LOCAL MULTILEVEL METHODS FOR ADAPTIVE NONCONFORMING FINITE ELEMENT METHODS

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Abstract. In this paper, we propose a local multilevel product algorithm and its additive version for linear systems arising from adaptive nonconforming finite element approximations of second order elliptic boundary value problems. The abstract Schwarz theory is applied to analyze the multilevel methods with Jacobi or Gauss-Seidel smoothers performed on local nodes on coarse meshes and global nodes on the finest mesh. It is shown that the local multilevel methods are optimal, i.e., the convergence rate of the multilevel methods is independent of the mesh sizes and mesh levels. Numerical experiments are given to confirm the theoretical results.

Introduction

Multigrid methods and other multilevel preconditioning methods for nonconforming finite elements have been studied by many researchers (cf. [4], [5], [6], [7], [14], [18], [19], [20], [21], [23], [26], [27], [28],[32], [34]). The BPX framework developed in [4] provides a unified convergence analysis for nonnested multigrid methods. Duan et al. [14] extended the result to general V-cycle nonnested multigrid methods, but only the case of full elliptic regularity was considered. Besides, Brenner [7] established a framework for the nonconforming V-cycle multigrid method under less restrictive regularity assumptions. All the above convergence results for nonconforming multigrid methods are based on the requirement of a sufficiently large number of smoothing steps at each level. For multilevel preconditioning methods, Oswald developed a hierarchical basis multilevel method [19] and a BPX-type multilevel preconditioner [20] for nonconforming finite elements. On the other hand, Vassilevski and Wang [26] presented multilevel algorithms with only one smoothing step per level. These multilevel algorithms may be considered as successive subspace correction methods (SSC) (cf. [30] for details). They are completely different from standard nonconforming multigrid methods [5]. By using the well-known Schwarz framework, a uniform convergence result has been obtained. The idea that the conforming finite element spaces are contained in their nonconforming counterparts is essential in the analysis of the multilevel algorithms (cf. [26] for details). In this paper, we will use this idea to design optimal multilevel methods for adaptive...
nonconforming element methods. We note that Hoppe and Wohlmuth [15] considered multilevel preconditioned conjugate gradient methods for nonconforming $P_1$ finite element approximations with respect to adaptively generated hierarchies of nonuniform meshes based on residual type a posteriori error estimators.

Recent studies (cf., e.g., [2], [10], [11], [17], [24]) indicate optimal convergence properties of adaptive conforming and nonconforming finite element methods. Therefore, in order to achieve an optimal numerical solution, it is imperative to study efficient iterative algorithms for the solution of linear systems arising from adaptive finite element methods (AFEM). Since the number of degrees of freedom $N$ per level may not grow exponentially with mesh levels, as Mitchell has pointed out in [16] for adaptive conforming finite element methods, the number of operations used for multigrid methods with smoothers performed on all nodes can be as bad as $O(N^2)$, and a similar situation may also occur in the nonconforming case.

For adaptive conforming finite element methods, Wu and Chen [29] have obtained uniform convergence for the multigrid $V$-cycle algorithm which performs Gauss-Seidel smoothing on newly generated nodes and those old nodes where the support of the associated nodal basis function has changed. To our knowledge, so far there does not exist an optimal multilevel method for nonconforming finite element methods on locally refined meshes. The reason is that the theoretical analysis for the local multilevel methods is rather difficult. Indeed, there are two difficulties which need to be overcome. First, the Xu and Zikatanov identity [31], on which the proof in [29] depends, can not be applied directly, because the multilevel spaces are nonnested in this situation. The second difficulty is how to establish the strengthened Cauchy-Schwarz inequality on nonnested multilevel spaces. In this paper, we will construct a special prolongation operator from the coarse space to the finest space, and obtain the key global strengthened Cauchy-Schwarz inequality. Two multilevel methods, the product and additive version, are proposed. Applying the well-known Schwarz theory (cf. [25]), we show that local multilevel methods for adaptive nonconforming finite element methods are optimal, i.e., the convergence rate of the multilevel algorithms is independent of mesh sizes and mesh levels.

The remainder of this paper is organized as follows: In section 2, we introduce some notations and briefly review nonconforming $P_1$ finite element methods. Section 3 is concerned with the study of condition number estimates of linear systems arising from adaptive nonconforming finite element methods by applying the techniques presented by Bank and Scott in [1]. The following section 4 is devoted to the derivation of a local multilevel product algorithm and its additive version. In section 5, we develop an abstract Schwarz theory based on three assumptions whose verification is carried out for local Jacobi and local Gauss-Seidel smoothers, respectively. Finally, in the last section we give some numerical experiments to confirm the theoretical analysis.

1. Notations and Preliminaries

Throughout this paper, we adopt standard notation from Lebesgue and Sobolev space theory (cf., e.g., [13]). In particular, we refer to $(\cdot, \cdot)$ as the inner product in $L^2(\Omega)$ and to $\| \cdot \|_{1, \Omega}$ as the norm in the Sobolev space $H^1(\Omega)$. We further use $A \lesssim B$, if $A \leq CB$ with a positive constant $C$ depending only on the shape regularity of the meshes. $A \approx B$ stands for $A \lesssim B \lesssim A$. We consider elliptic boundary value problems in polyhedral domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$. However, for the
sake of simplicity the analysis of local multilevel methods will be restricted to the 2D case.

Given a bounded, polygonal domain $\Omega \subset \mathbb{R}^2$, we consider the following second order elliptic boundary value problem

$$\mathcal{L}u := -\text{div}(a(x)\nabla u) = f \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega. \quad (1.1)$$

The choice of a homogeneous Dirichlet boundary condition is made for ease of presentation only. Similar results are valid for other types of boundary conditions and equation (1.1) with a lower order term as well. We further assume that the coefficient functions in (1.1) satisfy the following properties:

(a) $a(\cdot)$ is a measurable function and there exist constants $\beta_1 \geq \beta_0 > 0$ such that

$$\beta_0 \leq a(x) \leq \beta_1 \quad \text{f.a.a. } x \in \Omega; \quad (1.3)$$

(b) $f \in L^2(\Omega)$.

The weak formulation of (1.1) and (1.2) is to find $u \in V := H^1_0(\Omega)$ such that

$$a(u, v) = (f, v), \quad v \in V, \quad (1.4)$$

where the bilinear form $a : V \times V \to \mathbb{R}$ is given by

$$a(u, v) = (a\nabla u, \nabla v), \quad u, v \in V. \quad (1.5)$$

Since the bilinear form (1.5) is bounded and $V$-elliptic, the existence and uniqueness of the solution of (1.4) follows from the Lax-Milgram theorem.

