Model Reduction by Proper Orthogonal Decomposition (POD)

Mathematical models for human tissue and blood flow both represent time dependent nonlinear partial differential equations in three space dimensions. Their numerical solution based on appropriate space/time discretizations requires computational times that even when using state-of-the-art algorithmic solvers are far from being acceptable for real time OR scenarios. A way to overcome this difficulty is to use reduced order models (ROMs) where the dimension of the ROM is by orders of magnitude less than the dimension of the full order model while still reflecting the essential dynamics of the underlying physiological processes. Suitable model order reduction techniques include balanced truncation (BT), proper orthogonal decomposition (POD), and reduced basis methods (RBM) (cf., e.g., [4, 6, 7, 17, 19, 26, 33, 36]). In this project, we will focus on POD in combination with the discrete empirical interpolation method (DEIM) [9, 10] which has been particularly designed for nonlinear problems and has been shown to result in substantial savings of computational time compared to classical POD.

1. POD-DEIM FOR TISSUE MODELING

Mechanical models for soft human tissue have been studied extensively in the literature (cf., e.g., [14, 20, 22] and the references therein). It is now common belief that appropriate models must be based on transversely isotropic nearly incompressible finite elasticity [12, 21, 23, 29, 30, 41].

We assume that the tissue occupies a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\Gamma = \Gamma_D \cup \Gamma_N, \Gamma_D \cap \Gamma_N = \emptyset$. Assuming further $\rho$ to be the density, $f$ a given volumetric force, $u_D$ a prescribed displacement on $\Gamma_D$, $g$ a given traction on $\Gamma_N$, initial conditions $u^0, v^0$, and denoting by $u, p$ the displacement field and the pressure, the dynamic hyperelastic model equations read as follows:

\begin{align}
\rho \frac{\partial^2 u}{\partial t^2} - \nabla \cdot \sigma(u, p) &= f \quad \text{in } Q := \Omega \times (0, T), \\
(J - 1) - \kappa^{-1} p &= 0 \quad \text{in } Q := \Omega \times (0, T), \\
u &= u_D \quad \text{on } \Sigma_D := \Gamma_D \times (0, T), \\
\sigma(u, p)n &= g \quad \text{on } \Sigma_N := \Gamma_N \times (0, T), \\
u(\cdot, 0) &= u^0, \quad \frac{\partial u}{\partial t}(\cdot, 0) = v^0 \quad \text{in } \Omega.
\end{align}

We note that $\sigma := J^{-1}FSF^T$ stands for the Cauchy stress tensor with $F := \text{Grad } \varphi$ being the deformation gradient of the function mapping the reference configuration onto the current configuration and $J := \det(F) > 0$. Moreover, $S$ refers to the second Piola-Kirchhoff stress tensor which is split according to $S = S^{\text{vol}} + S^{\text{iso}}$ into its volumetric and isochoric part

$$S^{\text{vol}} := pJC^{-1}, \quad S^{\text{iso}} := 2J^{-2/3} \frac{\partial W_s^{\text{iso}}}{\partial C^{\text{iso}}} : (I - \frac{1}{3} C^{\text{iso}} \otimes C^{\text{iso}}).$$

Here, $C$ is the right Cauchy-Green tensor and $C^{\text{iso}}$ refers to the isochoric part $C^{\text{iso}} = J^{-2/3}C$. Finally, $W_s^{\text{iso}}$ is the isochoric part of the nonlinear strain energy function
modeled by means of the polyconvex function

\[ W_{iso}^s := c_1 \left( \exp(c_2(I_1 - 3)^2 + c_3(I_4 - 1)^2) - 1 \right), \]

where \( I_1, I_4 \) are computable invariants and \( c_i, 1 \leq i \leq 3 \), are material parameters that have to be identified for individual patients based on measurements of the displacement field obtained, e.g., by imaging techniques from ultrasound data as in [5].

POD has been used to create ROMs of nonlinear elastodynamical systems in [15, 16] and ROMs of nonlinear human tissues in [27, 28], whereas the application of POD-DEIM to nonlinear biomechanical systems is still unexplored territory. Our main objective is to provide POD-DEIM based ROMs from patient specific full order models that can be used by surgeons in real time OR scenarios.

2. POD-DEIM for Blood Flow Simulation

The flow of human blood in mid-size and large arteries can be modeled by Newtonian fluid dynamics in terms of the incompressible Navier-Stokes equations (cf., e.g., [32] and the references therein). We assume that the artery occupies a bounded domain \( \Omega \subset \mathbb{R}^3 \) with boundary \( \Gamma := \Gamma_{in} \cup \Gamma_{out} \cup \Gamma' \), where \( \Gamma_{in}, \Gamma_{out} \) stand for the inflow and outflow boundaries and \( \Gamma' := \Gamma \setminus (\Gamma_{in} \cup \Gamma_{out}) \). Denoting by \( \rho, \nu, v, v^0 \) the density and viscosity of the blood, the inflow velocity and the initial velocity, and by \( v, p, \sigma \) the velocity field, the pressure, and the stress \( \sigma := \nu \nabla v - pI \), the incompressible Navier-Stokes equations read

\[
\begin{align*}
(2.2a) & \quad \rho \left( \frac{\partial v}{\partial t} + (v \cdot \nabla)v \right) - \nabla \cdot \sigma(v, p) = f \quad \text{in } Q := \Omega \times (0, T), \\
(2.2b) & \quad \nabla \cdot v = 0 \quad \text{in } Q := \Omega \times (0, T), \\
(2.2c) & \quad v = v_{in} \quad \text{on } \Sigma_{in} := \Gamma_{in} \times (0, T), \\
(2.2d) & \quad \sigma(u, p)n = 0 \quad \text{on } \Sigma_{out} := \Gamma_{out} \times (0, T), \\
(2.2e) & \quad n \cdot v = 0 \quad \text{on } \Sigma' := \Gamma' \times (0, T), \\
(2.2f) & \quad v(\cdot, 0) = v^0 \quad \text{in } \Omega.
\end{align*}
\]

Model order reduction techniques such as BT, POD, and RBM have been widely used for the Navier-Stokes, Oseen, and the Stokes equations (cf., e.g., [1, 2, 3, 13, 24, 34, 40]). The application of RBM to real-time blood flow simulations based on the steady incompressible Navier-Stokes equations has been considered recently in [25].

In this project, full order models for patient specific geometries obtained from biomedical imaging techniques will be used to create POD-DEIM based ROMs suitable for real-time simulations in OR.

REFERENCES


