Chapter 2 Sobolev spaces

In this chapter, we give a brief overview on basic results of the theory of Sobolev spaces and their associated trace and dual spaces.

2.1 Preliminaries

Let Ω be a bounded domain in Euclidean space $\mathbb{R}^d$. We denote by $\overline{\Omega}$ its closure and refer to $\Gamma = \partial\Omega := \overline{\Omega} \setminus \Omega$ as its boundary. Moreover, we denote by $\Omega_e := \mathbb{R}^d \setminus \overline{\Omega}$ the associated exterior domain.

We consider functions $u : \Omega \to \mathbb{R}$ and denote by

$$D^\alpha u \;.=\; \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d}}$$

its partial derivatives of order $|\alpha|$. Here, $\alpha = (\alpha_1, ..., \alpha_d)^T \in \mathbb{N}_0^d$ is a multiindex of modulus $|\alpha| = \sum_{i=1}^d \alpha_i$.

We define by $C^m(\Omega), m \in \mathbb{N}_0$, the linear space of continuous functions on Ω whose partial derivatives $D^\alpha u, |\alpha| \leq m$, exist and are continuous. $C^m(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{C^m(\Omega)} \;:=\; \max_{0 \leq |\alpha| \leq m} \sup_{x \in \Omega} |D^\alpha u(x)| \;.$$

We refer to $C^{m,\alpha}(\Omega), m \in \mathbb{N}_0, 0 < \alpha < 1$, as the linear space of functions in $C^m(\Omega)$ whose m-th order partial derivatives are Hölder continuous, i.e., for all $\beta \in \mathbb{N}_0^d$ with $|\beta| = m$ there exist constants $\Gamma_\beta > 0$ such that for all $x, y \in \Omega$

$$|D^\beta u(x) - D^\beta u(y)| \;\leq\; \Gamma_\beta \; |x - y|^\alpha \;.$$

We note that $C^{m,\alpha}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{C^{m,\alpha}(\Omega)} \;:=\; \|u\|_{C^m(\Omega)} \;+\; \max_{|\beta|=m} \sup_{x,y \in \overline{\Omega}} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\alpha} \;.$$

Moreover, $C_0^m(\Omega)$ and $C_0^{m,\alpha}(\Omega)$ are the subspaces of functions with compact support in Ω).

Finally, $C^\infty(\Omega)$ stands for the set of functions with continuous partial derivatives of any order and $C_0^\infty(\Omega)$ denotes the set of $C^\infty(\Omega)$ functions with compact support in Ω.

In the sequel, we will mainly deal with **Lipschitz domains** which are defined as follows.

Definition 2.1 Lipschitz domain

A bounded domain $\Omega \subset \mathbb{R}^d$ with boundary Γ is said to be a **Lipschitz domain**, if there exist constants $\alpha > 0, \beta > 0$, and a finite number of **local coordinate systems** $(x_1^r, x_2^r, ..., x_d^r), 1 \leq r \leq R$, and **local Lipschitz continuous mappings**

$$a_r \ : \ \{\hat{x}^r = (x_2^r, ..., x_d^r) \in \mathbb{R}^{d-1} \mid |x_i^r| \leq \alpha \ , \ 2 \leq i \leq d\} \to \mathbb{R}$$

such that

$$\Gamma \ = \ \bigcup_{r=1}^{R} \{(x_1^r, \hat{x}^r) \mid x_1^r = a_r(\hat{x}^r) \ , \ |\hat{x}^r| < \alpha\} \ ,$$
$$\{(x_1^r, \hat{x}^r) \mid a_r(\hat{x}^r) < x_1^r < a_r(\hat{x}^r) + \beta \ , \ |\hat{x}^r| < \alpha\} \subset \Omega \ , \ 1 \leq r \leq R \ ,$$
$$\{(x_1^r, \hat{x}^r) \mid a_r(\hat{x}^r) - \beta < x_1^r < a_r(\hat{x}^r) \ , \ |\hat{x}^r| < \alpha\} \subset \Omega_e \ , \ 1 \leq r \leq R \ , \ .$$

In particular, the geometrical interpretation of the conditions is that both Ω and Ω_e are locally situated on exactly one side of the boundary Γ.

Definition 2.2 C^m-domain and $C^{m,\alpha}$-domain

A Lipschitz domain $\Omega \subset \mathbb{R}^d$ is a **C^m-domain** (**$C^{m,\alpha}$-domain**), if the functions $a_r, 1 \leq r \leq R$, in Definition 2.1 are C^m-functions ($C^{m,\alpha}$-functions).