Throughout this paper, we work with families of shape regular meshes $\{T_i, i = 0, 1, ..., J\}$, where $T_0$ is an intentionally chosen coarse initial triangulation, the others are obtained by adaptive procedures, refined by the newest vertex bisection algorithm. It has been proved that there exists a constant $\theta > 0$ such that

$$\theta_T \geq \theta \quad , \quad T \in T_i, \ i = 1, 2, ..., \quad (1.6)$$

where $\theta_T$ is the minimum angle of the element $T$. The set of edges on $T_i$ is denoted by $\mathcal{E}_i$, and the set of interior and boundary edges by $\mathcal{E}^0_i$ and $\mathcal{E}^\partial_i$, respectively. Correspondingly, let $M_i$ denote all the middle points of $\mathcal{E}_i$ and $M^0_i$ be the middle points of $\mathcal{E}^0_i$. We refer to $\mathcal{N}_i$ as the set of interior nodes of $T_i$. For any $E \in \mathcal{E}_i$, $h_{E}$ and $m_{E}$ denote the length and the midpoint of $E$. The patch $\omega_{i,E}, E \in \mathcal{E}^0_i$, is the union of two elements in $T_i$ sharing $E$. For any $T \in T_i$, $h_{T}$ and $x_{T}$ stand for the diameter and the barycenter of $T$.

We denote by $V_J$ the lowest order nonconforming Crouzeix-Raviart finite element space with respect to $T_J$, i.e.,

$$V_J = \{v_J \in L^2(\Omega) \mid v_J|_T \in P_1(T), T \in T_J, \int_E [v_J]ds = 0, E \in \mathcal{E}_J\}. \quad (1.7)$$

Here, $[v_J]|_E$ refers to the jump of $v_J$ across $E \in \mathcal{E}^0_J$ and is set to zero for $E \in \mathcal{E}^\partial_J$. Moreover, we define the conforming $P_1$ finite element space by

$$V_J^c = \{v_J^c \in V \mid v_J^c|_T \in P_1(T), T \in T_J\}. \quad \text{Moreover, we define the conforming } P_1 \text{ finite element space by}$$

The nonconforming finite element approximation of (1.4) is to find $u_J \in V_J$ such that

$$a_J(u_J, v_J) = (f, v_J), \quad v_J \in V_J, \quad (1.7)$$
where \( a_J(\cdot, \cdot) \) stands for the mesh-dependent bilinear form

\[
a_J(u_J, v_J) = \sum_{T \in T_J} (a \nabla u_J, \nabla v_J)_{0,T}.
\]

Existence and uniqueness of the solution \( u_J \) again follows from the Lax-Milgram theorem. In the sequel, we refer to \( \| \cdot \|_{1,J} \) as the mesh-dependent energy norm

\[
\| v_J \|_{1,J}^2 = \sum_{T \in T_J} \| v_J \|_{1,T}^2.
\]

For brevity, we will drop the subscript \( J \) from some of the above quantities, if no confusion is possible, e.g., we will write \( h_T \) instead of \( h_{J,T} \) and \( a(\cdot, \cdot) \) instead of \( a_J(\cdot, \cdot) \).

2. Condition number estimate

The computation of the solution \( u_J \) of (1.7) always requires to solve a matrix equation using a particular basis for \( V_J \). Suppose that \( \{ \phi_i, i = 1, ..., N \} \) is a given basis for \( V_J \), where \( N \) is the dimension of \( V_J \), and define the matrix \( A \) and the vector \( F \) according to

\[
A_{ij} := a(\phi_i, \phi_j) \quad \text{and} \quad F_i := (f, \phi_i), \quad i, j = 1, ..., N.
\]

Then, equation (1.7) is equivalent to the linear algebraic system

\[
AX = F,
\]

where \( u_J = \sum_{i=1}^N u_i \phi_i \) and \( X = (u_i) \).

In this section, we will not restrict ourselves to the two-dimensional case, but consider domains \( \Omega \subset \mathbb{R}^n, n \geq 2 \). We will specify conditions on \( V_J \) and the basis \( \{ \phi_i, i = 1, ..., N \} \) that will allow us to establish upper bounds for the condition number of \( A \).

We assume that \( T_J \) contains at most \( \alpha_1 n/2 \) elements, with \( \alpha_1 \) denoting a fixed constant. The following estimates hold true (cf., e.g., [13]):

\[
\| v \|^2_{1,T} \lesssim h_{T}^{n-2} \| v \|^2_{L^{\infty}(T)} \lesssim \| v \|^2_{L^{2n/(n-2)}(T)}, \quad T \in T_J, \quad v \in V_J, \quad n \geq 3.
\]

In the special case of two dimensions \( (n = 2) \), we supplement the following inequality to the latter one in (2.2),

\[
\| v \|^2_{L^{\infty}(T)} \lesssim h_{T}^{-2/p} \| v \|^2_{L^p(T)}, \quad T \in T_J, \quad v \in V_J, \quad 1 \leq p \leq \infty.
\]

Under the assumptions on the domain \( \Omega \), there exists a continuous embedding \( H^1(\Omega) \hookrightarrow L^p(\Omega) \). For \( n \geq 3 \), Sobolev’s inequality

\[
\| v \|_{L^{2n/(n-2)}(\Omega)} \leq C \| v \|_{1,\Omega}, \quad v \in H^1(\Omega).
\]

holds true. In two dimensions, we have a more explicit estimate (cf., e.g., [1])

\[
\| v \|_{L^p(\Omega)} \leq C \sqrt{p} \| v \|_{1,\Omega}, \quad v \in H^1(\Omega), \quad p < \infty.
\]

As far as the basis \( \{ \phi_i, i = 1, ..., N \} \) of \( V_J \) is concerned, we assume that it is a local basis:

\[
\max_{1 \leq i \leq N} \text{cardinality}(T \in T_J : \text{supp}(\phi_i) \cap T \neq \emptyset) \leq \alpha_2.
\]
Finally, we impose a more important assumption with regard to the scaling of the basis:

\[(2.7)\quad h_T^{-2}||v||^2_{L^\infty(T)} \lesssim \sum_{\supp(\phi_i) \cap T \neq \emptyset} v_i^2 \lesssim h_T^{-2}||v||^2_{L^\infty(T)}, \quad T \in T_{J},\]

where \(v = \sum_{i=1}^{N} v_i \phi_i\) and \((v_i)\) is arbitrary. For instance, if \(\{\psi_i, i = 1, \ldots, N\}\) denote the Crouzeix-Raviart P1 nonconforming basis functions, we define a new scaled basis \(\{\phi_i, i = 1, \ldots, N\}\) by

\[\phi_i := h_i^{(2-n)/2} \psi_i,\]

where \(h_i\) is the diameter of the support of \(\psi_i\). Then, the new basis satisfies assumption (2.7). We also impose the same assumption (2.7) for the conforming finite element basis, when utilized in the sequel.

