Unified Framework for an A Posteriori Error Analysis
of Non-Standard Finite Element Approximations
of H(curl)-Elliptic Problems

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Computational and Applied Mathematics

Rice University

Houston, October 14, 2009
Partially supported by

NSF Grants No. DMS-0511611, DMS-0707602, DMS-0810176, DMS-0811173 and DFG Research Center MATHEON
Non-Standard FEM for $H(\text{curl})$-Elliptic BVPs

- **Unified framework**: Mixed formulation of $H(\text{curl})$-elliptic bvps
- **Error analysis**: Residual-type a posteriori error estimators
- **Nonconformity**: Incorporation of the consistency error

Applications

- **Application 1**: Interior Penalty Discontinuous Galerkin methods
- **Application 2**: Mortar edge element approximations
The Loop in Adaptive Finite Element Methods (AFEM)

Adaptive Finite Element Methods (AFEM) consist of successive loops of the cycle

\[
\text{SOLVE} \implies \text{ESTIMATE} \implies \text{MARK} \implies \text{REFINE}
\]

**SOLVE:** Numerical solution of the FE discretized problem

**ESTIMATE:** Residual and hierarchical a posteriori error estimators
- Error estimators based on local averaging
- Goal oriented weighted dual approach
- Functional type a posteriori error bounds

**MARK:** Strategies based on the max. error or the averaged error
- Bulk criterion for AFEMs

**REFINE:** Bisection or ‘red/green’ refinement or combinations thereof
Convergence Analysis of AFEM (Standard FEM)

Morin/Siebert/Veeser [2007]

Convergence Analysis of AFEM (Nonstandard FEM)

A Posteriori Error Estimates for DG Methods

Becker/Hansbo/Larson [2003] Energy norm estimates
Karakashian/Pascal [2003] Residual-type estimates for IPDG
Houston/Perugia/Schötzau [2007] IPDG for H(curl)-problems
Carstensen/Gudi/Jensen [2008] Unified framework for DG Methods

Convergence Analysis of Adaptive IPDG Methods

Ainsworth [2007] Bonito/Nochetto [2008]
Unified Framework:
Mixed Formulation of $H(\text{curl})$-Elliptic BVPs
Unified Framework: Mixed Formulation of H(curl)-Elliptic BVPs

Consider the H(curl)-elliptic boundary value problem
\[
\text{curl } \mu^{-1} \text{curl } u + \sigma u = f \quad \text{in } \Omega \subset \mathbb{R}^3,
\]
\[
\gamma_t(u) := u \wedge n = 0 \quad \text{on } \Gamma = \partial \Omega.
\]
The elliptic PDE can be written as the following first order system
\[
\mu \ p - \text{curl } u = 0 \ , \quad \text{curl } p + \sigma u = f.
\]
The weak formulation amounts to the computation of \((u, p) \in V \times Q\), where \(V := H_0(\text{curl}; \Omega)\), \(Q := L^2(\Omega)\), such that
\[
\begin{align*}
    a(p, q) - b(u, q) &= \ell_1(q) \quad , \quad q \in Q, \\
    b(v, p) - c(u, v) &= \ell_2(v) \quad , \quad v \in V,
\end{align*}
\]
where the bilinear forms and the functionals are given by
\[
\begin{align*}
    a(p, q) := \int_\Omega \mu \ p \cdot q \ dx \ , \quad b(u, q) := \int_\Omega \text{curl } u \cdot q \ dx, \\
    c(u, v) := \int_\Omega \sigma \ u \cdot v \ dx \ , \quad \ell_1(q) := 0 \ , \quad \ell_2(v) := \int_\Omega f \cdot v \ dx.
\end{align*}
\]
Unified Framework: Mixed Formulation of H(curl)-Elliptic BVPs

In operator-theoretic form, the saddle point problem can be written as

\[(*) \quad A(u, p) = \ell_1 + \ell_2,\]

where the operator \(A := V \times Q \to (V \times Q)^*\) is defined by means of

\((A(u, p)) := a(p, q) - b(u, q) + b(v, p) + c(u, v).\)

**Theorem 1.** The operator \(A := V \times Q \to (V \times Q)^*\) is a continuous, linear and bijective operator. Hence, for any \((\ell_1, \ell_2) \in Q^* \times V^*\) there exists a unique solution \((u, p) \in V \times Q\) of (*)