2.2 Sobolev spaces

We refer to $L^p(\Omega), p \in [1, \infty)$, as the linear space of $p-th$ order integrable functions on Ω and to $L^\infty(\Omega)$ as the linear space of essentially bounded functions which are Banach spaces with respect to the norms

$$\|v\|_{p,\Omega} \ := \ \Big(\int_\Omega |v(x)|^p \ dx \Big)^{1/p} \ ,$$
$$\|v\|_{\infty,\Omega} \ := \ \text{ess} \ \ \text{sup}_{x\in\Omega} \ |v(x)| \ .$$

Note that for $p = 2$, the space $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$(v, w)_{0,\Omega} \ := \ \int_\Omega vw \ dx \ .$$

Sobolev spaces are based on the concept of weak (distributional) derivatives:

Definition 2.3 Weak derivatives

Let $u \in L^1(\Omega)$ and $\alpha \in \mathbb{N}_0^d$. The function u is said to have a **weak derivative** $D_w^\alpha u$, if there exists a function $v \in L^1(\Omega)$ such that

$$\int_\Omega u D^\alpha \varphi \, dx \ = \ (-1)^{|\alpha|} \int_\Omega v\varphi \, dx \quad , \quad \varphi \in C_0^\infty(\Omega) \ .$$

We then set $D_w^\alpha u := v$.

The notion 'weak derivative' suggests that it is a generalization of the classical concept of differentiability and that there are functions which are weakly differentiable, but not differentiable in the classical sense. We give an example.

Example 2.1 Example of a weakly differentiable function

Let $d = 1$ and $\Omega := (-1, +1)$. The function $u(x) := |x|, x \in \Omega$, is not differentiable in the classical sense. However, it admits a weak derivative $D_w^1 u$ given by

$$D_w^1 u \ = \ \begin{cases} -1 & , \quad x < 0 \\ +1 & , \quad x > 0 \end{cases} \ .$$

Indeed, for $\varphi \in C_0^\infty(\Omega)$ we obtain by partial integration

$$\int_{-1}^{+1} u(x) D^1 \varphi(x) \, dx \ = \ \int_{-1}^{0} u(x) D^1 \varphi(x) \, dx \ + \ \int_{0}^{+1} u(x) D^1 \varphi(x) \, dx \ =$$

$$= - \int_{-1}^{0} D_w^1 u(x) \varphi(x) \, dx + (u\varphi)|_{-1}^{0} - \int_{0}^{+1} D_w^1 u(x) \varphi(x) \, dx + (u\varphi)|_0^1 =$$

$$= \int_{-1}^{+1} D_w^1 u(x) \varphi(x) \, dx \ - \ [u(0)] \varphi(0) \ ,$$

where $[u(0)] := u(0+) - u(0-)$ is the jump of u in $x = 0$. But u is continuous and hence, $[u(0)] = 0$ which allows to conclude.

Definition 2.4 The Sobolev spaces $W^{m,p}(\Omega), p \in [1, \infty]$

The linear space $W^{m,p}(\Omega)$ given by

$$W^{m,p}(\Omega) \ := \ \{ \ u \in L^p(\Omega) \mid D_w^\alpha u \in L^p(\Omega) \ , \ |\alpha| \leq m \ \} \tag{2.1}$$

is called a **Sobolev space**. It is a Banach space with respect to the norm

$$(2.2) \qquad \|v\|_{m,p,\Omega} := \Big(\sum_{|\alpha|\leq m} \|D_w^\alpha v\|_{p,\Omega}^p \Big)^{1/p} \quad , \quad p \in [1,\infty) \ ,$$

$$(2.3) \qquad \|v\|_{m,\infty,\Omega} := \max_{|\alpha|\leq m} \|D_w^\alpha v\|_{\infty,\Omega} \ .$$

Note that $W^{m,2}(\Omega)$ is a Hilbert space with respect to the inner product

$$(2.4) \qquad (u,v)_{m,2,\Omega} := \sum_{|\alpha|\leq m} \int_\Omega D_w^\alpha u D_w^\alpha v \ dx \ .$$

The associated norm will be denoted by $\|\cdot\|_{m,2,\Omega}$. We will simply write $(\cdot,\cdot)_{m,\Omega}$ and $\|\cdot\|_{m,\Omega}$, if the context clearly indicates that the $W^{m,2}(\Omega)$-inner product and the $W^{m,2}(\Omega)$-norm are meant.