For the analysis of the condition number, we propose a prolongation operator from \(V_J\) to \(\hat{V}_{J+1}\), where \(\hat{V}_{J+1}\) is the conforming finite element space based on \(T_{J+1}\). \(T_{J+1}\) is an auxiliary triangulation, only used in the analysis, which is obtained from \(T_J\) by subdividing each \(T \in T_J\) into \(2^n\) simplices by joining the midpoints of the edges. We refer to \(T\) as an element in \(T_J\) with vertices \(x_k, k = 1, \ldots, n + 1\), and denote the midpoints of its edges by \(m_1, \ldots, m_s\), where \(s\) is the number of edges of \(T\), e.g., \(s = 3\) if \(n = 2\).

In case \(n = 2\), the prolongation operator \(I_{J+1}^{J+1}: V_J \rightarrow \hat{V}_{J+1}\) is defined by

\[I_{J+1}^{J+1}(m_l) = v(m_l), \quad l = 1, \ldots, 3, \quad I_{J+1}^{J+1}(x_k) = \beta_k, \quad k = 1, \ldots, 3,\]

where \(\beta_k\) is the average of \(v\) in \(x_k\). Moreover, \(I_{J+1}^{J+1}(x_k) = 0\), if \(x_k\) is located on the Dirichlet boundary. The stability analysis of \(I_{J+1}^{J+1}\) has been derived when \(T_{J+1}\) is obtained from \(T_J\) by the above bisection algorithm. In the AFEM procedures we use the newest vertex bisection algorithm. The associated stability analysis of \(I_{J-1}^{J-1}, i = 1, \ldots, J\), will be given in the appendix of this paper.

As in the case \(n \geq 3\), we define \(I_{J+1}^{J+1}: V_J \rightarrow \hat{V}_{J+1}\) according to

\[I_{J+1}^{J+1}(m_l) = \alpha_l, \quad l = 1, \ldots, s, \quad I_{J+1}^{J+1}(x_k) = \beta_k, \quad k = 1, \ldots, n + 1,\]

where \(\alpha_l\) and \(\beta_k\) are the averages of \(v\) at \(m_l\) and \(x_k\) respectively. \(I_{J+1}^{J+1}(x_k) = 0\) or \(I_{J+1}^{J+1}(m_l) = 0\), if \(x_k\) or \(m_l\) is situated on the Dirichlet boundary. The associated stability analysis of \(I_{J+1}^{J+1}\) can be obtained analogously.

We now give bounds on the condition number of the matrix \(A := (a(\phi_i, \phi_j))\), where \(\{\phi_i, i = 1, \ldots, N\}\) is the scaled basis for \(V_J\) satisfying the above assumptions.

In the general case \(n \geq 3\), we have the following result.

\textbf{Theorem 2.1.} Suppose that the nonconforming finite element space \(V_J\) satisfies (2.2) and the basis \(\{\phi_i, i = 1, \ldots, N\}\) satisfies (2.6) and (2.7). Then, the \(\ell_2\)-condition number \(K_{2}(A)\) of \(A\) is bounded by

\[(2.8) \quad K_{2}(A) \lesssim N^{2/n}.\]

\textbf{Proof.} We set \(v = \sum_{i=1}^{N} v_i \phi_i\), then

\[a(v, v) = X^tAX,\]

where \(X = (v_i)\). By a similar technique as in the proof of Theorem 4.1 in [1], we have

\[a(v, v) \lesssim X^tX.\]
On the other hand, we apply the prolongation operator $I_{j+1}^j$ to $v$, and set

$$I_{j+1}^j v = \sum_{x_i \in \mathcal{N}_{j+1}(\tilde{T})} I_{j+1}^j v(x_i) \tilde{\phi}_{i,j+1},$$

where $\{\tilde{\phi}_{i,j+1}\}$ is the conforming finite element basis of $\tilde{V}_{j+1}$. By Hölder’s inequality, Sobolev’s inequality, and the stability of $I_{j+1}^j$, we derive a complementary inequality according to

$$X^t X \leq \sum_{T \in \mathcal{T}_{j+1}} \left( \sum_{\{\phi_i\}_{i=1}^N \cap T \neq \emptyset} I_{j+1}^j v^2(x_i) \right) \leq \sum_{T \in \mathcal{T}_{j+1}} h_T^{n-2} ||I_{j+1}^j v||^2_{L^\infty(T)} \lesssim \sum_{T \in \mathcal{T}_{j+1}} ||I_{j+1}^j v||^2_{L^{2n/(n-2)}(T)} \lesssim N^{2/n} ||I_{j+1}^j v||^2_{L^{2n/(n-2)}(\Omega)} \lesssim N^{2/n} ||v||^2_{1,\Omega} \lesssim N^{2/n} a(v,v).$$

Using the above estimates, we obtain

$$N^{-2/n} X^t X \lesssim X^t A X \lesssim X^t X,$$

which implies that

$$N^{-2/n} \lesssim \lambda_{\min}(A) \quad \text{and} \quad \lambda_{\max}(A) \lesssim 1.$$

Recalling

$$K_2(A) = \lambda_{\max}(A)/\lambda_{\min}(A),$$

the above two estimates yield (2.8).

\[ \square \]

In the special case $n = 2$, a similar result can be deduced as follows.