**Proof.** For \(v := 3u\) and \(q := 2p - \mu^{-1}\text{curl } u\) we find

\[(A(u, p))(v, q) = (A(v, q))(u, p) = 2\mu \|p\|_{0, \Omega}^2 + 3\sigma \|u\|_{0, \Omega}^2 + \mu^{-1} \|\text{curl } u\|_{0, \Omega}^2.\]

This implies ellipticity on \(\text{Ker } B^* \times Q\) and the inf-sup condition which results in bijectivity.
Unified Framework: Residual-Based A Posteriori Error Estimation

Corollary 1. Let $(\tilde{u}_h, \tilde{p}_h) \in V \times Q$ be an approximation of the solution $(u, p) \in V \times Q$ of $(\ast)$. Then, there holds

$$\|(u - \tilde{u}_h, p - \tilde{p}_h)\|_{V \times Q} \lesssim \|\text{Res}_1\|_{Q^*} + \|\text{Res}_2\|_{V^*},$$

where the residuals $\text{Res}_1$ and $\text{Res}_2$ are given by

$$\text{Res}_1(q) := \ell_1(q) - a(\tilde{p}_h, q) + b(\tilde{u}_h, q), \quad q \in Q,$$

$$\text{Res}_2(v) := \ell_2(v) - b(v, \tilde{p}_h) - c(\tilde{u}_h, v), \quad v \in V.$$

Remark 1. It is easy to see that the residual $\text{Res}_1$ is given by

$$\|\text{Res}_1\|_{Q^*} \lesssim \|\tilde{p}_h - \mu^{-1} \text{curl} \tilde{u}_h\|_{0, \Omega}$$

and thus measures how well $(\tilde{u}_h, \tilde{p}_h)$ approximates the equation $\mu \ p - \text{curl} \ u$. The evaluation of $\text{Res}_2$ is more involved.
Unified Framework: Residual-Based A Posteriori Error Estimation

Let $\mathcal{T}_h(\Omega)$ be a shape-regular simplicial triangulation of $\Omega$ with $\mathcal{F}_h(D)$ denoting the set of faces of $\mathcal{T}_h(\Omega)$ in $D \subseteq \overline{\Omega}$. We denote by

$$\text{Nd}_1(\Omega; \mathcal{T}_h(\Omega)) := \{v_h \in V \mid v_h|_T \in \text{Nd}_1(T), T \in \mathcal{T}_h(\Omega)\}, \quad \text{Nd}_1(T) := \{v \mid v(x) = a + b \wedge x, \ a, b \in \mathbb{R}^3\},$$

the edge element space w.r.t. the curl-conforming edge elements of Nédélec’s first family.

Let $\eta_T, T \in \mathcal{T}_h(\Omega)$ and $\eta_F, F \in \mathcal{F}_h(\Omega)$ be the element and face residuals

$$\eta_T := h_T \|f - \sigma \tilde{u}_h - \text{curl} \ \tilde{p}_h\|_{0,T} + h_T \|\text{div} (f - \sigma \tilde{u}_h)\|_{0,T},$$

$$\eta_F := h_F^{1/2} \|\pi_t(\tilde{p}_h)|_F\|_{0,F} + h_F^{1/2} \|n_F \cdot [f - \sigma \tilde{u}_h]|_F\|_{0,F},$$

where $\pi_t(\cdot) := n \wedge (\cdot \wedge n)$ denotes the tangential trace components and $[\cdot]|_F$ stands for the jump across the interface $F \in \mathcal{F}_h(\Omega)$.

**Theorem 2.** Under the assumption $\text{Nd}_1(\Omega; \mathcal{T}_h(\Omega)) \subset \text{Ker Res}_2$ there holds

$$\|\text{Res}_2\|_{V^*}^2 \lesssim \eta_h^2 := \sum_{T \in \mathcal{T}_h(\Omega)} \eta_T^2 + \sum_{F \in \mathcal{F}_h(\Omega)} \eta_F^2.$$
Proof. We use Schöberl’s quasi-interpolation operator $\Pi_{h}^{Nd} : V \to \text{Nd}_1(\Omega; T_h(\Omega))$:

