Heat Diffusion Equation Problem II

**PROBLEM:** Derive the solution to

\[
\frac{\partial u}{\partial t}(x,t) = \kappa \frac{\partial^2 u}{\partial x^2}(x,t) \text{ for } t \geq 0 \text{ and } 0 \leq x \leq L,
\]

(1)

\[
\frac{\partial u}{\partial x}(0,t) = 0 \text{ for } t \geq 0,
\]

(2)

\[
\frac{\partial u}{\partial x}(L,t) = 0 \text{ for } t \geq 0, \quad \text{and}
\]

(3)

\[
u(x,0) = f(x) \text{ for } 0 \leq x \leq L
\]

(4)

where each of \(\kappa\) and \(L\) is a positive number. Find the solution when

\[
f(x) = x^2(L - x)^2 \text{ for } 0 \leq x \leq L.
\]

(5)

**SOLUTION:** Suppose that \(u\) is an elementary separated solution to (1). This means

\[
u(x,t) = \phi(x)G(t)
\]

for some pair of one-place functions \(\phi\) and \(G\). Inserting this into (1), we have

\[
\phi(x)G'(t) = \kappa \phi''(x)G(t).
\]

(6)

Assuming for now that \(u(x,t) \neq 0\),

and dividing each side of (6) by \(\phi(x)G(t)\), we have

\[
\frac{\phi(x)G'(t)}{\phi(x)G(t)} = \kappa \frac{\phi''(x)G(t)}{\phi(x)G(t)},
\]

so

\[
\frac{G'(t)}{G(t)} = \kappa \frac{\phi''(x)}{\phi(x)}.
\]

This holds for all \(t \geq 0\) and \(x\) with \(0 \leq x \leq L\), so there is a constant \(C\) such that

\[
\frac{G'(t)}{G(t)} = C = \kappa \frac{\phi''(x)}{\phi(x)}
\]

(7)

for all \(t \geq 0\) and \(x\) with \(0 \leq x \leq L\). As a matter of notational convenience and so that we can more easily make use of our earlier work on two-point boundary value problems, we let

\[
\lambda = -\frac{C}{\kappa}.
\]

From (7) we then have

\[-\phi''(x) = \lambda \phi(x) \text{ for all } x \in [0, L]
\]

(8)

and

\[G'(t) = -\kappa \lambda G(t) \text{ for all } t \geq 0.
\]

(9)
It is worth noting that if
\[ u(x, t) = \varphi(x)G(t) \]
and (8) and (9) hold, then
\[
\frac{\partial u}{\partial t}(x, t) = \varphi(x)G'(t) = -\varphi(x)\kappa \lambda G(t) \\
= \kappa \varphi''(x)G(t) = \kappa \frac{\partial^2 u}{\partial x^2}(x, t)
\]
so the PDE (1)
\[
\frac{\partial u}{\partial t}(x, t) = \kappa \frac{\partial^2 u}{\partial x^2}(x, t)
\]
will be satisfied, and we no longer need to assume that \( u(x, t) \neq 0 \). Continuing with our assumption that
\[ u(x, t) = \varphi(x)G(t) \]
we have from conditions (2) and (3) (which stated that \( \frac{\partial u}{\partial x}(0, t) = 0 = \frac{\partial u}{\partial x}(L, t) \)) that
\[ \varphi'(0) = 0 \tag{10} \]
and
\[ \varphi'(L) = 0. \tag{11} \]
The Sturm-Louville problem consisting of (8), (10), and (11) (which we repeat here)
\[
-\varphi'' = \lambda \varphi \text{ on } [0, L] \\
\varphi'(0) = 0, \text{ and} \\
\varphi'(L) = 0
\]
is one which we have studied. A proper listing of eigenvalues and eigenfunctions for this problem is
\[
\{\lambda_k\}_{k=0}^\infty \text{ and } \{\varphi_k\}_{k=0}^\infty
\]
where
\[ \lambda_0 = 0, \]
\[ \lambda_k = \left(\frac{k\pi}{L}\right)^2 \text{ for } k = 1, 2, \ldots \]

\[ \varphi_0(x) = 1 \]

and
\[ \varphi_k(x) = \cos \frac{k\pi}{L}x \text{ for all } x \text{ in } [0, L] \text{ and } k = 1, 2, \ldots. \]
When
\[ \lambda = \lambda_k \]
the solutions to
\[ G'(t) = -\kappa \lambda G(t) \text{ for all } t \geq 0. \tag{9} \]
are constant multiples of $G_k$ where
\[ G_0(t) = 1 \]
and
\[ G_k(t) = e^{-\kappa \lambda_k t} \]
for $k = 1, 2, 3, \ldots$.