**Theorem 2.2.** Suppose that the nonconforming finite element space $V_j$ satisfies (2.2) and (2.3), and that the basis $\{\phi_i, i = 1, \ldots, N\}$ satisfies (2.6) and (2.7). Then, the $\ell_2$-condition number $K_2(A)$ of $A$ is bounded by

$$K_2(A) \lesssim N(1 + |\log(Nh_{\text{min}}^2(\mathcal{E}_j))|).$$

**Proof.** As in the proof of the above theorem, it suffices to show that

$$N(1 + |\log(Nh_{\text{min}}^2(\mathcal{E}_j))|)^{-1} X^t X \lesssim X^t A X \lesssim X^t X.$$

We set $v = \sum_{i=1}^N v_i \phi_i, X = (v_i)$ and $a(v,v) = X^t A X$. Then, $a(v,v) \lesssim X^t X$ holds true as in Theorem 5.1 in [1].

As far as the lower bound in (2.10) is concerned, as in the proof of Theorem 2.1 we have ($p > 2$)

$$X^t X \leq \sum_{T \in \mathcal{T}_{j+1}} \left( \sum_{\{\phi_i\}_{i=1}^N \cap T \neq \emptyset} I_{j+1}^j v^2(x_i) \right) \leq \sum_{T \in \mathcal{T}_{j+1}} ||I_{j+1}^j v||^2_{L^\infty(T)} \lesssim \sum_{T \in \mathcal{T}_{j+1}} h_T^{-4/p} ||I_{j+1}^j v||^2_{L^p(T)} \lesssim \left( \sum_{T \in \mathcal{T}_{j+1}} h_T^{-4/(p-2)} \right)^{1/p} ||I_{j+1}^j v||^2_{L^p(T)} \lesssim \left( \sum_{T \in \mathcal{T}_{j+1}} h_T^{-2/(p-2)} \right)^{1/p} ||v||^2_{1,T} \lesssim \left( \sum_{T \in \mathcal{T}_{j+1}} h_T^{-2/(p-2)} \right)^{1/p} p a(v,v) \lesssim N(Nh_{\text{min}}^2(\mathcal{E}_j))^{-2/p} p a(v,v).$$

The special choice $p = \max\{2, |\log(Nh_{\text{min}}^2(\mathcal{E}_j))|\}$ allows to conclude.  

\[ \square \]
For a fixed triangulation, the conforming P1 finite element space is contained in the nonconforming P1 finite element space. Hence, the sharpness of the bounds in Theorem 2.2 can be verified by the same example as in [1].

3. LOCAL MULTILEVEL METHODS

The above section clearly shows that for the solution of a large scale problem the convergence of standard iterations such as Gauss-Seidel or CG will become very slow. This motivates the construction of more efficient iterative algorithms for those algebraic systems resulting from adaptive nonconforming finite element approximations.

We will derive our local multilevel methods for adaptive nonconforming finite element discretizations based on the Crouzeix-Raviart elements. As a prerequisite, we again use the prolongation operator \( I_{i-1} : V_{i-1} \rightarrow V_i \) defined as in section 2. Now, \( T_i \) represents a refinement of \( T_{i-1} \) by the newest vertex bisection algorithm, \( I_{i-1}^* \) defines the values of \( I_{i-1}^* v \) at the vertices of elements of level \( i \), yielding a continuous piecewise linear function on \( T_i \). \( I_{i-1}^* v \) being a function in \( V_i \), it naturally represents a function in the finest space \( V_J \). Hence, the operator \( I_{i-1} \) given by

\[
I_{i-1} v := I_{i-1}^* v, \quad v \in V_{i-1},
\]
defines an intergrid operator from \( V_{i-1} \) to \( V_J \).

For \( 0 \leq i \leq J \), we define \( A_i : V_i \rightarrow V_i \) by means of

\[
(A_i v, w) = a_i(v, w), \quad w \in V_i.
\]

We also define projections \( P_i, P_0^i : V_J \rightarrow V_i \) according to

\[
a_i(P_i v, w) = a(v, P_i w), \quad (P_0^i v, w) = (v, P_i w), \quad v \in V_J, w \in V_i.
\]

For any node \( z \in N_i \), we use the notation \( \varphi_i^z \) to represent the associated nodal conforming basis function of \( V_i^c \). Let \( \tilde{N}_i^c \) be the set of new nodes and those old nodes where the support of the associated basis function has changed, i.e.,

\[
\tilde{N}_i^c = \{ z \in N_i : z \in N_i \setminus N_{i-1} \text{ or } z \in N_{i-1} \text{ but } \varphi_i^z \neq \varphi_i^{z-1} \}.
\]

Let \( \tilde{M}_i \) represent the set of midpoints on which local smoothers are performed:

\[
\tilde{M}_i := \{ m_{i,E} \in M_i : m_{i,E} \in M_i(\tilde{T}_i) \},
\]

where \( \tilde{T}_i = \bigcup_{z \in \tilde{N}_i^c} \{ \text{supp}(\varphi_i^z) \} \).

For convenience, we set \( \tilde{M}_i = \{ m_i^k, k = 1, \ldots, \tilde{n}_i \} \), where \( \tilde{n}_i \) is the cardinality of \( \tilde{M}_i \), and refer to \( \phi_i^k = \phi_i^{m_i^k} \) as the Crouzeix-Raviart nonconforming finite element basis function associated with \( m_i^k \). Then, for \( k = 1, \ldots, \tilde{n}_i \) let \( P_i^k, Q_i^k : V_i \rightarrow V_i^k = \text{span}\{\phi_i^k\} \) be defined by

\[
a_i(P_i^k v, \phi_i^k) = a_i(v, \phi_i^k), \quad (Q_i^k v, \phi_i^k) = (v, \phi_i^k), \quad v \in V_i,
\]

and let \( A_i^k : V_i^k \rightarrow V_i^k \) be defined by

\[
(A_i^k v, \phi_i^k) = a_i(v, \phi_i^k), \quad v \in V_i^k.
\]

It is easy to see that the following relationship holds true:

\[
(3.1) \quad A_i^k P_i^k = Q_i^k A_i.
\]

We assume that the local smoothing operator \( R_i : V_i \rightarrow V_i \) is nonnegative, symmetric or nonsymmetric with respect to the inner product \((\cdot, \cdot)\). It will be
Figure 1. Coarse mesh (left), fine mesh (right) and illustration of $\tilde{M}_i$: the big nodes on the right refer to $\tilde{N}_i$, the small nodes refer to $\tilde{M}_i$, $i = 1, \ldots, J - 1$.

precisely defined and further studied in section 4. For $i = 1, \ldots, J - 1$, $R_i$ is only performed on local midpoints $\tilde{M}_i$ (we refer to Figure 1 for an illustration). $R_0$ is solved directly, i.e., $R_0 = A_0^{-1}$. On the finest level, $R_J$ is carried out on all midpoints $M^0_J$, i.e., $\tilde{u}_J = \#M^0_J$. For simplicity, we set $A = A_J$ and denote by $I_J$ and $P_J$ the identity operator on the finest space $V_J$. We set $S_i := I_i R_i A_i P_i$, $i = 0, 1, \ldots, J$.