For any $v \in V$ there exist $\varphi \in H^1_0(\Omega)$ and $z \in H^1_0(\Omega)^3$ such that

$$v - \Pi_{h}^{Nd}v = \nabla \varphi + z.$$

Then, there holds

$$\text{Res}_2(v) = \text{Res}_2(v - \Pi_{h}^{Nd}v) = \text{Res}_2(\nabla \varphi) + \text{Res}_2(z),$$

and the assertion follows from the approximation properties

$$\|\varphi\|_{0,T} \lesssim h_T \|v\|_{0,\omega_{T}^{Nd}}, \quad \|\varphi\|_{0,F} \lesssim h_F^{1/2} \|\text{curl } v\|_{0,\omega_{F}^{Nd}},$$

$$\|z\|_{0,T} \lesssim h_T \|\text{curl } v\|_{0,\omega_{T}^{Nd}}, \quad \|\gamma_1(z)\|_{0,F} \lesssim h_F^{1/2} \|\text{curl } v\|_{0,\omega_{F}^{Nd}},$$

where $\omega_{T}^{Nd}$ and $\omega_{F}^{Nd}$ are patches associated with $T$ and $F$. 
Interior Penalty Discontinuous Galerkin Methods
for H(curl)-Elliptic Boundary Value Problems
Interior Penalty Discontinuous Galerkin Method

Given a shape regular simplicial triangulation $\mathcal{T}_h(\Omega)$, we choose

$$V_h := \prod_{T \in \mathcal{T}_h(\Omega)} P_k(T)^3, \quad Q_h := \prod_{T \in \mathcal{T}_h(\Omega)} P_k(T)^3, \quad k \in \mathbb{N}.$$ 

The Interior Penalty Discontinuous Galerkin Method for the $H(\text{curl})$-elliptic BPV reads:

Find $u_h \in V_h$ such that for all $v_h \in V_h$ there holds

$$\left(\mu^{-1}\text{curl } u_h, \text{curl } v_h\right)_{0,T} - \sum_{F \in \mathcal{F}_h(\Omega)} \left(\gamma_t(u_h), [\gamma_t(v_h)]_F\right)_{0,F}$$

$$- \sum_{F \in \mathcal{F}_h(\Omega)} \left(\gamma_t(u_h), [\gamma_t(v_h)]_F\right)_{0,F} + \alpha \sum_{F \in \mathcal{F}_h(\Omega)} \left(\gamma_t(u_h), [\gamma_t(v_h)]_F\right)_{0,F} = \ell_1(\mu^{-1}\text{curl } v_h) + \ell_2(v_h),$$

where $\alpha > 0$ is an appropriately chosen penalty parameter.
Interior Penalty Discontinuous Galerkin Method

**Theorem 3.** Assume that $u_h \in V_h$ solves $(IPDG_1)$. Then, there exists $p_h \in Q_h$ such that the pair $(u_h, p_h) \in V_h \times Q_h$ solves $(IPDG_2)$:

\[
\begin{align*}
    a(p_h, q_h) - b(u_h, q_h) &= \ell_1(q_h) + L_1(u_h, q_h), \quad q_h \in Q_h, \\
    b(v_h, p_h) + c(u_h, v_h) &= \ell_2(v_h) + L_2(u_h, v_h), \quad v_h \in V_h,
\end{align*}
\]

where $L_1(u_h, q_h)$ and $L_2(u_h, v_h)$ are given by

\[
\begin{align*}
    L_1(u_h, q_h) &:= \sum_{F \in \mathcal{F}_h(\Omega)} \left( \{\pi_t(\mu^{-1} \text{curl } q_h)\}_F, [\gamma_t(u_h)]_F \right)_{0,F}, \\
    L_2(u_h, v_h) &:= \sum_{F \in \mathcal{F}_h(\Omega)} \left( \{\pi_t(\mu^{-1} \text{curl } u_h)\}_F, [\gamma_t(v_h)]_F \right)_{0,F} - \alpha \sum_{F \in \mathcal{F}_h(\Omega)} \left( [\gamma_t(u_h)]_F, [\gamma_t(v_h)]_F \right)_{0,F}.
\end{align*}
\]