Let us recall the original problem
\[
\begin{align*}
\frac{\partial u}{\partial t} (x, t) &= \kappa \frac{\partial^2 u}{\partial x^2} (x, t) \text{ for } t \geq 0 \text{ and } 0 \leq x \leq L, \\
\frac{\partial u}{\partial x} (0, t) &= 0 \text{ for } t \geq 0, \\
\frac{\partial u}{\partial x} (L, t) &= 0 \text{ for } t \geq 0, \text{ and} \\
u(x, 0) &= f(x) \text{ for } 0 \leq x \leq L.
\end{align*}
\]

The problem consisting of (1), (2), and (3) is linear and homogeneous, so of $\{A_k\}_{k=0}^n$ is a finite sequence of numbers and
\[ u(x, t) = \sum_{k=0}^n A_k \varphi_k(x) G_k(t), \]
then $u$ will be a solution to (1), (2), and (3). Thus we hope that the solution to the problem consisting of (1) through (4) will be of the form
\[ u(x, t) = \sum_{k=0}^{\infty} A_k \varphi_k(x) G_k(t) \]
for some perhaps infinite sequence of constants $\{A_k\}$. Noting that $G_k(0) = 1$, we see that condition (4),
\[ u(x, 0) = f(x) \text{ for } x \text{ in } [0, L], \]
implies
\[ f = \sum_{k=0}^{\infty} A_k \varphi_k. \]
Since $\{\varphi_k\}_{k=0}^{\infty}$ is an orthogonal sequence of non zero functions this implies
\[ A_k = \frac{\langle f, \varphi_k \rangle}{\langle \varphi_k, \varphi_k \rangle} \]
for $k = 0, 1, \ldots$ where the inner product is defined by
\[ \langle \alpha, \beta \rangle = \int_0^L \alpha(x)\beta(x) dx. \]
For this sequence $\{\varphi_k\}$,
\[ \langle \varphi_0, \varphi_0 \rangle = \int_0^L (1)^2 dx = L \]
and
\[ <\varphi_k, \varphi_k> = \int_0^L (\cos \frac{k\pi x}{L})^2 dx = \frac{L}{2} \text{ for } k = 1, 2, \ldots. \]

In summary, the solution to the original problem (1) through (4) is \( u \) where
\[
u(x, t) = A_0 + \sum_{k=1}^{\infty} A_k \cos \frac{k\pi x}{L} e^{-\kappa \left( \frac{k\pi x}{L} \right)^2 t}
\]
in which
\[
A_0 = \frac{1}{L} \int_0^L f(x) dx
\]
and
\[
A_k = \frac{2}{L} \int_0^L f(x) \cos \frac{k\pi x}{L} dx \text{ for } k = 1, 2, \ldots.
\]

If \( f \) is given by
\[
f(x) = x^2(L - x)^2 \text{ for } 0 \leq x \leq L
\]
then
\[
A_0 = \frac{1}{L} \int_0^L x^2(L - x)^2 dx = \frac{L^4}{30}
\]
and
\[
A_k = \frac{2}{L} \int_0^L x^2(L - x)^2 \cos \frac{k\pi x}{L} dx
\]
for \( k = 1, 2, \ldots. \)

Remembering that
\[
\sin k\pi = 0 \text{ and } \cos k\pi = (-1)^k
\]
and integrating by parts, we find that
\[
A_k = -\frac{24L^4}{\pi^4 k^4} (1 + (-1)^k)
\]
for \( k = 1, 2, \ldots. \) So
\[
u(x, t) = \frac{L^4}{30} - \frac{24L^4}{\pi^4} \sum_{k=1}^{\infty} \frac{1}{k^4} (1 + (-1)^k) \cos \frac{k\pi x}{L} e^{-\kappa \left( \frac{k\pi x}{L} \right)^2 t}.
\]