

NOTE

We give an axiomatic description of a system (denoted by \mathbb{R}) that is commonly known as the real number system. We will then show how the natural numbers (\mathbb{N}), integers (\mathbb{Z}), and rational numbers (\mathbb{Q}) fit into it. We also show how the complex numbers (\mathbb{C}) can be built up from it.

In as much as the so called real numbers are no more nor less real than any other numbers, we will, for now at least, just call them numbers.

Proofs of the theorems that follow will be addressed as we move through the course.

NUMBERS

The Number System \mathbb{R}

We assume the existence of certain things called numbers along with an operation called addition and an operation called multiplication.

When each of x and y is a number, the value of addition at x and y is denoted by

$$x + y$$

and the value of multiplication is denoted by

$$xy \text{ or } x \cdot y.$$

ADDITION AND MULTIPLICATION

Note. The operations of addition and multiplication are actually functions of a certain type. Because of this if each of a , b , and c is a number and

$$a = b \text{ then } a + c = b + c \text{ and } ac = bc$$

(There will be more about functions later in the course.)

EQUALITY

Note. The symbol

=

stands for the word *is* with the meaning *is the same as*. Thus

if $A = B$, then $B = A$.

if $A = B$ and $B = C$ then $A = C$,

and

$$A = A.$$

AXIOM I

Axiom I. Addition is commutative. If each of x and y is a number, then

$$x + y = y + x.$$

AXIOM II

Axiom II. Addition is associative. If each of x , y , and z is a number, then

$$(x + y) + z = x + (y + z).$$

AXIOM III

Definition. An additive identity is a number E with the property that if x is a number, then

$$x + E = x.$$

Axiom III. There is an additive identity.

UNIQUENESS

Theorem 1. There is only one additive identity.

Proof. Axiom III shows that there is at least one additive identity. To show that there is at most one, suppose that each of E and F is an additive identity. Since E is an additive identity and F is a number,

$$F + E = F.$$

Since F is an additive identity and E is a number,

$$E + F = E.$$

Since addition is commutative,

$$E + F = F + E.$$

Thus

$$E = F.$$

ZERO

Note. Since there is only one additive identity, we may give it a special name and notation.

Definition. The additive identity is called *zero* and is denoted by 0.

Note. If x is a number then,

$$x + 0 = x \text{ and } 0 + x = x.$$

AXIOM IV

Definition. If x is a number, an additive inverse for x is a number E such that $x + E = 0$.

Axiom IV. If x is a number, there is an additive inverse for x .

UNIQUENESS

Theorem 2. If x is a number, there is only one additive inverse for x .

Proof. Axiom IV shows that if x is a number, then there is an additive inverse for x . To show that there is at most one, suppose that x is a number and each of E and F is an additive inverse for x . Then

$$x + E = 0 \text{ and } x + F = 0.$$

Since

$$E + 0 = E + 0,$$

it follows that

$$E + (x + E) = E + (x + F).$$

Using associativity and commutativity it follows that

$$(E + x) + E = (E + x) + F \text{ and } (x + E) + E = (x + E) + F,$$

so

$$0 + E = 0 + F.$$

Thus

$$E = F$$

MINUS X

Note. When x is a number, its additive inverse is denoted by $-x$ and is called *minus x*.

THEOREM.

Theorem 3. If x is a number then $-(-x) = x$.

Theorem 4.

$$-0 = 0$$

SUBTRACTION

Definition. When each of x and y is a number, $x - y$ is defined to be $x + (-y)$.

AXIOM V

Axiom V. Multiplication is commutative. If each of x and y is a number, then

$$xy = yx$$

AXIOM VI

Axiom VI. Multiplication is associative. If each of x , y , and z is a number, then

$$(xy)z = x(yz) \text{ or } (x \cdot y) \cdot z = x \cdot (y \cdot z)$$

AXIOM VII

Definition. A multiplicative identity is a number E with the property that if x is a number, then $x \cdot E = x$.

Axiom VII. There is a multiplicative identity.

UNIQUENESS

Theorem 5. There is only one multiplicative identity.

ONE

Note. Since there is only one multiplicative identity, we may give it a special name and

notation.

Definition. The multiplicative identity is called *one* and is denoted by 1.

Note. If x is a number, then

$$x \cdot 1 = x \text{ and } 1 \cdot x = x.$$

AXIOM VIII

Axiom VIII. The numbers zero and one are different.

$$0 \neq 1$$

AXIOM IX

Definition. If x is a number, a multiplicative inverse for x is a number E such that $x \cdot E = 1$.

Axiom IX. If x is a number different from zero, then there is a multiplicative inverse for x .

Theorem 6. If x is a number different from zero there is only one multiplicative inverse for x .

RECIPROCAL

Note. When x is a number different from zero, the multiplicative inverse of x is called the *reciprocal* of x or *one over x* and is denoted by $1/x$.

DIVISION.

Definition. When each of x and y is a number and $y \neq 0$,

$$x/y \text{ or } \frac{x}{y}$$

is defined to be $x \cdot (1/y)$.

AXIOM X

Axiom X. Multiplication distributes over addition, If each of x , y , and z is a number, then

$$x(y + z) = xy + xz.$$

THEOREMS

Theorem 7. If x is a number, then $0 \cdot x = 0$.

Theorem 8. If x is a nonzero number then $1/(1/x) = x$

Theorem 9. The multiplicative inverse of 1 is 1.

Theorem 10. If x is a number, then $(-1) \cdot x = -x$

Theorem 11. $(-1)(-1) = 1$

Theorem 12. If each of x and y is a number, then $(-x)(-y) = xy$

ALGEBRA

Note. At this point, all the valid algebraic manipulations involving addition, subtraction, multiplication and division with which you are already familiar can be justified.

