A METHOD FOR SOLVING NONHOMOGENEOUS
DIFFERENTIAL EQUATIONS

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We give a formula that can be used to solve any regular nth order nonhomogeneous linear differential equation with a continuous right side provided that a fundamental sequence (i.e. a linearly independent n-tuple of solutions to the corresponding homogeneous equation) and certain antiderivatives can be found. We begin with the second order case.

Theorem 1. Suppose that \( L \) is a second order regular linear differential operator over the interval \( J \) with
\[
Ly(x) = a(x)y''(x) + b(x)y'(x) + c(x)y(x).
\]
Suppose that \((u_1, u_2)\) is a fundamental pair for \( L \) and that \( f \) is a continuous function defined on \( J \). Let \( w \) be the Wronskian of \((u_1, u_2)\), let \( I_1 \) be an antiderivative of \( \frac{u_1 f}{aw} \),
\[
I_1(x) = \int u_1(x) \frac{f(x)}{a(x)w(x)} \, dx,
\]
and let \( I_2 \) be an antiderivative of \( \frac{u_2 f}{aw} \),
\[
I_2(x) = \int u_2(x) \frac{f(x)}{a(x)w(x)} \, dx.
\]
Finally, let \( v \) be given by
\[
v = u_2 I_1 - u_1 I_2.
\]
It follows that
\[
Lv = f.
\]

It is worth noting that \( v \) can be given on one line by the following formula.
\[
v(x) = u_2(x) \int u_1(x) \frac{f(x)}{a(x)w(x)} \, dx - u_1(x) \int u_2(x) \frac{f(x)}{a(x)w(x)} \, dx
\]
Only one antiderivative is needed in each case, so leave off the “\( +C \)" when finding the integrals.

Proof. Let \( v \) be as indicated. Then
\[
v' = u_2 I_1' + u_2 I_1' - u_1 I_2' - u_1 I_2'
= u_2 I_1 + u_2 \frac{u_1 f}{aw} - u_1 I_2 - u_1 \frac{u_2 f}{aw}
= u_2 I_1 - u_1 I_2'
\]
and since \( w = u_1u_2' - u_1'u_2 \),
\[
v'' = u_2''I_1 + u_2'u_1' - u_1''I_2 - u_1'u_2'
= u_2''I_1 + u_2'u_1' + u_2'\frac{u_1f}{aw} - u_1'u_2' - u_1\frac{u_2f}{aw}
= u_2''I_1 - u_1''I_2 + \frac{(u_1u_2' - u_1'u_2)}{aw}
= u_2''I_1 - u_1''I_2 + \frac{f}{a}.
\]

Thus
\[
Lv = av'' + bv' + cv
= a \left( u_2''I_1 - u_1''I_2 + \frac{f}{a} \right)
+ b(u_2'I_1 - u_1'I_2)
+ c(u_2I_1 - u_1I_2).
\]

So
\[
Lv = (au_2'' + bu_2' + cu_2)I_1 - (au_1'' + bu_1' + cu_1)I_2 + f
= (Lu_2)I_1 - (Lu_1)I_2 + f
= 0 \cdot I_1 - 0 \cdot I_2 + f
= f.
\]

**Example 1.** Find all functions \( y \) such that
\[
y''(x) + y(x) = \tan x \quad \text{for} \quad 0 < x < \frac{\pi}{2}.
\]

**Solution.** We are looking for all \( y \) such that
\[
Ly = f \quad \text{on} \quad J
\]
where \( Ly = y'' + y, \ f(x) = \tan x, \) and \( J = (0, \frac{\pi}{2}) \). The leading coefficient function is the constant function with value 1, and a fundamental sequence for \( L \) is \((u_1, u_2)\) where \( u_1(x) = \cos x \) and \( u_2(x) = \sin x \). The Wronskian \( w \) is given by
\[
w(x) = \det \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} = 1.
\]

\[
I_1(x) = \int \frac{\cos x \tan x}{(1)(1)} \, dx = -\cos x
\]
and
\[
I_2(x) = \int \frac{\sin x \tan x}{(1)(1)} \, dx = \int \frac{\sin^2 x}{\cos x} \, dx = \int \frac{1 - \cos^2 x}{\cos x} \, dx
= \int (\sec x - \cos x) \, dx = \ln(\sec x + \tan x) - \sin x.
\]

So a function \( v \) such that \( Lv = f \) is given by
\[
v(x) = (\sin x)(-\cos x) - (\cos x)(\ln(\sec x + \tan x) - \sin x)
= -\cos x \ln(\sec x + \tan x).
\]
Since \( Ly = f \) if and only if \( y = u + v \) for some \( u \) such that \( Lu = 0 \), we find that \( y \) is a solution to the problem if and only if

\[
y(x) = c_1 \cos x + c_2 \sin x - \cos x \ln(\sec x + \tan x)
\]

for some pair of numbers \((c_1, c_2)\) and all \( x \) with \( 0 < x < \frac{\pi}{2} \).