Conversely, if $(u_h, p_h) \in V_h \times Q_h$ solves $(IPDG_2)$, then $u_h \in V_h$ solves $(IPDG_1)$. 
Proof. Suppose that \( u_h \in V_h \) solves IPDG\(_1\). Then
\[
b(u_h, q_h) + \ell_1(q_h) + L_1(u_h, q_h) , \quad q_h \in Q_h
\]
defines a bounded linear functional on \( Q_h \). Let \( a(p_h, \cdot) \) be the unique Riesz representation of this functional. It follows that \( (u_h, p_h) \) solves the first equation of IPDG\(_2\).
Choosing \( q_h := \mu^{-1} \text{curl } v_h \) as a test function in the first equation of IPDG\(_2\) results in
\[
(\ast) \quad a(p_h, \mu^{-1} \text{curl } v_h) = b(v_h, p_h).
\]
Combining this equation with IPDG\(_1\) shows that \( (u_h, p_h) \) solves the second equation of IPDG\(_2\).

Conversely, assume that \( (u_h, p_h) \in V_h \times Q_h \) solves IPDG\(_2\). Choosing again \( q_h := \mu^{-1} \text{curl } v_h \) as a test function in the first equation of IPDG\(_2\) gives \((\ast)\). Together with the second equation in IPDG\(_2\) this shows that \( u_h \) solves IPDG\(_1\).
IPDG: Construction of an Approximation \((\widetilde{u}_h, \widetilde{p}_h) \in V \times Q\)

The IPDG method represents a nonconforming approach, since \(V_h\) is not a subspace of \(V\). Hence, there is a consistency error

\[
\kappa_h := \inf_{\tilde{v}_h \in V} J(\tilde{v}_h) \quad J(\tilde{v}_h) := \left( \sum_{T \in T_h(\Omega)} (\|u_h - \tilde{v}_h\|_{0,T}^2 + \|\text{curl } u_h - \text{curl } \tilde{v}_h\|_{0,T}^2)^{1/2}. \right.
\]

We define \(\widetilde{u}_h \in V\) as the unique minimizer, i.e., \(J(\tilde{u}_h) = \inf_{\tilde{v}_h \in V} J(\tilde{v}_h)\), and set \(\tilde{p}_h := \mu^{-1}\text{curl } \tilde{u}_h\).

**Lemma 1.** Let \(v_h \in \text{Nd}_1(\Omega; \mathcal{T}_h(\Omega))\). Then, there holds

\[
\text{Res}_2(v_h) = c(u_h - \tilde{u}_h, v_h) + d(u_h - \tilde{u}_h, v_h),
\]

where \(d(u_h - \tilde{u}_h, v_h)\) is given by

\[
d(u_h - \tilde{u}_h, v_h) := \sum_{F \in \mathcal{F}_h(\Omega)} \left( \{\pi_t(u_h - \tilde{u}_h)\}_F, [\gamma_t(\mu^{-1}\text{curl } v_h)]_F \right)_{0,F}. \]
Proof. Since $[\gamma_t(v_h)]_F = 0, F \in \mathcal{F}_h(\Omega)$, we have $L_2(u_h,v_h) = 0$, whence

$$\text{Res}_2(v_h) = c(u_h - \tilde{u}_h,v_h) + \sum_{T \in \mathcal{T}_h(\Omega)} \left( \mu^{-1} \text{curl} (u_h - \tilde{u}_h), \text{curl} v_h \right)_{0,T} - L_1(u_h, \mu^{-1} \text{curl} v_h).$$

Observing $\text{curl curl} \ v_h|_T = 0, T \in \mathcal{T}_h(\Omega)$, and $[\gamma_t(\tilde{u}_h)]_F = 0, F \in \mathcal{F}_h(\Omega)$, Stokes’ theorem yields

$$\sum_{T \in \mathcal{T}_h(\Omega)} \left( \mu^{-1} \text{curl} (u_h - \tilde{u}_h), \text{curl} v_h \right)_{0,T} = \sum_{F \in \mathcal{F}_h(\Omega)} \left( \{\pi_t(u_h - \tilde{u}_h)\}_F, [\gamma_t(\mu^{-1} \text{curl} v_h)]_F \right)_{0,F}$$

$$+ \sum_{F \in \mathcal{F}_h(\Omega)} \left( \{\pi_t(\mu^{-1} \text{curl} v_h)\}_F, [\gamma_t(u_h)]_F \right)_{0,F} = d(u_h - \tilde{u}_h,v_h) + L_1(u_h,v_h).$$
IPDG: Reliability of the Residual-Type Error Estimator

