

# Deriving d'Alembert's Solution to the Wave Equation

In what follows,  $c$  is a positive number.

1. Suppose that

$$u(x, t) = f(x + ct) + g(x - ct)$$

for some pair of twice differentiable one-place functions  $f$  and  $g$  and all  $(x, t)$ . Show that

$$\frac{\partial^2 u}{\partial t^2}(x, t) = c^2 \frac{\partial^2 u}{\partial x^2}(x, t)$$

for all  $(x, t)$ .

**Solution.**

$$\frac{\partial u}{\partial t}(x, t) = f'(x + ct) \cdot c + g'(x - ct) \cdot (-c) \quad (1)$$

$$\frac{\partial^2 u}{\partial t^2}(x, t) = f''(x + ct) \cdot c \cdot c + g''(x - ct) \cdot (-c) \cdot (-c) \quad (2)$$

$$\frac{\partial^2 u}{\partial t^2}(x, t) = c^2[f''(x + ct) + g''(x - ct)] \quad (3)$$

$$\frac{\partial u}{\partial t}(x, t) = f'(x + ct) + g'(x - ct) \quad (4)$$

$$\frac{\partial^2 u}{\partial x^2}(x, t) = f''(x + ct) + g''(x - ct) \quad (5)$$

The result follows from (3) and (5).

2. Suppose that

$$\frac{\partial^2 u}{\partial t^2}(x, t) = c^2 \frac{\partial^2 u}{\partial x^2}(x, t) \quad (1)$$

for all  $(x, t)$ . Show that

$$u(x, t) = f(x + ct) + g(x - ct)$$

for some pair of twice differentiable one-place functions  $f$  and  $g$  and all  $(x, t)$ .

**Solution.** Let

$$S(x, t) = (x + ct, x - ct)$$

and

$$T(x, t) = \left(\frac{x+t}{2}, \frac{x-t}{2c}\right)$$

for all  $(x, t)$ . Computation shows that

$$S(T(x, t)) = T(S(x, t)) = (x, t)$$

for all  $(x, t)$ .

Let

$$w(x, t) = u(T(x, t))$$

so that

$$u(x, t) = w(S(x, t)) = w(x + ct, x - ct)$$

for all  $(x, t)$ . Letting  $p$  stand for  $x + ct, x - ct$  it follows that

$$\frac{\partial u}{\partial t}(x, t) = \frac{\partial w}{\partial x}(p) \cdot c + \frac{\partial w}{\partial t}(p) \cdot (-c) \quad (2)$$

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x, t) &= \left[ \frac{\partial^2 w}{\partial x^2}(p) \cdot c + \frac{\partial^2 w}{\partial t \partial x}(p) \cdot (-c) \right] \cdot c \\ &\quad + \left[ \frac{\partial^2 w}{\partial x \partial t}(p) \cdot c + \frac{\partial^2 w}{\partial t^2}(p) \cdot (-c) \right] \cdot (-c) \end{aligned} \quad (3)$$

$$\frac{\partial^2 u}{\partial t^2}(x, t) = \left[ \frac{\partial^2 w}{\partial x^2}(p) + \frac{\partial^2 w}{\partial t^2}(p) \right] \cdot c^2 - 2 \frac{\partial^2 w}{\partial t \partial x}(p) \cdot c^2 \quad (4)$$

$$\frac{\partial u}{\partial x}(x, t) = \frac{\partial w}{\partial x}(p) + \frac{\partial w}{\partial t}(p) \quad (5)$$

$$\frac{\partial^2 u}{\partial x^2}(x, t) = \left[ \frac{\partial^2 w}{\partial x^2}(p) + \frac{\partial^2 w}{\partial t \partial x}(p) \right] + \left[ \frac{\partial^2 w}{\partial x \partial t}(p) + \frac{\partial^2 w}{\partial y^2}(p) \right] \quad (6)$$

$$\frac{\partial^2 u}{\partial x^2}(x, t) = \left[ \frac{\partial^2 w}{\partial x^2}(p) + \frac{\partial^2 w}{\partial t^2}(p) \right] + 2 \frac{\partial^2 w}{\partial t \partial x}(p) \quad (7)$$

From (1), (4), and (7) it follows that

$$-4c^2 \frac{\partial^2 w}{\partial t \partial x}(p) = 0$$

so

$$\frac{\partial^2 w}{\partial t \partial x}(x + ct, x - ct) = 0$$

or

$$\frac{\partial^2 w}{\partial y \partial x}(S(x, t)) = 0$$

for all  $(x, t)$ . Thus, since  $T(x, t)$  is a point in the  $x$ - $t$  plane whenever  $(x, t)$  is,

$$\frac{\partial^2 w}{\partial y \partial x}(S(T(x, t))) = 0$$

for all  $(x, t)$ . So

$$\frac{\partial^2 w}{\partial t \partial x}(x, t) = 0$$

for all  $(x, t)$ . From this it follows that

$$\frac{\partial w}{\partial x}(x, t) = h(x)$$

for some one-place function  $h$  and from this it follows that

$$w(x, t) = f(x) + g(t)$$

where  $f$  is an anti-derivative of  $h$  and  $g$  is a one-place function. Since

$$u(x, t) = w(x + ct, x - ct)$$

it follows that

$$u(x, t) = f(x + ct) + g(x - ct)$$

for all  $(x, t)$ .