A natural question is whether $C^m(\Omega)$ is dense in $W^{m,p}(\Omega)$. To this end, we consider the linear space

$$C^{m,*}(\Omega) := \{ \ u \in C^m(\Omega) \mid \|u\|_{m,p,\Omega} < \infty \ \} \ .$$

It is a normed linear space with respect to $|\cdot|_{m,p,\Omega}$, but not complete, i.e., a pre-Banach space. We denote its completion with respect to the $\|\cdot\|_{m,p,\Omega}$-norm by $H^{m,p}(\Omega)$, i.e.,

$$(2.5) \qquad H^{m,p}(\Omega) := \overline{C^{m,*}(\Omega)}^{\|\cdot\|_{m,p,\Omega}} \ .$$

We note that for $p = \infty$ we have $H^{m,\infty}(\Omega) = C^m(\Omega)$ and hence, $W^{m,\infty}(\Omega) \neq H^{m,\infty}(\Omega)$. However, a famous result by Meyers/Serrin (cf., e.g., [1]) states that for a Lipschitz domain and $p \in [1,\infty)$ the spaces $W^{m,p}(\Omega)$ and $H^{m,p}(\Omega)$ are equivalent.

Theorem 2.1 Characterization of Sobolev spaces

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and $m \in \mathbb{N}_0$. Then, for $p \in [1,\infty)$ there holds

$$(2.6) \qquad W^{m,p}(\Omega) = H^{m,p}(\Omega) \ .$$

Remark 2.1 Remark on Meyers/Serrin

The result (2.6) actually holds true for any open subset $\Omega \subset \mathbb{R}^d$. However, $C^m(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$, if Ω satisfies the so-called **segment property** (cf., e.g., [1]). In particular, Lipschitz domains have this property.

Theorem 2.1 asserts that for Lipschitz domains and $p \in [1,\infty)$ the space $C^m(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$. A natural question to ask is whether or not functions in $W^{m,p}(\Omega), m \geq 1,$ belong to the Banach space $L^q(\Omega)$, or even are continuous. As we shall see, the latter only holds true, if m is sufficiently large (**Sobolev imbedding theorem**). However, let us first give an example which shows that in general we can not expect such a result:

Example 2.2 Example of a weakly differentiable, but not essentially bounded function

Let $d \geq 2$, $\Omega := \{x \in \mathbb{R}^d \mid |x| < \frac{1}{2}\}$ and consider the function

$$u(x) \; := \; \ln(|\ln(|x|)) \; .$$

The function u has square-integrable first-order weak derivatives

$$D^\alpha u(x) \; = \; \frac{x^\alpha}{|x|^2 \, \ln(|x|)} \quad , \quad |\alpha| = 1 \; ,$$

since in view of

$$|D^\alpha u(x)|^d \; \leq \; \rho(|x|) \; := \; \frac{1}{|x|^d \, |\ln(|x|)|^d} \quad , \quad |\alpha| = 1 \; ,$$

it possesses a square-integrable majorant.
On the other hand, u obviously is not essentially bounded.

Theorem 2.2 Sobolev imbedding theorems

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $m \in \mathbb{N}_0$ and $p \in [1, \infty]$. Then, the following mappings represent continuous imbeddings

$$W^{m,p}(\Omega) \; \to \; L^{p^*}(\Omega) \quad , \quad \frac{1}{p^*} = \frac{1}{p} - \frac{m}{d} \; , \; \text{if } m < \frac{d}{p} \; , \tag{2.7}$$

$$W^{m,p}(\Omega) \; \to \; L^q(\Omega) \quad , \quad q \in [1, \infty) \; , \; \text{if } m = \frac{d}{p} \; , \tag{2.8}$$

$$W^{m,p}(\Omega) \; \to \; C^{0,m-\frac{d}{p}}(\overline{\Omega}) \quad , \quad \text{if } \frac{d}{p} < m < \frac{d}{p} + 1 \; , \tag{2.9}$$

$$W^{m,p}(\Omega) \; \to \; C^{0,\alpha}(\overline{\Omega}) \quad , \quad 0 < \alpha < 1 \; , \; \text{if } m = \frac{d}{p} + 1 \; , \tag{2.10}$$

$$W^{m,p}(\Omega) \; \to \; C^{0,1}(\overline{\Omega}) \quad , \quad \text{if } m > \frac{d}{p} + 1 \; . \tag{2.11}$$

Proof. For a proof, we refer to [1]. □

In the following chapters, we will frequently take advantage of **compact imbeddings** of Sobolev spaces.

