Two-point Boundary Value Problem I

**PROBLEM:** Consider the following Sturm-Liouville problem in which $L$ is a positive number.

(i) $-\varphi''(x) = \lambda \varphi(x)$ for $0 \leq x \leq L$,
(ii) $\varphi(0) = 0$, and
(iii) $\varphi(L) = 0$.

(a) Use the Rayleigh Quotient to show that all eigenvalues are positive.

(b) Find a proper listing of eigenvalues and eigenfunctions. For this problem, this is a pair of sequences $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ where the sequence of numbers $\{\lambda_k\}$ is increasing and lists each eigenvalue, and the sequence of real valued functions $\{\varphi_k\}$ is such that $\varphi_k$ is an eigenfunction corresponding to $\lambda_k$ for each $k$.

**SOLUTION:** Suppose that $\lambda$ is an eigenvalue and $\varphi$ is a corresponding real valued eigenfunction. Multiplying each side of (i) by $\varphi(x)$ and integrating from 0 to $L$, we have

$$- \int_0^L \varphi''(x) \varphi(x) dx = \lambda \int_0^L \varphi(x) \varphi(x) dx.$$

Integrating by parts on the left, we have

$$- \left( [\varphi'(x) \varphi(x)]_{x=0}^{x=L} - \int_0^L \varphi'(x) \varphi'(x) dx \right) = \lambda \int_0^L \varphi(x) \varphi(x) dx$$

or

$$[\varphi'(0) \varphi(0)] - [\varphi'(L) \varphi(L)] + \int_0^L (\varphi'(x))^2 dx = \lambda \int_0^L (\varphi(x))^2 dx.$$

Since $\varphi(0) = 0$ and $\varphi(L) = 0$,

it follows that

$$\lambda = \frac{\int_0^L (\varphi'(x))^2 dx}{\int_0^L (\varphi(x))^2 dx}$$

and from this it follows that

$$\lambda \geq 0.$$

If $\lambda$ were zero, then

$$\int_0^L (\varphi'(x))^2 dx = 0.$$

This would imply

$$\varphi'(x) = 0 \text{ for all } x \text{ in } [0, L].$$

This in turn, would imply

$$\varphi(x) = C \text{ for all } x \text{ in } [0, L]$$

for some constant $C$ which must be non zero since $\varphi$ is an eigenfunction. This cannot happen because of (ii) which states that

$$\varphi(0) = 0.$$
Thus
\[ \lambda > 0. \]

**All eigenvalues are positive.**

Suppose that \( \lambda \) is an eigenvalue and \( \varphi \) is a corresponding eigenfunction. We have just seen that \( \lambda > 0 \). Let
\[ \rho = \sqrt{\lambda}. \]
Since (i) states
\[ -\varphi''(x) = \lambda \varphi(x) \quad \text{for} \quad 0 \leq x \leq L, \]
we see that \( \varphi \) is a solution of the constant coefficient second order linear ODE
\[ \varphi'' + \rho^2 \varphi = 0 \quad \text{on} \quad [0, L], \]
so
\[ \varphi(x) = c_1 \cos \rho x + c_2 \sin \rho x \]
for some pair of numbers \( c_1 \) and \( c_2 \) and all \( x \) in \([0, L]\). Condition (ii) states that \( \varphi(0) = 0 \) so
\[ 0 = c_1 \cos 0 + c_2 \sin 0 \quad \text{implying} \quad c_1 = 0. \]
Thus
\[ \varphi(x) = c_2 \sin \rho x \]
for some \( c_2 \) which must be different from zero since \( \varphi \) is an eigenfunction. Condition (iii) states that
\[ \varphi(L) = 0 \]
so
\[ \sin \rho L = 0. \]
From this it follows that
\[ \rho L = k\pi \quad \text{or} \quad \rho = \frac{k\pi}{L} \quad \text{or} \quad \lambda = \left( \frac{k\pi}{L} \right)^2 \]
for some positive integer \( k \), and using that same integer \( k \),
\[ \varphi(x) = c_2 \sin \frac{k\pi}{L} x \quad \text{for all} \quad x \in [0, L] \]
and some number \( c_2 \neq 0 \). It is easy to verify by direct substitution into (i), (ii), and (iii) that if \( \lambda \) and \( \varphi \) are so given, then \( \lambda \) is an eigenvalue and \( \varphi \) is a corresponding eigenfunction.

We are working with a Sturm-Liouville problem so all eigenspaces are one-dimensional. In order to account for all eigenfunctions we need only one for each eigenvalue. Sumerizing our results, we see that a **proper listing of eigenvalues and eigenfunctions for this problem** is
\[ \{\lambda_k\}_{k=1}^{\infty} \quad \text{and} \quad \{\varphi_k\}_{k=1}^{\infty} \]
where
\[ \lambda_k = \left( \frac{k\pi}{L} \right)^2 \quad \text{for} \quad k = 1, 2, \ldots \]
and
\[ \varphi_k(x) = \sin \frac{k\pi}{L} x \quad \text{for all} \quad x \in [0, L] \quad \text{and} \quad k = 1, 2, \ldots. \]