Now, we scale $S_i$ as follows:

$$T_i := \mu_{J,i} S_i, \quad i = 0, 1, \ldots, J,$$

where $\mu_{J,i} > 0$ is a parameter, independent of mesh sizes and mesh levels, chosen to satisfy

$$a(T_i v, T_i v) \leq \omega_i a(T_i v, v), \quad v \in V_J, \quad w_i < 2.$$

We will also drop the subscript $J$ from $\mu_{J,i}$ since no confusion is possible in the convergence analysis.

With the sequences of operators $\{T_i, i = 0, 1, \ldots, J\}$, we can now state the local multilevel algorithm for adaptive nonconforming finite element methods as follows.

**Algorithm 3.1. Local multilevel product algorithm (LMPA)**

Given an arbitrarily chosen initial iterate $u^0 \in V_J$, we seek $u^n \in V_J$ as follows:

(i) Let $v_0 = u^{n-1}$. For $i = 0, 1, \ldots, J$, compute $v_{i+1}$ by

$$v_{i+1} = v_i + T_i (u_J - v_i).$$

(ii) Set $u^n = v_{J+1}$.

**Algorithm 3.2. Local multilevel additive algorithm (LMAA)**

Let $T = \sum_{i=0}^{J} T_i$ and let $u_J$ be the exact solution of (1.7). Find $\tilde{u}_J \in V_J$ such that

$$T \tilde{u}_J = \tilde{f},$$

where $\tilde{f} = \sum_{i=0}^{J} T_i u_J$.

In view of the operator equation $A_i P_i = P_i^0 A$, we have

$$A_i P_i = P_i^0 A_i.$$
the function \( \tilde{f} \) in (3.4) is formally defined by the exact finite element solution \( u_J \) which can be computed directly, and so does the iteration (3.3).

Obviously, there exists a unique solution \( \tilde{u}_J \) of (3.4) coinciding with \( u_J \) for (1.7).

The conjugate-gradient method can be used to solve the new problem, if \( T \) is symmetric. We can also apply the conjugate-gradient method to the symmetric version of \textbf{LMAA} (\textbf{SLMAA}) by solving

\[
\frac{(T + T^*)}{2} \tilde{u}_J = \hat{f}
\]

instead of (3.4), where \( \hat{f} = \sum_{i=0}^{J} \frac{(T_i + T_i^*)}{2} u_J \) and \( T^* \) denotes the adjoint operator of \( T \) with respect to the inner product \( a(\cdot, \cdot) \).

4. Convergence theory

In this section, we provide an abstract theory concerned with the convergence of local multilevel methods for linear systems arising from adaptive nonconforming finite element methods. We will use the well-known Schwarz theory developed in [25], [30] and [35] to analyze the algorithms.

Let \( \{T_i, i = 0, 1, \ldots, J\} \) be a sequence of operators from the finest space \( V_J \) to itself. The abstract theory provides an estimate for the norm of the error operator

\[
E = (I - T_J) \cdots (I - T_1)(I - T_0) = \prod_{i=0}^{J}(I - T_i),
\]

where \( I \) is the identity operator in \( V_J \). The convergence estimate for the algorithm \textbf{LMPA} is then obtained by the norm estimate for \( E \). The abstract theory can be invoked due to the following assumptions.

(A1). Each operator \( T_i \) is nonnegative with respect to the inner product \( a(\cdot, \cdot) \), and there exists a positive constant \( \omega_i < 2 \), which depends on \( \mu_i \), such that

\[
a(T_i v, T_i v) \leq \omega_i a(T_i v, v), \quad v \in V_J.
\]

(A2). Stability: There exists a constant \( K_0 \) such that

\[
a(v, v) \leq K_0 \mu a(T v, v), \quad v \in V_J,
\]

where \( \mu = \min_{0 \leq i \leq J} \{\mu_i\} \).

(A3). Global strengthened Cauchy-Schwarz inequality: There exists a constant \( K_1 \) such that

\[
\sum_{i=0}^{J} \sum_{j=0}^{i-1} a(T_i v, T_j u) \leq K_1 \left( \sum_{i=0}^{J} a(T_i v, v) \right)^{1/2} \left( \sum_{j=0}^{J} a(T_j u, u) \right)^{1/2}, \quad v, u \in V_J.
\]

As in the proof of (4.1) in [33], it is easy to show that the following inequality holds true for the algorithms \textbf{LMPA} and \textbf{LMAA} with local smoothers chosen as Jacobi or Gauss-Seidel iterations (especially \( K_2 = 1 \) in the Jacobi case):

\[
\sum_{i=0}^{J} a(T_i v, u) \leq K_2 \left( \sum_{i=0}^{J} a(T_i v, v) \right)^{1/2} \left( \sum_{i=0}^{J} a(T_i u, u) \right)^{1/2}, \quad v, u \in V_J.
\]
Theorem 4.1. Let the assumptions A1-A3 be satisfied. Then, for the algorithm 3.1 the norm of the error operator $E$ can be bounded as follows (cf. [25], [30], [35])

$$a(Ev, Ev) \leq \delta a(v, v), \quad v \in V_J,$$

where $\delta = 1 - \frac{a(2-\omega)}{K_0(K_1+K_2)^2}, \omega = \max_{0 \leq i \leq J} \{\omega_i\}$.

For the additive multilevel algorithm 3.2, the following theorem provides a spectral estimate for the operator $T = \sum_{i=0}^J T_i$ when $T$ is symmetric with respect to the inner product $a(\cdot, \cdot)$.

Theorem 4.2. If $T$ is symmetric with respect to $a(\cdot, \cdot)$ and assumptions A1-A3 hold true, then we have (cf. [25], [30], [35])

$$\frac{\mu}{K_0} a(v, v) \leq a(Tv, v) \leq (2K_1 + \omega) a(v, v), \quad v \in V_J.$$

When $T$ is nonsymmetric with respect to $a(\cdot, \cdot)$, similar analysis can be done for the spectral estimate of the symmetric part $T^+T^*$.  