Theorem 2.3 Kondrasov imbedding theorems

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $m \in \mathbb{N}_0$ and $p \in [1,\infty]$. Then, the following mappings are compact imbeddings

$$W^{m,p}(\Omega) \rightarrow L^q(\Omega) \quad , \quad 1 \leq q \leq p^* \, , \, \frac{1}{p^*} = \frac{1}{p} - \frac{m}{d} \, , \text{ if } m < \frac{d}{p} \, , \tag{2.12}$$

$$W^{m,p}(\Omega) \rightarrow L^q(\Omega) \quad , \quad q \in [1,\infty) \, , \text{ if } m = \frac{d}{p} \, , \tag{2.13}$$

$$W^{m,p}(\Omega) \rightarrow C^0(\overline{\Omega}) \, , \text{ if } m > \frac{d}{p} \, . \tag{2.14}$$

Proof. The interested reader is referred to [1]. □

We finally deal with **restrictions** and **extensions** of Sobolev functions:

If $u \in W^{m,p}(\mathbb{R}^d), p \in [1,\infty],$, then for any domain $\Omega \subset \mathbb{R}^d$ the restriction Ru of u to Ω

$$Ru(x) \ := \ u(x) \quad \text{f.a.a. } x \in \Omega$$

belongs to $W^{m,p}(\Omega)$ and $R : W^{m,p}(\mathbb{R}^d) \rightarrow W^{m,p}(\Omega)$ is a bounded linear operator.

Conversely, it is in general not possible to continuously extend a function $u \in W^{m,p}(\Omega)$ to a function in $W^{m,p}(\mathbb{R}^d)$ as the following example shows:

Example 2.3 Counterexample (Extension of Sobolev functions)

Let $d = 2 \, , \ \Omega ;= \{x = (x_1,x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1, |x_2| < x_1^r\}, r > 1,$ and consider the function

$$u(x) \ := \ x_1^{-\frac{\varepsilon}{p}} \quad , \quad 0 < \varepsilon < r \, .$$

We note that Ω has a cusp in the origin and hence, is no Lipschitz domain.

For $\varepsilon < r + 1 - p$, we have that $u \in W^{1,p}(\Omega)$, since

$$\sum_{|\alpha|=1} \int_\Omega |D^\alpha u|^p \, dx \ = \ C_{\varepsilon,p} \int_0^1 x_1^{-\varepsilon-p+r} \, dx_1 \, .$$

On the other hand, u is not in $L^\infty(\Omega)$. Choosing ε such that $p > 2$ is possible, we see that a **Lipschitz domain** is **necessary** for the Sobolev imbedding theorem to hold true. Since the Sobolev imbedding theorem is valid on $\mathbb{R}^d$, we can not extend u to a function in $W^{1,2}(\mathbb{R}^2)$.

However, in case of a Lipschitz domain we have the validity of the following Sobolev extension theorem.

Theorem 2.3 Sobolev extension theorem

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $m \in \mathbb{N}_0$, and $p \in [1,\infty]$. Then, there exists a bounded linear extension operator $E : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^d)$, i.e., $Eu = u$ for all $u \in W^{m,p}(\Omega)$ and there exists a constant $C \geq 0$ such that for all $u \in W^{m,p}(\Omega)$

$$\|Eu\|_{m,p,\Omega} \;\leq\; C\,\|u\|_{m,p,\mathbb{R}^d} \;.$$

Proof. We refer to [3]. □

We recall that the **algebraic and topological dual** V^* of a Hilbert space V is the linear space of all bounded linear functionals on V which is itself a Hilbert space with respect to the norm

$$\|\ell\|_{V^*} \;:=\; \sup_{v\neq 0} \frac{|\ell(v)|}{\|v\|_V} \;.$$

We may thus define the dual spaces of the Sobolev spaces $W^{m,p}(\Omega), p \in [1,\infty]$.

Definition 2.5 Sobolev spaces with negative index

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, let m be a negative integer and suppose $p \in [1,\infty]$. Then, the Sobolev space $W^{m,p}(\Omega)$ is defined as the dual space $(W^{-m,q}(\Omega))^*$, where q is conjugate to p, i.e., $\frac{1}{q} + \frac{1}{p} = 1$.