**Theorem 4.** Let \((\tilde{u}_h, \tilde{p}_h) \in V \times Q\) be the approximation obtained by the solution \((u_h, p_h) \in V_h \times Q_h\) of the IPDG method. Then, there holds
\[
\|(u - \tilde{u}_h, p - \tilde{p}_h)\|_{V \times Q} \lesssim \eta_h + \kappa_h.
\]

**Proof.** Lemma 1 tells us to consider the functional
\[
\text{Res}_3 := \text{Res}_2 - c(u_h - \tilde{u}_h, \cdot) - d(u_h - \tilde{u}_h, \cdot),
\]
which, by construction, satisfies
\[
\text{Nd}_1(\Omega; T_h(\Omega)) \subset \text{Ker Res}_3.
\]
Hence, Theorem 2 gives
\[
\|\text{Res}_3\|_{V^*}^2 \lesssim \eta_h^2,
\]
and we obtain
\[
\|\text{Res}_2\|_{V^*}^2 \lesssim \eta_h^2 + \sum_{T \in T_h(\Omega)} \left(\|u_h - \tilde{u}_h\|_{0,T}^2 + \|\text{curl} (u_h - \tilde{u}_h)\|_{0,T}^2 \right) \leq \eta_h^2 + \kappa_h^2.
\]
IPDG: Upper Bound for the Consistency Error

Remark 2. According to Houston/Perugia/Schötzau [2007], the consistency error $\kappa_h$ can be bounded from above by

$$\kappa_h^2 \lesssim \alpha \sum_{F \in \mathcal{F}_h(\Omega)} h_F^{-1} \| [\gamma_t(u_h)]_F \|_{0,F}^2.$$
Mortar Edge Element Approximations of
H(curl)-Elliptic Boundary Value Problems
Mortar Edge Element Approximations of H(curl)-Elliptic BVPs

We consider a non overlapping decomposition of \( \Omega \) into \( N \) mutually disjoint subdomains

\[
\overline{\Omega} = \bigcup_{j=1}^{N} \Omega_j \quad \text{with} \quad \Omega_j \cap \Omega_k \neq \emptyset \quad \text{for all} \quad 1 \leq j < k \leq N.
\]

We assume the decomposition to be geometrically conforming, i.e., two adjacent subdomains either share a face, an edge, or a vertex. The skeleton \( S \) of the decomposition

\[
S = \bigcup_{m=1}^{M} \gamma_m
\]

consists of the interfaces \( \gamma_1, \ldots, \gamma_M \) between all adjacent subdomains \( \Omega_j \) and \( \Omega_k \). We refer to \( \gamma_{m(j)} \) as the mortar associated with subdomain \( \Omega_j \), while the other face, which geometrically occupies the same place, is denoted by \( \delta_{m(j)} \) and is called the nonmortar.
Mortar Edge Element Approximations of $H(\text{curl})$-Elliptic BVPs

The mortar edge element approximation is based on individual shape-regular simplicial triangulations $\mathcal{T}_1, \ldots, \mathcal{T}_N$ of the subdomains $\Omega_1, \ldots, \Omega_N$ regardless the situation on the skeleton $S$ of the decomposition. In particular, the interfaces inherit two different non-matching triangulations.

The discretization of

$$H_{0, \partial \Omega_i \cap \partial \Omega}(\text{curl}; \Omega_j) := \{ u \in H(\text{curl}; \Omega_j) \mid \gamma_t(u)_{\partial \Omega_i \cap \partial \Omega} = 0 \}$$

with curl-conforming edge elements of Nédélec’s first family considers the edge element spaces $\text{Nd}_{1, \Gamma}(\Omega_j; \mathcal{T}_j)$ of vector fields with vanishing tangential trace on $\Gamma \cap \partial \Omega_j$. We define

$$X_h := \{ v_h \in L^2(\Omega) \mid v_h|_{\Omega_j} \in \text{Nd}_{1, \Gamma}(\Omega_j; \mathcal{T}_j), \ 1 \leq j \leq N \}. $$

equipped with the norm

$$\| v_h \|_{X_h} := \left( \| v_h \|_X^2 + \| [\gamma_t(v_h)|_S]_{+1/2,h,S} \|_{+1/2,h,S}^2 \right)^{1/2} \text{ for all } v_h \in X_h,$$

where $\| \cdot \|_{+1/2,h,S}$ is given by

$$\| [\gamma_t(v_h)|_S]_{+1/2,h,S} \|_{+1/2,h,S} := \left( \sum_{m=1}^M \| [\gamma_t(v_h)|_{\gamma_m}]_{+1/2,\gamma_m} \|_{+1/2,\gamma_m}^2 \right)^{1/2}.$$
Mortar Edge Element Approximations of $H(\text{curl})$-Elliptic BVPs