Remark 5.1. It should be pointed out that the convergence result for LMPA or for the preconditioned conjugate gradient method by LMAA depends on the parameter $\mu$, which will be observed in our numerical experiments. The convergence rate deteriorates for decreasing $\mu$.

Next, we will apply the above convergence theory to LMPA and LMAA by verifying assumptions A1-A3 for the adaptive nonconforming finite element method. There are two classes of smoothers $R_i$, Jacobi and Gauss-Seidel iterations, which will be considered separately.

4.1. Local Jacobi smoother. First, for $v \in V_J$ we consider the decomposition

$$v = \sum_{i=0}^J v_i , \quad v_J = v - \tilde{v}, \quad v_i = (\Pi_i - \Pi_{i-1}) \tilde{v}, \quad i = 0, 1, \ldots, J - 1,$$

where $\tilde{v} = \tilde{\Pi}_{J-1} v$ and $\tilde{\Pi}_{J-1} v$ represents a local regularization of $v$ in $V_{J-1}^c$ (c.f. [9]), e.g., by a Clément-type interpolation. $\Pi_i : V_{J-1}^c \rightarrow V_i^c$ stands for the Scott-Zhang interpolation operator [22].

The local Jacobi smoother is defined as an additive smoother (cf. [3]):

$$R_i := \gamma \sum_{k=1}^{\hat{n}_i} (A_i^k)^{-1} Q_i^k,$$

where $\gamma$ is a suitably chosen positive scaling factor. Due to (3.1), we have

$$T_0 = \mu_0 I_0 P_0 \quad T_i = \mu_i I_i R_i A_i P_i = \mu_i \gamma I_i \sum_{k=1}^{\hat{n}_i} P_i^k P_i, \quad i = 1, \ldots, J.$$


Lemma 4.1. Let $T_i, i \geq 0$, be defined by (4.4). Then, we have

$$a(T_i v, Tv) \leq \omega_i a(T_i v, v), \quad v \in V_J,$$

Moreover, $T_i$ is symmetric and nonnegative in $V_J$. Therefore, assumption A1 is satisfied.
Proof. Following (4.4), for \(v, w \in V_j\) we deduce
\[
a(T_i v, w) = a(\mu_i I_i R_i A_i P_i v, w) = a(\mu_i R_i A_i P_i v, P_i w) = (\mu_i R_i A_i P_i v, A_i P_i w).
\]
In view of the definition of \(R_i\) in (4.3), we can easily see that \(R_i\) is symmetric and nonnegative in \(V_i\). Hence, \(T_i\) is symmetric and nonnegative in \(V_j\).

It is easy to show that the stated result holds true for \(T_0\). Actually, we have
\[
a(T_0 v, T_0 v) \leq \mu_0^2 C_0 a_0 (P_0 v, P_0 v) = \mu_0 C_0 a(T_0 v, v).
\]
Let \(\omega_0 = \mu_0 C_0\). We choose \(\mu_0 < 2/C_0\) such that \(\omega_0 < 2\).

For \(T_i, i \geq 1\), we set
\[
K^k_i = \{P_i^m : \text{supp}(I_i P_i^k v) \cap \text{supp}(I_i P_i^m v) \neq \emptyset, \ v \in V_i\}
\]
and
\[
\gamma_{k,m} = \begin{cases} 1 & \text{if } \text{supp}(I_i P_i^k v) \cap \text{supp}(I_i P_i^m v) \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases}
\]
The cardinality of \(K^k_i\) is bounded by a constant depending only on the minimum angle \(\theta\) in (1.6). For \(v \in V_i, i = 1, \ldots, J\), Hölder’s inequality implies
\[
(4.5) \quad \sum_{k,m=1}^{\tilde{n}_i} |a(I_i P_i^k v, I_i P_i^m v)| \leq \sum_{k,m=1}^{\tilde{n}_i} \gamma_{k,m} |a(I_i P_i^k v, I_i P_i^m v)|
\]
\[
\leq \sum_{k,m=1}^{\tilde{n}_i} \gamma_{k,m} |a(I_i P_i^k v, I_i P_i^k v)| \leq C_i \sum_{k=1}^{\tilde{n}_i} a(I_i P_i^k v, I_i P_i^k v).
\]

Taking advantage of the definition of \(T_i\) in (4.4), (4.5), and the stability of \(I_i\), for \(v \in V_j\) we have
\[
a(T_i v, T_i v) = \mu_i^2 \gamma^2 \left(\sum_{k=1}^{\tilde{n}_i} I_i P_i^k v, \sum_{k=1}^{\tilde{n}_i} I_i P_i^k v\right)
\]
\[
\leq \mu_i^2 \gamma^2 \sum_{k,m=1}^{\tilde{n}_i} |a(I_i P_i^k v, I_i P_i^m v)|
\]
\[
\leq \mu_i^2 \gamma^2 C_i \sum_{k=1}^{\tilde{n}_i} a(I_i P_i^k v, I_i P_i^k v)
\]
\[
\leq \mu_i^2 \gamma^2 C_0 C_i \sum_{k=1}^{\tilde{n}_i} a(I_i P_i^k v, I_i P_i^k v)
\]
\[
= \mu_i^2 \gamma^2 C_0 C_i \sum_{k=1}^{\tilde{n}_i} a(I_i P_i^k v, v) = \mu_i \gamma C_0 C_i a(T_i v, v).
\]

The proof is completed by setting \(\omega_i = \mu_i \gamma C_0 C_i\) and choosing
\[
(4.6) \quad 0 < \gamma < 1 \quad \text{and} \quad 0 < \mu_i < \frac{2}{\gamma C_0 C_i}
\]
such that \(\omega_i < 2\).

We remark that due to the fact that \(I_J\) is the identity we may choose \(\mu_J = 1\) and \(0 < \gamma < 1\) such that \(\omega_J = \gamma C_J < 2\). \(\square\)
4.1.2. Verification of assumption A2.