Remark 2.2 The Dirac δ-function

The Sobolev spaces $W^{m,p}(\Omega), m < 0,$ are proper subspaces of $L^p(\Omega)$. For instance, for $m < -d + \frac{d}{p}$, if $p < \infty$, and $m \leq -d$, if $p = \infty$, they contain the Dirac δ-function considered as a linear functional

$$\begin{aligned} \delta \;:\; W^{-m,p}(\Omega) \quad &\to \quad \mathbb{R} \\ u \quad &\longmapsto \quad \delta_x(u) \;, \end{aligned}$$

where x is some given point in Ω.

2.3 Sobolev spaces with broken index

For $\Omega = \mathbb{R}^d$, we define **Sobolev spaces with broken index** $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}_+$ by using the **Fourier transform** $\hat{v}$ of a function $v \in C_0^\infty(\mathbb{R}^d)$

$$\hat{v}(\xi) \ := \ (\frac{1}{2\pi})^{d/2} \int\limits_{\mathbb{R}^d} \exp(-i\xi \cdot x) v(x) \ dx \ .$$

The associated **Sobolev norm** is defined according to

$$\|v\|_{s,\mathbb{R}^d} \ := \ \|(1 + |\cdot|^2)^{s/2} \ \hat{v}(\cdot)\|_{0,\mathbb{R}^d} \ , \tag{2.15}$$

and we set

$$H^s(\mathbb{R}^d) \ := \ \overline{C_0^\infty(\mathbb{R}^d)}^{\|\cdot\|_{s,\mathbb{R}^d}} \ . \tag{2.16}$$

If $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain, the space $H^s(\Omega)$ can be either defined implicity by means of

$$\|v\|_{s,\Omega} \ := \inf_{z = Ev \in H^s(\mathbb{R}^d)} \|z\|_{s,\mathbb{R}^d} \ , \tag{2.17}$$

where Ev is the **extension** of v to $H^s(\mathbb{R}^d)$, or - for $s = m + \lambda, m \in \mathbb{N}_0, 0 \leq \lambda < 1,$ - explicitly by

$$\|v\|_{s,\Omega} \ := \ \Big((\frac{1}{\text{diam}(\Omega)})^{2s} \ \|v\|_{0,\Omega}^2 \ + \ |v|_{s,\Omega}^2 \Big)^{1/2} \ , \tag{2.18}$$

where $|\cdot|_{\lambda,\Omega}$ stands for the **seminorm**

$$|v|_{s,\Omega}^2 \ := \sum_{1 \leq |\alpha| \leq m} \|D^\alpha v\|_{0,\Omega}^2 \ + \ \sum_{|\alpha| = m} \int\limits_\Omega \int\limits_\Omega \frac{|D^\alpha v(x) - D^\alpha v(y)|^2}{|x - y|^{d+2\lambda}} \ dx \ dy \ . \tag{2.19}$$

If $\Sigma \subseteq \Gamma = \partial\Omega$, we define the space$H^s(\Sigma)$ as follows

$$H^s(\Sigma) \ := \ \{v \in L^2(\Sigma) \mid \|v\|_{s,\Sigma} < \infty\} \ , \tag{2.20}$$

equipped with the norm

$$\|v\|_{s,\Sigma} \ := \ \Big((\frac{1}{\text{diam}(\Sigma)})^{2s} \ \|v\|_{0,\Sigma}^2 \ + \ |v|_{s,\Sigma}^2 \Big)^{1/2} \ , \tag{2.21}$$

where

$$|v|_{s,\Sigma}^2 \ := \sum_{1 \leq |\alpha| \leq m} \|D^\alpha v\|_{0,\Sigma}^2 \ + \ \sum_{|\alpha| = m} \int\limits_\Sigma \int\limits_\Sigma \frac{|D^\alpha v(x) - D^\alpha v(y)|^2}{|x - y|^{d-1+2\lambda}} \ d\sigma_x \ d\sigma_y \ . \tag{2.22}$$

In case $\Sigma \subset \Gamma$ and $v \in H^s(\Sigma), s < 1$,, we define $\tilde{v}$ as the **extension by zero** of v, i.e.,

$$\tilde{v}(x) \; := \; \begin{cases} v(x) & , \quad x \in \Sigma \\ 0 & , \quad x \in \Gamma \setminus \Sigma \end{cases} . \tag{2.23}$$