Due to the occurrence of nonconforming edges on the interfaces between adjacent subdomains, there is a lack of continuity across the interfaces: neither the tangential traces $\gamma_t(v_h)$ nor the tangential trace components $\pi_t(v_h)$ can be expected to be continuous. We note that $\gamma_t(v_h)|_{\delta_m(j)} \in RT_0(\delta_m(j);T_{\delta_m(j)})$ and $\pi_t(v_h)|_{\delta_m(j)} \in Nd_1(\delta_m(j);T_{\delta_m(j)})$. Therefore, continuity can be enforced either in terms of the tangential traces or the tangential trace components. If we choose the tangential traces, the multiplier space $M_h(S)$ can be constructed according to

$$M_h(S) := \prod_{m=1}^M M_h(\delta_m(j))$$

with $M_h(\delta_m(j))$ chosen such that

$$RT_{0,0}(\delta_m(j);T_{\delta_m(j)}) \subset M_h(\delta_m(j)) \quad , \quad \dim M_h(\delta_m(j)) = \dim RT_{0,0}(\delta_m(j);\delta_m(j)).$$

equipped with the mesh-dependent norm

$$\|\mu_h\|_{M_h(S)} := \left(\sum_{m=1}^M \|\mu_h|_{\delta_m(j)}\|_{-1/2,m,\delta_m(j)}\right)^{1/2}.$$
Mortar Edge Element Approximations of $H(\text{curl})$-Elliptic BVPs

The mortar edge element approximation requires the solution of the saddle point problem:

\[ A_h(u_h, v_h) + B_h(v_h, \lambda_h) = \ell(v_h), \quad v_h \in X_h, \]
\[ B_h(u_h, \mu_h) = 0, \quad \mu_h \in M_h(S), \]

where the bilinear forms \( A_h(\cdot, \cdot) : X_h \times X_h \to \mathbb{R} \) and \( B_h(\cdot, \cdot) : X_h \times M_h(S) \to \mathbb{R} \) are given by

\[ A_h(u_h, v_h) := \sum_{j=1}^{N} \int_{\Omega_j} \left( \mu^{-1} \text{curl} u_h \cdot \text{curl} v_h + \sigma u_h \cdot v_h \right) \, dx, \]
\[ B_h(u_h, \mu_h) := \langle \mu_h, [\gamma_t(u_h)] \rangle_{-1/2, S}. \]

With the minimizer \( \tilde{u}_h \in V \) of the consistency error \( \kappa_h \) as given before and \( \tilde{p}_h := \mu^{-1} \text{curl} \, \tilde{u}_h \) we find

\[ \|(u - \tilde{u}_h, p - \tilde{p}_h)\|_{V \times Q} \lesssim \|\text{Res}_2\|_{V^*}, \]

where
\[ \text{Res}_2(v) = \sum_{i=1}^{N} \text{Res}_2^{(i)}(v), \quad \text{Res}_2^{(i)}(v) := (f, v)_{0, \Omega_i} - (\mu^{-1} \text{curl} \, \tilde{u}_h, \text{curl} v)_{0, \Omega_i} - (\sigma \, \tilde{u}_h, v)_{0, \Omega_i}. \]
Mortar Edge Element Approximation: Residual-Type Error Estimator

We introduce the following residual-type a posteriori error estimator

\[ \eta_h^2 := \sum_{i=1}^{N} \left( \sum_{T \in \mathcal{T}_i} \eta_T^2 + \sum_{F \in \mathcal{F}_h(\Omega_i)} \eta_F^2 \right), \]

where the element residuals \( \eta_T \) and the face residuals \( \eta_F \) are given by

\[ \eta_T := h_T \| f - \text{curl} \mu^{-1} \text{curl} u_h - \sigma u_h \|_{0,T} + h_T \| \text{div} (\sigma u_h) \|_{0,T}, \]
\[ \eta_F := h_F^{1/2} \| [\pi_t(p_h)]_F \|_{0,F} + h_F^{1/2} \| n_F \cdot [\sigma u_h]_F \|_{0,F}. \]

