag{3}
\]

\[
\frac{\partial \nu}{\partial t}(x,y,0) = \beta(x,y) \text{ for } 0 \leq x \leq L, \text{ and } 0 \leq y \leq H. \tag{4}
\]

**Solution.** Suppose that

\[
u(x,t) = \varphi(x,y) h(t). \tag{5}
\]

From (1) it follows that

\[
\varphi(x,y) h''(t) = c^2 \left( \frac{\partial^2 \varphi}{\partial x^2}(x,y) + \frac{\partial^2 \varphi}{\partial y^2}(x,y) \right) h(t) \tag{5}
\]

for \(0 \leq x \leq L, \ 0 \leq x \leq H \) and all \(t \) in \( \mathbb{R}. \)

Assuming for now that \( \varphi(x,y) h(t) \neq 0 \) and dividing each side of (5) by \( c^2 \) times this quantity, we have

\[
\nabla^2 \varphi(x,y) = \frac{1}{c^2} \frac{h''(t)}{h(t)} \text{ for } 0 \leq x \leq L, \ 0 \leq x \leq H \text{ and all } t \text{ in } \mathbb{R}. \tag{6}
\]

Letting \(-\lambda\) be the common constant value for the left and right sides of (6), it follows that

\[
\nabla^2 \varphi(x,y) = \lambda \varphi(x,y) \text{ for } 0 \leq x \leq L \text{ and } 0 \leq y \leq H \tag{7}
\]

and

\[
h''(t) + c^2 \lambda h(t) = 0 \text{ for all } t \text{ in } \mathbb{R}. \tag{8}
\]

It should be noted that if (7) and (8) hold and \( u(x,y,t) = \varphi(x,y) h(t) \) then (1) will hold and the assumption that \( \varphi(x,y) h(t) \neq 0 \) is not needed. If \( u \) is not the zero function ( (3) and (4) will not hold if it is), it follows from (2) that

\[
\varphi(x,y) = 0 \text{ for all } (x,y) \text{ on the boundary of the rectangle } [0,L] \times [0,H]. \tag{9}
\]
A proper listing of eigenvalues and eigenfunctions (See 'A Two-dimensional Rectangular Eigenvalue Problem') for (7) and (9) is given by

$$\{\lambda_{kj}\}_{k=1,j=1}^{\infty} \text{ and } \{\varphi_{kj}\}_{k=1,j=1}^{\infty}$$

where

$$\lambda_{kj} = \left(\frac{k\pi}{L}\right)^2 + \left(\frac{j\pi}{H}\right)^2$$

and

$$\varphi_{kj}(x, y) = \sin\frac{k\pi x}{L} \sin\frac{j\pi y}{H}$$

for $0 \leq x \leq L, 0 \leq y \leq H, k = 1, 2, \ldots$, and $j = 1, 2, \ldots$.

When $\lambda = \lambda_{kj}$ the solutions to (8) are linear combinations of $h_{1kj}$ and $h_{2kj}$ where

$$h_{1kj}(t) = \cos \sqrt{\lambda_{kj}} ct, \text{ and } h_{2kj}(t) = \sin \sqrt{\lambda_{kj}} ct \text{ for } k = 1, 2, \ldots \text{ and } j = 1, 2, \ldots.$$}

Considering the possible combinations, we expect the solution to (1)-(4) to be given by

$$u(x, y, t) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \varphi_{kj}(x, y) (A_{kj} h_{1kj}(t) + B_{kj} h_{2kj}(t))$$

and

$$\frac{\partial u}{\partial t}(x, y, t) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \varphi_{kj}(x, y) (A_{kj} h'_{1kj}(t) + B_{kj} h'_{2kj}(t))$$

Note that $h_{1kj}(0) = 1, h_{2kj}(0) = 0, h'_{1kj}(0) = 0$, and $h'_{2kj}(0) = \sqrt{\lambda_{kj}} c$ for for $k = 1, 2, \ldots \text{ and } j = 1, 2, \ldots$.

Condition (3) will hold if and only if

$$\alpha = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} A_{kj} \varphi_{kj}$$

and this will hold if and only if

$$A_{kj} = \frac{\langle \alpha, \varphi_{kj} \rangle}{\langle \varphi_{kj}, \varphi_{kj} \rangle} \text{ for } k = 0, 1, \ldots \text{ and } j = 1, 2, \ldots.$$}

Condition (4) will hold if and only if

$$\beta = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} B_{kj} \sqrt{\lambda_{kj}} c \varphi_{kj}$$

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and this will hold if and only if

\[ B_{kj} = \frac{1}{\sqrt{\lambda_{kj}c}} \frac{< \beta, \varphi_{kj} >}{< \varphi_{kj}, \varphi_{kj} >} \text{ for } k = 1, 2, \ldots \text{ and } j = 1, 2, \ldots \]

Thus the solution is given by

\[ u(x, y, t) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \sin \frac{k\pi x}{L} \sin \frac{j\pi y}{H} \left( A_{kj} \cos \sqrt{\lambda_{kj}c}t + B_{kj} \sin \sqrt{\lambda_{kj}c}t \right) \]

where

\[ A_{kj} = \frac{4}{LH} \int_{0}^{L} \int_{0}^{H} \alpha(x, y) \sin \frac{k\pi x}{L} \sin \frac{j\pi y}{H} dy dx \text{ for } k = 1, 2, \ldots \text{ and } j = 1, 2, \ldots \]

and

\[ B_{k} = \frac{4}{LH \sqrt{\lambda_{kj}c}} \int_{0}^{L} \int_{0}^{H} \beta(x, y) \sin \frac{k\pi x}{L} \sin \frac{j\pi y}{H} dy dx \text{ for } k = 1, 2, \ldots \text{ and } j = 1, 2, \ldots \]

in which

\[ \lambda_{kj} = \left( \frac{k\pi}{L} \right)^2 + \left( \frac{j\pi}{H} \right)^2. \]