3. Suppose that  $\psi$  is differentiable,  $\varphi$  is twice differentiable, and

$$u(x, t) = \frac{1}{2} [\varphi(x + ct) + \varphi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds \quad (1)$$

for all  $(x, t)$ . Show that

$$\frac{\partial^2 u}{\partial t^2}(x, t) = c^2 \frac{\partial^2 u}{\partial x^2}(x, t) \quad (2)$$

for all  $(x, t)$ ,

$$u(x, 0) = \varphi(x)$$

for all  $x$ , and

$$\frac{\partial u}{\partial t}(x, 0) = \psi(x)$$

for all  $x$ .

**Solution.** From (1) it follows that

$$\begin{aligned} u(x, t) &= \frac{1}{2} [\varphi(x + ct) + \varphi(x - ct)] + \frac{1}{2c} \int_0^{x+ct} \psi(s) ds - \frac{1}{2c} \int_0^{x-ct} \psi(s) ds \\ &= \left[ \frac{1}{2} \varphi(x + ct) + \frac{1}{2c} \int_0^{x+ct} \psi(s) ds \right] + \left[ \frac{1}{2} \varphi(x - ct) - \frac{1}{2c} \int_0^{x-ct} \psi(s) ds \right] \quad (3) \end{aligned}$$

showing that  $u(x, t)$  is a one-place function at  $x + ct$  plus a one-place function at  $x - ct$ . Thus (2) follows from Part 1. Also

$$u(x, 0) = \frac{1}{2} \varphi(x) + \frac{1}{2} \varphi(x) + 0 = \varphi(x)$$

for all  $x$ . From (3) we have

$$\frac{\partial u}{\partial t}(x, t) = \frac{c}{2} [\varphi'(x + ct) - \varphi'(x - ct)] + \frac{c}{2c} \psi(x + ct) + \frac{c}{2c} \psi(x - ct)$$

so

$$\frac{\partial u}{\partial t}(x, 0) = \psi(x)$$

for all  $x$ .

4. Suppose that

$$\frac{\partial^2 u}{\partial t^2}(x, t) = c^2 \frac{\partial^2 u}{\partial x^2}(x, t) \quad (1)$$

for all  $(x, t)$ ,

$$u(x, 0) = \varphi(x) \quad (2)$$

for all  $x$ , and

$$\frac{\partial u}{\partial t}(x, 0) = \psi(x) \quad (3)$$

for all  $x$ . Show that

$$u(x, t) = \frac{1}{2} [\varphi(x + ct) + \varphi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds$$

**Solution.** From Part 2 it follows that

$$u(x, t) = f(x + ct) + g(x - ct)$$

for some pair of one-place functions  $f$  and  $g$ . Thus

$$\frac{\partial u}{\partial t}(x, t) = f'(x + ct) \cdot c - g'(x - ct) \cdot c.$$

From (2) and (3) we have

$$\varphi(x) = f(x) + g(x) \quad (4)$$

and

$$\psi(x) = cf'(x) + cg'(x).$$

Then

$$\varphi'(x) = f'(x) + g'(x) \quad (5)$$

and

$$\frac{1}{c}\psi(x) = f'(x) - g'(x) \quad (6)$$

for all  $x$ . Solving (5) and (6) for  $f'$  and  $g'$  we have

$$f'(x) = \frac{1}{2} \left( \varphi'(x) + \frac{1}{c}\psi(x) \right)$$

and

$$g'(x) = \frac{1}{2} \left( \varphi'(x) - \frac{1}{c}\psi(x) \right)$$

for all  $x$ . Integrating from 0 to  $z$  we have

$$f(z) - f(0) = \frac{1}{2}\varphi(z) - \frac{1}{2}\varphi(0) + \frac{1}{2c} \int_0^z \psi(s) ds \quad (7)$$

and

$$g(z) - g(0) = \frac{1}{2}\varphi(z) - \frac{1}{2}\varphi(0) - \frac{1}{2c} \int_0^z \psi(s) ds. \quad (8)$$

Replacing  $z$  with  $x + ct$  in (7) and by  $x - ct$  in (8), adding and using (4) to get  $-f(0) - g(0) = -\varphi(0)$ , we get

$$f(x + ct) + g(x - ct) + \frac{1}{2c} \int_0^{x+ct} \psi(s) ds - \frac{1}{2c} \int_0^{x-ct} \psi(s) ds.$$

Thus

$$u(x,t) = \frac{1}{2} [\varphi(x + ct) + \varphi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds$$

for all  $(x, t)$ .