Wave Equation Problem I

**PROBLEM:** Suppose that each of \( c \) and \( L \) is a positive number. Derive the solution to

\[
\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \quad \text{for } 0 \leq x \leq L \text{ and all } t \in \mathbb{R},
\]

(1)

\[
u(0,t) = 0 \text{ for all } t \in \mathbb{R},
\]

(2)

\[
u(L,t) = 0 \text{ for all } t \in \mathbb{R},
\]

(3)

\[
u(x,0) = f(x) \text{ for } 0 \leq x \leq L, \text{ and}
\]

(4)

\[
\frac{\partial u}{\partial t}(x,0) = g(x) \text{ for } 0 \leq x \leq L.
\]

(5)

Then find the solution when

\[
f(x) = x(L-x) \text{ for } 0 \leq x \leq L
\]

and

\[
g(x) = \left| x - \frac{L}{2} \right| - \frac{L}{2} \quad \text{for } 0 \leq x \leq L.
\]

**SOLUTION:** Suppose that \( u \) is an elementary separated solution to (1). This means

\[
u(x,t) = \varphi(x) h(t)
\]

for some pair of one-place functions \( \varphi \) and \( h \). Inserting this into (1), we have

\[
\varphi(x) h''(t) = c^2 \varphi''(x) h(t).
\]

(6)

Assuming for now that

\[
u(x,t) \neq 0,
\]

and dividing each side of (6) by \( \varphi(x) h(t) \), we have

\[
\frac{\varphi(x) h''(t)}{\varphi(x) h(t)} = \frac{c^2 \varphi''(x) h(t)}{\varphi(x) h(t)},
\]

so

\[
\frac{h''(t)}{h(t)} = c^2 \frac{\varphi''(x)}{\varphi(x)}.
\]

This holds for all \( t \) and all \( x \) with \( 0 \leq x \leq L \), so there is a constant \( K \) such that

\[
\frac{h''(t)}{h(t)} = K = \frac{c^2 \varphi''(x)}{\varphi(x)}
\]

(7)

for all \( t \) and all \( x \) with \( 0 \leq x \leq L \). As a matter of notational convenience and so that we can more easily make use of our earlier work on two-point boundary value problems, we let

\[
\lambda = -\frac{K}{c^2} \quad \text{so} \quad K = -c^2 \lambda.
\]

From (7) we then have

\[
-\varphi''(x) = \lambda \varphi(x) \text{ for all } x \text{ in } [0,L]
\]

(8)
and
\[h''(t) = -\lambda c^2 h(t) \text{ for all } t.\] (9)

It is worth noting that if
\[u(x,t) = \varphi(x)h(t)\]
and (8) and (9) hold, then
\[
\frac{\partial^2 u}{\partial t^2}(x,t) = \varphi(x)h''(t) = -\lambda c^2 \varphi(x)h(t)
= c^2 \varphi''(x)h(t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t)
\]
so the PDE (1)
\[
\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t)
\]
will be satisfied, and we no longer need to assume that \(u(x,t) \neq 0\). Continuing with our assumption that
\[u(x,t) = \varphi(x)h(t)\]
We have from conditions (2) and (3) (which stated that \(u(0,t) = 0 = u(L,t)\)) that either \(h(t) = 0\) for all \(t\) which we reject because of (4) and (5) (which stated that \(u(x,0) = f(x)\) and \(\frac{\partial u}{\partial t}(x,0) = g(x)\)) or
\[
\varphi(0) = 0 \quad (10)
\]
and
\[
\varphi(L) = 0 \quad (11)
\]
which we must accept.

The Sturm-Louville problem consisting of (8), (10), and (11) (which we repeat here)
\[-\varphi'' = \lambda \varphi \text{ on } [0,L] \\
\varphi(0) = 0, \text{ and} \\
\varphi(L) = 0\]
is one which we have studied. A proper listing of eigenvalues and eigenfunctions for this problem is
\[
\{\lambda_k\}_{k=1}^{\infty} \text{ and } \{\varphi_k\}_{k=1}^{\infty}
\]
where
\[
\lambda_k = \left(\frac{k\pi}{L}\right)^2 \text{ for } k = 1, 2, \ldots
\]
and
\[
\varphi_k(x) = \sin \frac{k\pi}{L}x \text{ for all } x \text{ in } [0, L] \text{ and } k = 1, 2, \ldots.
\]
The equation (9)
\[h''(t) = -c^2 \lambda h(t)\]
is equivalent to
\[ h''(t) + c^2 \lambda h(t) = 0. \] (12)

When \( \lambda > 0 \) as it must be because all eigenvalues for the problem (8), (10), and (11) are positive, a linearly independent pair of solutions to (12) is the pair whose values at \( t \) are \( \cos \sqrt{\lambda} ct \) and \( \sin \sqrt{\lambda} ct \).

Thus when \( \lambda = \lambda_k \) the solutions to (9) are linear combinations of the functions \( h_{1k} \) and \( h_{2k} \) where
\[ h_{1k}(t) = \cos \sqrt{\lambda_k} ct \] and \( h_{2k}(t) = \sin \sqrt{\lambda_k} ct \).