We define $H^s_{00}(\Sigma)$ according to

$$H^s_{00}(\Sigma) \; := \; \{ v \in L^2(\Sigma) \mid \tilde{v} \in H^s(\Gamma) \} \; , \tag{2.24}$$

equipped with the norm

$$\|v\|_{H^s_{00}(\Sigma)} \; := \; \Big(\|v\|^2_{0,\Sigma} \; + \; |v|_{H^s_{00}(\Sigma)} \Big)^{1/2} \; , \tag{2.25}$$

where

$$|v|^2_{H^s_{00}(\Sigma)} \; := \; |v|^2_{s,\Sigma} \; + \; \int\limits_{\Sigma} \frac{v^2(x)}{\mathrm{dist}(x, \partial\Sigma)} \; d\sigma \tag{2.26}$$

and

$$|v|^2_{s,\Sigma} \; := \; \int\limits_{\Sigma} \int\limits_{\Sigma} \frac{|v(x) - v(y)|^2}{|x - y|^{d-1+2s}} \; d\sigma_x \; d\sigma_y \; . \tag{2.27}$$

For $s < 0$, we define $\tilde{H}^s(G), G \in \{\Sigma, \Omega\}$, as the dual of $H^{-s}(G)$, equipped with the **negative norm**

$$|v|_{s,G} \; := \; \sup_{w \in H^{-s}(G), w \neq 0} \frac{< v, w >_G}{\|w\|_{-s,G}} \; , \tag{2.28}$$

where $< \cdot, \cdot >_G$ stands for the dual pairing.
Moreover, if $-1 < s < 0$, we define $H^s(G), G \in \{\Sigma, \Omega\}, \Sigma \subset \Gamma$, as the dual of $H^{-s}_{00}(G)$. For $s = -1$, we further define $H^{-1}(G)$ as the dual of $H^1_0(G)$, whereas for $\Sigma = \Gamma$ and $-1 \leq s < 0$ we define $H^s(\Gamma)$ as the dual of $H^{-s}(\Gamma)$.
For details we refer to [2].

2.4 Trace spaces

For a bounded domain $\Omega \subset \mathbb{R}^d$ with boundary Γ and a function $u \in C(\overline{\Omega})$, it makes sense to define the restriction of u to the boundary Γ simply by considering the pointwise restriction. In view of the Sobolev imbedding theorem, however, the **pointwise restriction** does not make sense for functions $u \in W^{m,2}(\Omega)$ unless m is sufficiently large. Therefore, we have to find appropriate means to specify the **trace** $u|_\Gamma$ of a function $u \in W^{m,2}(\Omega)$. The following example suggests how to proceed.

Example 2.4 Trace of smooth functions

Let $\Omega \subset \mathbb{R}^2$ be the unit disk which in polar coordinates reads as follows

$$\Omega \ := \ \{ \ (r,\theta) \mid 0 \le r < 1 \ , \ 0 \le \theta < 2\pi \ \} \ .$$

For $u \in C^1(\overline{\Omega})$, the restriction to $\partial\Omega$ can be expressed according to

$$\begin{aligned} u(1,\theta)^2 \ &= \ \int\limits_0^1 \frac{\partial}{\partial r}\Big(r^2 u(r,\theta)^2\Big) \ dr \ = \ \int\limits_0^1 2\Big(r^2 u u_r + r u^2\Big)(r,\theta) \ dr \ = \\ &= \ \int\limits_0^1 2\Big(r^2 u \nabla u \cdot \frac{(x_1,x_2)}{r} + r u^2\Big)(r,\theta) \ dr \ \le \\ &\le \ \int\limits_0^1 2\Big(r^2 |u||\nabla u| + r u^2\Big)(r,\theta) \ dr \ \le \\ &\le \ \int\limits_0^1 2\Big(|u||\nabla u| + u^2\Big)(r,\theta) \ r \ dr \ . \end{aligned}$$

Integrating the previous inequality over θ and applying the Cauchy-Schwarz inequality as well as Young's inequality results in

$$\begin{aligned} \|u\|_{0,\partial\Omega}^2 \ &= \ \int\limits_{\partial\Omega} u^2 \ d\theta \ \le \ 2 \int\limits_{\Omega} \Big(|u||\nabla u| + u^2\Big) \ dx_1 \ dx_2 \ \le \\ &\le \ 2 \Big(\int\limits_{\Omega} u^2 \ dx\Big)^{1/2} \Big(\int\limits_{\Omega} |\nabla u|^2 \ dx\Big)^{1/2} \ + \ 2 \int\limits_{\Omega} u^2 \ dx \ \le \\ &\le \ \sqrt[4]{8} \ \|u\|_{0,\Omega}^{1/2} \ \|u\|_{1,\Omega}^{1/2} \ . \end{aligned}$$

The example can be easily generalized to give the following result.