**Theorem 5.** Let \( \eta_h \) be the error estimator and \( \kappa_h \) be the consistency error as given above. Then, there holds

\[ \|(u - \bar{u}_h, p - \bar{p}_h)\|_{V \times Q} \lesssim \eta_h + \kappa_h. \]
Proof. For $v_h \in \prod_{i=1}^{N} \text{Nd}_{1,0}(\Omega_i; T_h)$ we have

$$\text{Res}_2(v_h) = \sum_{i=1}^{N} \text{Res}_2^{(i)}(v_h), \quad \text{Res}_2^{(i)}(v_h) := (\sigma(u_h - \tilde{u}_h), v_h)_{0,\Omega_i} + (\mu^{-1}\text{curl}_h(u_h - \tilde{u}_h), \text{curl} v_h)_{0,\Omega_i}.$$ 

Therefore, we define

$$\text{Res}_3 := \sum_{i=1}^{N} \text{Res}_3^{(i)}, \quad \text{Res}_3^{(i)} := \text{Res}_2^{(i)} - \left( (\sigma(u_h - \tilde{u}_h, \cdot)_{0,\Omega_i} + (\mu^{-1}(\text{curl}_h(u_h - \tilde{u}_h), \text{curl} \cdot)_{0,\Omega_i}) \right),$$

and note that $\text{Nd}_{1,0}(\Omega_i; T_h) \subset \text{Ker Res}_3^{(i)}, 1 \leq i \leq N$. A subdomainwise application of Theorem 2 yields

$$\|\text{Res}_3\|_{V^*} \lesssim \eta_h.$$ 

Hence, it follows that

$$\|\text{Res}_2\|_{V^*} \lesssim \eta_h + \|u_h - \tilde{u}_h\|_{0,\Omega} + \|\text{curl} u_h - \text{curl} \tilde{u}_h\|_{0,\Omega} \leq \eta_h + \kappa_h.$$
Mortar Edge Element Approximation: Upper bound for the Consistency Error

Remark 3. An upper bound for the consistency error $\kappa_h$ can be derived using the techniques from Hoppe [2006]. In particular, we obtain

$$\kappa_h^2 \lesssim \sum_{i=1}^{N} \sum_{F \in \mathcal{F}_h \setminus \mathcal{F}_m} \left( \eta^2_F + \hat{\eta}^2_F \right),$$

with additional face residuals

$$\hat{\eta}_F := h_F^{1/2} \| \lambda_h - \{ \pi_t(p_h) \} \|_{0,F} + h_F^{1/2} \| \lambda_h - \{ n_F \cdot \sigma u_h \} \|_{0,F} + h_F^{-1/2} \| [\gamma_t(u_h)] \|_{0,F}.$$

Here, $\lambda_h \in H^{-1/2}(\gamma_m)$ satisfies

$$\langle \lambda_h, \text{grad}\varphi \rangle_{-1/2,\gamma_m} = - \langle \lambda_h, \varphi \rangle_{-1/2,\gamma_m} \quad \text{for all} \quad \varphi \in H^{1/2}(\gamma_m).$$
Numerical Example I: Geometric Edge Singularity

Data:
\[ \Omega := (-1, +1)^3 \setminus [0, 1]^2 \times [-1, +1] \]
\[ \chi = \kappa = 1 \]

Exact solution:
\[ j = \text{grad}(r^{2/3} \sin(\frac{2}{3}\Phi)) \]

Solution is in \( H(\text{curl}, \Omega) \), but not in \( H^1(\Omega)^3 \)
Numerical Results: 3D Eddy Currents Equations with an Edge Singularity

Grid after 5 (left) and 7 (right) refinement steps ( $\Theta_i = 0.4, 1 \leq i \leq 2$)
Numerical Results: 3D Eddy Currents Equations with an Edge Singularity

Left: True error (straight line), error estimator (dashed line), and data oscillations (dotted line) \(( \Theta_i = 0.4, 1 \leq i \leq 2)\); Right: Adaptive refinement (dashed line) versus uniform refinement (straight line)