Let us recall the original problem.
\[ \frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \quad \text{for} \quad 0 \leq x \leq L \quad \text{and} \quad t \in \mathbb{R}, \] (1)
\[ u(0,t) = 0 \quad \text{for all} \quad t \in \mathbb{R}, \] (2)
\[ u(L,t) = 0 \quad \text{for all} \quad t \in \mathbb{R}, \] (3)
\[ u(x,0) = f(x) \quad \text{for} \quad 0 \leq x \leq L, \quad \text{and} \] (4)
\[ \frac{\partial u}{\partial t}(x,0) = g(x) \quad \text{for} \quad 0 \leq x \leq L. \] (5)

The problem consisting of (1), (2), and (3) is linear and homogeneous, so if \( \{A_k\}_{k=1}^n \) and \( \{B_k\}_{k=1}^n \) are finite sequences of numbers and
\[ u(x,t) = \sum_{k=1}^n \varphi_k(x)[A_k h_{1k}(t) + B_k h_{2k}(t)], \]
then \( u \) will be a solution to (1), (2), and (3). Thus we hope that the solution to the problem consisting of (1) through (5) will be of the form
\[ u(x,t) = \sum_{k=1}^\infty \varphi_k(x)[A_k h_{1k}(t) + B_k h_{2k}(t)] \] (13)
for some perhaps infinite sequences of constants \( \{A_k\}_{k=1}^\infty \) and \( \{B_k\}_{k=1}^\infty \).

Condition (4)
\[ u(x,0) = f(x) \quad \text{for} \quad x \in [0,L], \]
implies
\[ f = \sum_{k=1}^\infty \varphi_k[A_k h_{1k}(0) + B_k h_{2k}(0)] = \sum_{k=1}^\infty [A_k \cos 0 + B_k \sin 0]\varphi_k = \sum_{k=1}^\infty A_k \varphi_k. \]

Since \( \{\varphi_k\}_{k=1}^\infty \) is an orthogonal sequence of non zero functions this implies
\[ A_k = \frac{\langle f, \varphi_k \rangle}{\langle \varphi_k, \varphi_k \rangle}. \]
so for $k = 1, 2, \ldots$ where the inner product is defined by

$$< \alpha, \beta > = \int_0^L \alpha(x)\beta(x)dx.$$ 

For this sequence $\{\varphi_k\}$,

$$< \varphi_k, \varphi_k > = \int_0^L (\sin \frac{k\pi x}{L})^2 dx = \frac{L}{2} \text{ for } k = 1, 2, \ldots.$$ 

Returning to (13) we expect

$$\frac{\partial u}{\partial t}(x, t) = \sum_{k=1}^{\infty} \varphi_k(x)[A_k h'_1(t) + B_k h'_2(t)].$$

Condition (5) implies

$$\frac{\partial u}{\partial t}(x, 0) = g(x) \text{ for all } x \text{ in } [0, L]$$

so

$$(\frac{k\pi c}{L})B_k = \frac{<g, \varphi_k>}{<\varphi_k, \varphi_k>} \text{ or } B_k = (\frac{2}{L})(\frac{L}{k\pi c}) \frac{<g, \varphi_k>}{<\varphi_k, \varphi_k>}$$

for $k = 1, 2, 3, \ldots$. In summary, the solution to the original problem (1) through (5) is $u$ where

$$u(x, t) = \sum_{k=1}^{\infty} [A_k \cos \frac{k\pi ct}{L} + B_k \sin \frac{k\pi ct}{L}] \sin \frac{k\pi x}{L}$$

in which

$$A_k = \frac{2}{L} \int_0^L f(x) \sin \frac{k\pi x}{L} dx \text{ for } k = 1, 2, \ldots$$

and

$$B_k = \frac{2}{k\pi c} \int_0^L g(x) \sin \frac{k\pi x}{L} dx \text{ for } k = 1, 2, \ldots$$

If $f$ is given by

$$f(x) = x(L - x) \text{ for } 0 \leq x \leq L$$

then

$$A_k = \frac{2}{L} \int_0^L x(L - x) \sin \frac{k\pi x}{L} dx$$

Remembering that

$$\sin k\pi = 0 \text{ and } \cos k\pi = (-1)^k$$
and integrating by parts twice, we find that
\[ A_k = \frac{4L^2}{\pi^3 k^3} (1 - (-1)^k). \]

If \( g \) is given by
\[ g(x) = \left| x - \frac{L}{2} \right| - \frac{L}{2} \text{ for } 0 \leq x \leq L \]
then
\[ B_k = \frac{2}{k \pi c} \int_0^L \left[ \left| x - \frac{L}{2} \right| - \frac{L}{2} \right] \sin \frac{k \pi x}{L} \, dx. \]
So
\[ B_k = -\frac{2}{k \pi c} \int_0^{L/2} x \sin \frac{k \pi x}{L} \, dx + \frac{2}{k \pi c} \int_{L/2}^L (x - L) \sin \frac{k \pi x}{L} \, dx. \]
Using integration by parts, we find that
\[ B_k = \frac{-4L^2 \sin \frac{1}{2} k \pi}{k^3 \pi^3 c} \]
so
\[ u(x, t) = \frac{4L^2}{\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^3} \left[ (1 - (-1)^k) \cos \frac{k \pi c t}{L} - \frac{\sin \frac{1}{2} k \pi}{c} \sin \frac{k \pi c t}{L} \right] \sin \frac{k \pi x}{L}. \]