Lemma 2.1 Trace of Sobolev space functions. Part I

Let $\Omega \subset \mathbb{R}^2$ be the unit disk and $u \in W^{1,2}(\Omega)$. Then, the trace $u|_{\partial\Omega}$ can be interpreted as a function in $L^2(\partial\Omega)$ satisfying

$$\|u\|_{0,\partial\Omega}^2 \ \le \ \sqrt[4]{8} \ \|u\|_{0,\Omega}^{1/2} \ \|u\|_{1,\Omega}^{1/2} \ . \tag{2.29}$$

Proof. Since $C^1(\overline{\Omega})$ is dense in $W^{1,2}(\Omega)$, there exists a sequence $(u_k)_{\mathbb{N}} \subset C^1(\overline{\Omega})$ such that $\|u - u_k\|_{1,\Omega} \le 1/k, k \in \mathbb{N}$. Applying the inequality derived in Example 2.4, we obtain

$$\begin{aligned} \|u_i - u_k\|_{0,\partial\Omega} &\leq \sqrt[4]{8}\ \|u_i - u_k\|_{0,\Omega}^{1/2}\ \|u_i - u_k\|_{1,\Omega}^{1/2} \leq \\ &\leq \sqrt[4]{8}\ \|u_i - u_k\|_{1,\Omega} \leq \sqrt[4]{8}\ \Big(\frac{1}{i} + \frac{1}{k}\Big) . \end{aligned}$$

Consequently, $(u_k)_{\mathbb{N}}$ is a Cauchy sequence in $L^2(\partial\Omega)$ and hence, there exists $v \in L^2(\partial\Omega)$ such that $u_k \to v\ (k \to \infty)$ in $L^2(\partial\Omega)$.
We define the **trace** of u on $\partial\Omega$ according to

$$u_{\partial\Omega} := v .$$

We have to show that $u_{\partial\Omega}$ is well defined, i.e., it does not depend on the particular choice of a sequence of $C^1(\overline{\Omega})$-functions. To this end, let $(v_k)_{\mathbb{N}}, v_k \in C^1(\overline{\Omega}), k \in \mathbb{N}$ be another sequence satisfying $\|u - u_k\|_{1,\Omega} \to 0\ (k \to \infty)$. We find

$$\begin{aligned} \|v - v_k\|_{0,\partial\Omega} &\leq \|v - u_k\|_{0,\partial\Omega} + \|u_k - v_k\|_{0,\partial\Omega} \leq \\ &\leq \|v - u_k\|_{0,\partial\Omega} + \sqrt[4]{8}\ \|u_k - v_k\|_{1,\Omega} \leq \\ &\leq \|v - u_k\|_{0,\partial\Omega} + \sqrt[4]{8}\ \Big(\|u_k - u\|_{1,\Omega} + \|u - v_k\|_{1,\Omega}\Big) \\ &\to 0\ (k \to \infty) . \end{aligned}$$

Finally, the validity of (2.29) follows by a density argument:

$$\begin{aligned} \|u\|_{0,\partial\Omega} &= \|v\|_{0,\partial\Omega} = \lim_{k\to\infty} \|u_k\|_{0,\partial\Omega} \leq \\ &\leq \lim_{k\to\infty} \sqrt[4]{8}\ \|u_k\|_{0,\partial\Omega}^{1/2}\ \|u_k\|_{1,\Omega}^{1/2} = \sqrt[4]{8}\ \|u\|_{0,\partial\Omega}^{1/2}\ \|u\|_{1,\Omega}^{1/2} . \quad \square \end{aligned}$$

The previous result can be generalized to arbitrary Lipschitz domains.

Theorem 2.4 Trace of Sobolev space functions. Part II
Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and $p \in [1, \infty]$. Then, the trace mapping

$$\begin{aligned} W^{1,p}(\Omega) &\to L^p(\partial\Omega) \\ u &\longmapsto u|_{\partial\Omega} \end{aligned}$$

is a bounded linear, injective mapping.
Proof. We refer to [2]. □

However, the trace mapping considered as a mapping from $W^{1,p}(\Omega)$ in $L^p(\partial\Omega)$ is not surjective, i.e., there exist functions $v \in L^p(\partial\Omega)$ such that we can not find $u \in W^{1,p}(\Omega)$ with $u|_{\partial\Omega} = v$.
In particular, for $p = 2$ we have the following result.