Lemma 4.2. Let \( \{T_i, i = 0, 1, \ldots, J\} \) be defined by (4.4). Then, there exists a constant \( K_0 \) such that

\[
a(v, v) \leq \frac{K_0}{\mu} a(Tv, v) \quad , \quad v \in V_J \quad , \quad \mu = \min_{0 \leq i \leq J} \{\mu_i\}.
\]

Proof. Due to the decomposition of \( K \) constant for the finest level, there holds

\[
\sum_{i=0}^{J} a(v_i, v_i) = \sum_{i=0}^{J} a(I_i v_i, v) = \sum_{i=0}^{J} a_i(v_i, P_i v).
\]

For \( i = 1, \ldots, J \), we have

\[
a_i(v_i, P_i v) = \sum_{k=1}^{\tilde{n}_i} a_i(v_i(m_i^k) \phi_i^k, P_i v) \leq \sum_{k=1}^{\tilde{n}_i} a_i(v_i(m_i^k) \phi_i^k, v_i(m_i^k) \phi_i^k v) \cdot \tilde{a}_i^{1/2}\(P_i P_i v, P_i P_i v\)
\]

\[
\leq (\sum_{k=1}^{\tilde{n}_i} a_i(v_i(m_i^k) \phi_i^k, v_i(m_i^k) \phi_i^k v))^{1/2} (\sum_{k=1}^{\tilde{n}_i} a(I_i P_i P_i v, v))^{1/2}.
\]

Following (4.7), we deduce

\[
a(v, v) = \sum_{i=0}^{J} a_i(v_i, P_i v)
\]

\[
\leq (a_0(v_0, v_0) + \sum_{i=1}^{J} \sum_{k=1}^{\tilde{n}_i} a_i(v_i(m_i^k) \phi_i^k, v_i(m_i^k) \phi_i^k))^{1/2}
\]

\[
\cdot (a(I_0 P_0 v, v) + \sum_{i=1}^{J} \sum_{k=1}^{\tilde{n}_i} a(I_i P_i P_i v, v))^{1/2}.
\]

Since \( a_i(\phi_i^k, \phi_i^k) \approx 1 \), we have

\[
a_i(v_i(m_i^k) \phi_i^k, v_i(m_i^k) \phi_i^k) \approx v_i^2(m_i^k).
\]

We note that the following inequality can be derived similarly as Lemma 3.3 in [29]

\[
\sum_{i=1}^{J} \sum_{k=1}^{\tilde{n}_i} v_i^2(m_i^k) \lesssim a(\bar{v}, \bar{v}) = a(\bar{\Pi}_{J-1} v, \bar{\Pi}_{J-1} v) \lesssim a(v, v).
\]

For the initial level, we have

\[
a_0(v_0, v_0) = a_0(\Pi_0 \bar{v}, \Pi_0 \bar{v}) \lesssim a(\bar{v}, \bar{v}) \lesssim a(v, v).
\]

For the finest level, there holds

\[
\sum_{k=1}^{\tilde{n}_J} v_J^2(m_J^k) \lesssim \sum_{k=1}^{\tilde{n}_J} (h_J^k)^{-2} \|v - \bar{\Pi}_{J-1} v\|_{L^2(\omega_J)}^2 \lesssim a(v, v),
\]
where $h_j^k = h_{j,E}, m_j^k \in E, E \in \mathcal{E}_j^k$. Hence, we have

\begin{equation}
(4.10) \quad a_0(v_0, v_0) + \sum_{i=1}^{J} \sum_{k=1}^{\tilde{n}_i} \varepsilon_i^2 (m_i^k) \lesssim a(v, v).
\end{equation}

Combining the above inequalities, we conclude that there exists a constant $\tilde{K}_0$ independent of mesh sizes and mesh levels such that

$$a(v, v) \leq \frac{\tilde{K}_0}{\mu_0} \left( \mu_0 a(I_0 P_0 v, v) + \sum_{i=1}^{J} \sum_{k=1}^{\tilde{n}_i} a(\mu_i I_i P_i^k P_i v, v) \right)$$

$$\leq \frac{\tilde{K}_0}{\mu_0} \sum_{i=0}^{J} a(T_i v, v) = \frac{\tilde{K}_0}{\mu_0} a(T v, v).$$

We thus obtain the stated result by setting $K_0 = \tilde{K}_0 / \gamma$. \hfill \Box

### 4.1.3. Verification of assumption A3

As a prerequisite to verify assumption A3, we provide the following key lemma which will be proved in the appendix.

**Lemma 4.3.** For $i = 1, \ldots, J$, let $T_i$ be a refinement of $T_{i-1}$ by the newest vertex bisection algorithm and denote by $\tilde{\Omega}_i^k$ the support of $I_j \phi_j^k$. Then, for $m_j^k \in \tilde{\mathcal{M}}_j$ we have

\begin{equation}
(4.11) \quad \sum_{i=1}^{J} \sum_{m_i^j \in \tilde{\mathcal{M}}_j, I_i \phi_i^j \neq 0 \text{ on } \mathcal{E}_{j+1}^k} \left( \frac{h_i^j}{h_j^k} \right)^{3/2} \lesssim 1,
\end{equation}

where $\mathcal{E}_{j+1}^k = \mathcal{E}_{j+1}(\tilde{\Omega}_j^k)$. Likewise, for $m_i^j \in \tilde{\mathcal{M}}_i$,

\begin{equation}
(4.12) \quad \sum_{j=1}^{J} \sum_{m_j^i \in \tilde{\mathcal{M}}_i, I_j \phi_j^i \neq 0 \text{ on } \mathcal{E}_{j+1}^k} \left( \frac{h_i^j}{h_j^k} \right)^{1/2} \lesssim 1.
\end{equation}

We are now in a position to verify assumption A3.

**Lemma 4.4.** There exists a constant $K_1$ independent of mesh sizes and mesh levels such that assumption A3 holds true.

**Proof.** In view of (4.4), we have

$$\sum_{i=1}^{J} \sum_{k=1}^{\tilde{n}_i} a(T_i v, T_j u) = \gamma^2 \sum_{j=1}^{J} \sum_{k=1}^{J} a(\mu_j I_j P^k P_j u, \sum_{i=1}^{J} \sum_{k=1}^{\tilde{n}_i} \mu_i I_i P_i^k P_i v)$$

$$= \gamma^2 \sum_{j=1}^{J} \sum_{k=1}^{\tilde{n}_j} a(\mu_j I_j P^k P_j u, \sum_{i=1}^{J} \sum_{k=1}^{\tilde{n}_i} \mu_i I_i P_i^k P_i v).$$