EXAMPLES

Example. If each of a , b , c , and d is a number, then

$$(a + b)(c + d) = ac + bc + ad + bd.$$

Example. If each of a , b , c , and d is a number, $b \neq 0$, and $d \neq 0$, then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

Example. If each of a , b , c , and x is a number, $a \neq 0$, and

$$ax + b = c, \text{ then } x = \frac{c - b}{a}.$$

POSITIVE NUMBERS - AXIOM XI

Note. Some numbers are called positive numbers.

Axiom XI. If x is a number, then exactly one of the following is true.

x is a positive number.

x is zero.

$-x$ is a positive number.

POSITIVE NUMBERS - AXIOM XII AND XIII

Axiom XII. If each of x and y is a positive number then $x + y$ is a positive number.

Axiom XIII. If each of x and y is a positive number then xy is a positive number.

DEFINITION

Definition. When each of x and y is a number, then $x < y$ means $y - x$ is a positive number.

$$x > y \text{ means } y < x.$$

$$x \leq y \text{ means } x < y \text{ or } x = y.$$

$$x \geq y \text{ means } y \leq x.$$

$$x < y < z \text{ means } x < y \text{ and } y < z.$$

$$x \leq y \leq z \text{ means } x \leq y \text{ and } y \leq z.$$

$$x < y \leq z \text{ means } x < y \text{ and } y \leq z.$$

$$x \leq y < z \text{ means } x \leq y \text{ and } y < z.$$

NOTE

Note. We identify numbers with points on a line. In this identification,

$$x < y$$

means that the point corresponding to x is to the left of the point corresponding to y .

THEOREM

Theorem 13. If each of x and y is a number, then exactly one of the following is true.

$$x < y.$$

$$x = y.$$

$$x > y.$$

THEOREMS

Theorem 14. If each of x , y , and z is a number $x < y$ and $y < z$ then $x < z$.

Theorem 15. The number one is a positive number.

THEOREM

Theorem 16. Suppose that each of x , y , z , and w is a number.

1. If $x < y$, then $x + z < y + z$.
2. If $0 < x$, then $0 < 1/x$.
3. If $x < 0$, then $1/x < 0$.
4. If $x < y$ and $0 < z$, then $xz < yz$.
5. If $x < y$ and $z < 0$, then $yz < xz$.
6. If $x < y$ and $z < w$, then $x + z < y + w$.
7. If $0 < x < y$ and $0 < z < w$, then $0 < xz < yw$.
8. If $x < 0$ and $y < 0$, then $x + y < 0$ and $xy > 0$.
9. If $x \neq 0$, then $0 < x^2$. (x^2 means $x \cdot x$.)
10. If $0 < x < y$, then $0 < 1/y < 1/x$.
11. If $x \leq y$ and $y \leq x$, then $x = y$.
12. If $x < y$, then $x < \frac{x+y}{2} < y$. ($2 = 1 + 1$)

Definition. Suppose that S is a set of numbers. The statement that p is a greatest member of S means that p is a member of S and if x is a member of S , then $x \leq p$. The statement that q is a least member of S means that q is a member of S and if x is a member of S , then $q \leq x$.

Theorem 17. If S is a set of numbers, then S has at most one greatest member, and S has at most one least member.

Note. If S is the set of all numbers x such that $0 \leq x \leq 1$, then 1 is the greatest member and 0 is the least member of S . If S is the set of all numbers x such that $0 < x < 1$, then S has no least member, and S has no greatest member.

AXIOM XIV

Axiom XIV. If A and B are sets of numbers such that

1. if x is a number, then x is in A or x is in B , and
 2. if x is in A and y is in B then $x < y$
- then A contains a greatest number or B contains a least number.

NOTE

Note. As noted, we identify numbers with points on a line. With Axioms I-XIV each number corresponds to a point and each point corresponds to a number. Without Axiom XIV, there would be many points that would not correspond to numbers. Axiom XIV is sometimes called the Completeness Axiom.

NOTE

Note. There is essentially only one system that satisfies Axioms I-XIV. This justifies calling such a system **the** real number system and calling its members real numbers. These numbers are no more nor less real than any other numbers. The terminology arose in connection with complex numbers.

Theorem 18. If x is a number and $x^2 = 0$, then $x = 0$. If $x > 0$ there is exactly one positive number z such that $z^2 = x$.

Definition. When $x \geq 0$, the symbol \sqrt{x} is called the *square root* of x or just *root* x . And is defined as follows: $\sqrt{0} = 0$ and when $x > 0$, then \sqrt{x} is the positive number z such that $z^2 = x$.

Theorem 19. When $x > 0$, there are exactly two numbers z such that $z^2 = x$. One is \sqrt{x} and the other is $-\sqrt{x}$.

Definition. When x is a number, the **absolute value** of x is denoted by $|x|$ and is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} .$$

Note. When numbers are identified with points on a line, $|x|$ corresponds to the distance between x and 0, and $|x - y|$ corresponds to the distance between x and y .

Theorem 20. Suppose that each of x and y is a number. Then

1. $|x| \geq 0$,
2. $|x| = 0$ if and only if $x = 0$,

3. $|-x| = |x|$,
4. $|xy| = |x||y|$,
5. $|x| \leq y$ if and only if $-y \leq x \leq y$,
6. $|x| < y$ if and only if $-y < x < y$,
7. $-|x| \leq x \leq |x|$,
8. $|x + y| \leq |x| + |y|$, and
9. $||x| - |y|| \leq |x - y|$.