Laplace Equation Problem V

**PROBLEM:** Suppose that each of $L$ and $H$ is a positive number and that $F$ is a continuous function defined on the boundary of the rectangle consisting of all $(x, y)$ where $0 \leq x \leq L$ and $0 \leq y \leq H$. Find the solution to

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0$$

(1)

for $0 \leq x \leq L$ and $0 \leq y \leq H$ such that

$$u(x, y) = F(x, y)$$

(2)

for all $(x, y)$ on the boundary of the rectangle.

**SOLUTION:** In order for (2) to hold at the corners of the rectangle and to improve the convergence of the series that occur in the solution, we begin by finding the function $v$ of the form

$$v(x, y) = ax + by + cxy + d$$

that has the same values as $F$ at the four corners of the rectangle. The system to solve for $(a, b, c, d)$ is

$$
\begin{align*}
    d &= F(0, 0) \\
    aL + d &= F(L, 0) \\
    aL + bH + cLH + d &= F(L, H) \\
    bH + d &= F(0, H)
\end{align*}
$$

or

$$
\begin{pmatrix}
0 & 0 & 0 & 1 \\
L & 0 & 0 & 1 \\
L & H & LH & 1 \\
0 & H & 0 & 1
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix}
= 
\begin{pmatrix}
F(0, 0) \\
F(L, 0) \\
F(L, H) \\
F(0, H)
\end{pmatrix}
$$
The coefficient matrix is nonsingular, so there will be a unique solution. Note that \( v \) satisfies Laplace equation

\[
\frac{\partial^2 v}{\partial x^2}(x, y) + \frac{\partial^2 v}{\partial y^2}(x, y) = 0
\]

for all \((x, y)\) in the plane.

Next we find the function \( w \) such that

\[
\frac{\partial^2 w}{\partial x^2}(x, y) + \frac{\partial^2 w}{\partial y^2}(x, y) = 0
\]

for \(0 \leq x \leq L\) and \(0 \leq y \leq H\) such that

\[
w(x, y) = F(x, y) - v(x, y)
\]

for all \((x, y)\) on the boundary of the rectangle. To find \( w \) we follow the procedure given in Laplace Equation Problem I to find \( w_1 \) such that

\[
\frac{\partial^2 w_1}{\partial x^2}(x, y) + \frac{\partial^2 w_1}{\partial y^2}(x, y) = 0 \quad \text{for } 0 \leq x \leq L \text{ and } 0 \leq y \leq H,
\]

\[
w_1(0, y) = 0 \quad \text{for } 0 \leq y \leq H,
\]

\[
w_1(L, y) = 0 \quad \text{for } 0 \leq y \leq H,
\]

\[
w_1(x, H) = 0 \quad \text{for } 0 \leq x \leq L, \text{ and}
\]

\[
w_1(x, 0) = F(x, 0) - v(x, 0) \quad \text{for } 0 \leq x \leq L,
\]

the procedure given in Laplace Equation Problem II to find \( w_2 \) such that

\[
\frac{\partial^2 w_2}{\partial x^2}(x, y) + \frac{\partial^2 w_2}{\partial y^2}(x, y) = 0 \quad \text{for } 0 \leq x \leq L \text{ and } 0 \leq y \leq H,
\]

\[
w_2(0, y) = 0 \quad \text{for } 0 \leq y \leq H,
\]

\[
w_2(L, y) = 0 \quad \text{for } 0 \leq y \leq H,
\]

\[
w_2(x, H) = F(x, H) - v(x, H) \quad \text{for } 0 \leq x \leq L, \text{ and}
\]

\[
w_2(x, 0) = 0 \quad \text{for } 0 \leq x \leq L,
\]

the procedure given in Laplace Equation Problem III to find \( w_3 \) such that

\[
\frac{\partial^2 w_3}{\partial x^2}(x, y) + \frac{\partial^2 w_3}{\partial y^2}(x, y) = 0 \quad \text{for } 0 \leq x \leq L \text{ and } 0 \leq y \leq H,
\]

\[
w_3(0, y) = F(0, y) - v(0, y) \quad \text{for } 0 \leq y \leq H,
\]

\[
w_3(L, y) = 0 \quad \text{for } 0 \leq y \leq H,
\]

\[
w_3(x, H) = 0 \quad \text{for } 0 \leq x \leq L, \text{ and}
\]

\[
w_3(x, 0) = 0 \quad \text{for } 0 \leq x \leq L,
\]
and the procedure given in Laplace Equation Problem IV to find \( w_4 \) such that

\[
\frac{\partial^2 w_4}{\partial x^2}(x, y) + \frac{\partial^2 w_4}{\partial y^2}(x, y) = 0 \text{ for } 0 \leq x \leq L \text{ and } 0 \leq y \leq H,
\]

\[
w_4(0, y) = 0 \text{ for } 0 \leq y \leq H,
\]

\[
w_4(L, y) = F(L, y) - v(L, y) \text{ for } 0 \leq y \leq H,
\]

\[
w_4(x, H) = 0 \text{ for } 0 \leq x \leq L, \text{ and }
\]

\[
w_4(x, 0) = 0 \text{ for } 0 \leq x \leq L.
\]

Then let

\[
w = w_1 + w_2 + w_3 + w_4.
\]

Finally, let

\[
u(x, y) = v(x, y) + w(x, y)
\]

for all \((x, y)\) with \(0 \leq x \leq L\) and \(0 \leq y \leq H\).