Theorem 2.5 Trace of Sobolev space functions. Part III
Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and $1/2 < s \leq 1$. Then there holds:

(i) There exists a **unique linear bounded trace mapping**

$$\begin{aligned} \gamma_0 \; : \; H^s(\Omega) \quad &\rightarrow \quad H^{s-1/2}(\Gamma) \\ v \quad &\longmapsto \quad \gamma_0(v) := v|_\Gamma \end{aligned} \tag{2.30}$$

such that

$$\begin{aligned} \|\gamma_0(v)\|_{s-1/2,\Gamma} \quad &\leq \quad C \; \|v\|_{s,\Omega} \; , \\ |\gamma_0(v)|_{s-1/2,\Gamma} \quad &\leq \quad C \; |v|_{s,\Omega} \; . \end{aligned} \tag{2.31}$$

(ii) The **trace operator** γ_0 has a **bounded right inverse**

$$\gamma_0^- \; : \; H^{s-1/2}(\Gamma) \quad \rightarrow \quad H^s(\Omega) \; , \tag{2.32}$$

i.e., $\gamma_0 \gamma_0^- w = w, w \in H^{s-1/2}(\Gamma)$, and

$$\begin{aligned} \|\gamma_0^-(w)\|_{s,\Omega} \quad &\leq \quad C \; \|w\|_{s-1/2,\Gamma} \; , \\ |\gamma_0^-(w)|_{s,\Omega} \quad &\leq \quad C \; |w|_{s-1/2,\Gamma} \; . \end{aligned} \tag{2.33}$$

(iii) In case $\Sigma \subset \Gamma$, the results in **(i)** and **(ii)** hold true with $H^{s-1/2}(\Gamma)$ replaced by $H_{00}^{s-1/2}(\Sigma)$ and $H^s(\Omega)$ replaced by $H^s_{\Gamma_D}(\Omega)$, where $\Gamma_D := \Gamma \setminus \Sigma$ and

$$H^s_{\Gamma_D}(\Omega) \; := \; \{v \in H^s(\Omega) \mid \gamma_0 v = 0 \text{ on } \Gamma_D\} \; . \tag{2.34}$$

Proof. We refer to [2]. □

Definition 2.6 Dirichlet data
For $u \in H^1(\Omega)$, the trace $\gamma_0 u$ is called the **Dirichlet data**.

Definition 2.7 Normal component trace mapping
Given a Lipschitz domain $\Omega \subset \mathbb{R}^d$ with boundary Γ, we denote by

$$\mathbf{n} \; = \; (n_1, ..., n_d)^T$$

the **unit exterior normal vector** which is well defined at almost all $x \in \Gamma$. For a vector field $\mathbf{q} \in C^\infty(\overline{\Omega})^d$, we refer to

$$\eta_{\mathbf{n}}(\mathbf{q}) \; := \; \mathbf{n} \cdot \mathbf{q}|_\Gamma \; . \tag{2.35}$$

as the **normal component trace mapping**. It can be extended by continuity to a linear continuous mapping

$$\eta_{\mathbf{n}} \; : \; H^1(\Omega)^d \; \rightarrow \; H^{1/2}(\Gamma) \; . \tag{2.36}$$

Definition 2.8 Normal derivative and Neumann data

Within the notations of Definition 2.7, for $u \in C^\infty(\overline{\Omega})$ the normal component trace of ∇u

$$\partial_{\mathbf{n}} u \ := \ \eta_{\mathbf{n}}(\nabla u) \ = \ \mathbf{n} \cdot \nabla u \tag{2.37}$$

is called the **normal derivative** of u.

The mapping $\partial_{\mathbf{n}}$ can be extended to a linear continuous mapping

$$\partial_{\mathbf{n}} \ : \ H^2(\Omega) \ \to \ H^{1/2}(\Gamma) \ . \tag{2.38}$$

In particular, for $u \in H^2(\Omega)$ the normal derivative $\partial_{\mathbf{n}} u$ is called the **Neumann data**.

References

[1] Adams, R.A. (1975), Sobolev spaces. Academic Press, New York.

[2] Grisvard, P. (1985), Elliptic Problems in Nonsmooth Domains. Pitman, Boston.

[3] Stein, E.M. (1970), Singular Integrals and Differentiability Properties of Funcrions. Princeton University Press, Princeton.