Sometimes when the method presented here is used, the function \( v \) can be seen to be of the form

\[
v = v_1 + v_2
\]

where \( v_2 \) is a solution to the corresponding homogeneous equation. In this case \( v_1 \) provides a simpler solution than \( v \) to the nonhomogeneous equation. This is because

\[
Lv_1 = L(v - v_2) = Lv - Lv_2 = f - 0 = f.
\]

**Example 2.** Find all functions \( y \) such that

\[
y''(x) - 3y'(x) + 2y(x) = 1 + e^{-x}
\]

for all \( x \) in \( \mathbb{R} \).

Solution. We are looking for all \( y \) such that

\[
Ly = f \text{ on } J
\]

where \( Ly = y'' - 3y' + 2y, \ f(x) = \frac{1}{1 + e^{-x}} \), and \( J = \mathbb{R} \). The leading coefficient function is the constant function with value 1, and a fundamental sequence for \( L \) is \((u_1, u_2)\) where \( u_1(x) = e^x \) and \( u_2(x) = e^{2x} \). The Wronskian \( w \) is given by

\[
w(x) = \det \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix} = e^{3x}.
\]

\[
I_1(x) = \int e^x \cdot \frac{1}{1 + e^{-x}} \, dx = \ln(1 + e^{-x}) - e^{-x}
\]

and

\[
I_2(x) = \int e^{2x} \cdot \frac{1}{1 + e^{-x}} \, dx = -\ln(1 + e^{-x}).
\]

So a function \( v \) such that \( Lv = f \) is given by

\[
v(x) = e^{2x}(\ln(1 + e^{-x}) - e^{-x}) + e^x \ln(1 + e^{-x})
\]

\[
= e^{2x} \ln(1 + e^{-x}) + e^x \ln(1 + e^{-x}) - e^x
\]

\[
= v_1(x) + v_2(x)
\]

where

\[
v_1(x) = e^{2x} \ln(1 + e^{-x}) + e^x \ln(1 + e^{-x}) \text{ and }
\]

\[
v_2(x) = -e^x.
\]

Since \( Lv_2 = -Lu_1 = 0 \), a simpler solution to the nonhomogeneous equation is \( v_1 \).

Since \( Ly = f \) if and only if \( y = u + v_1 \) for some \( u \) such that \( Lu = 0 \), we find that \( y \) is a solution to the problem if and only if

\[
y(x) = c_1 e^x + c_2 e^{2x} + e^{2x} \ln(1 + e^{-x}) + e^x \ln(1 + e^{-x})
\]

for some pair of numbers \((c_1, c_2)\) and all \( x \).
The formula that we have given for the second order is a special case of the one presented in the following theorem.

**Theorem 2.** Suppose that $L$ is an $n$th order regular linear differential operator over the interval $J$ with

$$Ly(x) = a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots + a_n(x)y(x).$$

Suppose that $(u_1, \ldots, u_n)$ is a fundamental sequence for $L$ and that $f$ is a continuous function defined on $J$. Let $w$ be the Wronskian of $(u_1, \ldots, u_n)$, and for $k = 1, \ldots, n$, let $w_k$ be the determinant of the matrix obtained by replacing the $k$th column of the Wronski matrix (the matrix used to find the Wronskian) of $(u_1, \ldots, u_n)$ with the column

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

For $k = 1, \ldots, n$, let $I_k$ be an antiderivative of $\frac{w_k f}{a_0 w}$,

$$I_k(x) = \int \frac{w_k(x)f(x)}{a_0(x)w(x)} \, dx.$$

Finally, let $v$ be given by

$$v(x) = \sum_{k=1}^{n} u_k(x)I_k(x).$$

It follows that

$$Lv = f.$$