Setting $\omega = \sum_{i=j+1}^{J} \sum_{k=1}^{\tilde{n}_i} \mu_i I_i P_i^k P_i v$, we have

$$a(\mu_j I_j P^k P_j u, \omega) = a_j(\mu_j P^k P_j u, P_j \omega)$$

$$\leq a_j^{1/2} (\mu_j P^k P_j u, \mu_j P^k P_j u) a_j^{1/2} (P_j \omega, P_j \omega),$$
whence

\[
(4.13) \quad \sum_{i=1}^{J} \sum_{j=1}^{l_{i}} a(T_{i}v, T_{j}u) \leq \gamma^{2}(\sum_{j=1}^{J} \sum_{k=1}^{l_{j}} a_{j}(\mu_{j}P_{j}^{k}P_{j}u, \mu_{j}P_{j}^{k}P_{j}u))^{1/2} \cdot \left(\sum_{j=1}^{J} \sum_{k=1}^{l_{j}} a_{j}(P_{j}^{k}P_{j}\omega, P_{j}^{k}P_{j}\omega)\right)^{1/2}.
\]

In view of (4.6), it is obvious that

\[
(4.14) \quad \gamma \mu_{j} < \frac{2}{C_{0}C_{j}} \lesssim 1, \quad 1 \leq j \leq J,
\]

and there also holds \( \gamma \mu_{J} = \gamma \lesssim \frac{2}{C_{J}} \lesssim 1 \). If we choose \( \mu_{J} = 1 \), then

\[
(4.15) \quad \gamma \sum_{j=1}^{J} \sum_{k=1}^{l_{j}} a_{j}(\mu_{j}P_{j}^{k}P_{j}u, \mu_{j}P_{j}^{k}P_{j}u) = \sum_{j=1}^{J} \gamma \mu_{j} a(\mu_{j}I_{j}P_{j}^{k}P_{j}u, u) \lesssim \sum_{j=1}^{J} a(T_{j}u, v).
\]

Next, it suffices to show that

\[
(4.16) \quad \gamma \sum_{j=1}^{J} \sum_{k=1}^{l_{j}} a_{j}(P_{j}^{k}P_{j}\omega, P_{j}^{k}P_{j}\omega) \lesssim \sum_{i=2}^{J} a(T_{i}v, v).
\]

Clearly, \( a_{j}(\phi_{j}^{k}, \phi_{j}^{k}) \approx 1 \). We note that

\[
P_{j}^{k}P_{j}I_{i}P_{i}^{d}P_{i}v = \frac{a_{j}(P_{j}I_{i}P_{i}^{d}P_{i}v, \phi_{j}^{k})}{a_{j}(\phi_{j}^{k}, \phi_{j}^{k})} \phi_{j}^{k} \approx a_{j}(P_{j}I_{i}P_{i}^{d}P_{i}v, \phi_{j}^{k})\phi_{j}^{k},
\]

which leads us to

\[
a_{j}(P_{j}^{k}P_{j}\omega, P_{j}^{k}P_{j}\omega) \approx (\sum_{j=1}^{J} \sum_{k=1}^{l_{j}} a_{j}(P_{j}\mu_{j}I_{i}P_{i}^{d}P_{i}v, \phi_{j}^{k}))^{2}.
\]

Similarly, \( P_{i}^{d}P_{i}v \approx a_{i}(P_{i}v, \phi_{i}^{d})\phi_{i}^{d} \). It follows that

\[
a_{j}(P_{j}\mu_{j}I_{i}P_{i}^{d}P_{i}v, \phi_{j}^{k}) = a(\mu_{i}I_{i}P_{i}^{d}P_{i}v, I_{j}\phi_{j}^{k}) = a(\mu_{i}P_{i}^{d}P_{i}v, P_{i}I_{j}\phi_{j}^{k})
\approx a_{i}(\mu_{i}P_{i}v, \phi_{i}^{d})\phi_{i}^{d}P_{i}I_{j}\phi_{j}^{k}) = a(I_{i}\phi_{i}^{d}, I_{j}\phi_{j}^{k})a(\mu_{i}P_{i}v, \phi_{i}^{d}).
\]

Since \( I_{j}\phi_{j}^{k} \) is conforming and piecewise linear on \( T_{j+1} \subset \Omega_{j}^{k} \), we obtain

\[
a(I_{i}\phi_{i}^{d}, I_{j}\phi_{j}^{k}) = \sum_{T \subset \Omega_{j}^{k}, \ T \subset T_{j+1}} \int_{T} a(x) \nabla I_{i}\phi_{i}^{d} \cdot \nabla I_{j}\phi_{j}^{k}
\]
\[= \sum_{T \subset \Omega_{j}^{k}, \ T \subset T_{j+1}} \int_{\partial T} a(x) \frac{\partial I_{j}\phi_{j}^{k}}{\partial n} I_{i}\phi_{i}^{d} - \sum_{T \subset \Omega_{j}^{k}, \ T \subset T_{j+1}} \int_{T} (\nabla a(x) \cdot \nabla I_{j}\phi_{j}^{k})I_{i}\phi_{i}^{d}.
\]
We set \( d_j^k = \max\{h_{j+1,T}^k : T \in \tilde{\Omega}_j^k, T \in T_{j+1}^k\} \). By the minimum angle property in (1.6) we have \( d_j^k \approx h_j^k \). Similarly, \( d_j^1 \approx h_j^1 \). Observing (1.3), \( |\partial_{l_j} \phi_j^k| \lesssim (d_j^k)^{-1} \) and

\[
(4.17) \quad a_i(\mu_i P_j v, \phi_j^k) = a_i(\mu_i P_j^1 P_j v, \phi_j^1)
\]

\[
\lesssim a_i^{1/2}(\mu_i P_j^1 P_j v, \mu_i P_j^1 P_j v) a_i^{1/2}(\phi_j^1, \phi_j^1) \lesssim a^{1/2}(\mu_i^2 I_i P_j^1 P_j v, v),
\]

we deduce

\[
(4.18) \quad \sum_{m_i \in M_i, I_i, \phi_j^k \neq \emptyset \text{ on } \tilde{\Omega}_j^k} a(I_i \phi_j^k, I_j \phi_j^k) a_i(\mu_i P_j v, \phi_j^1)
\]

\[
\lesssim \sum_{m_i \in M_i, I_i, \phi_j^k \neq \emptyset \text{ on } E_j^k} \frac{h_j^k}{h_j^k} a^{1/2}(\mu_i^2 I_i P_j^1 P_j v, v)
\]

\[
+ \sum_{m_i \in M_i, I_i, \phi_j^k \neq \emptyset \text{ on } \tilde{\Omega}_j^k} \frac{(h_j^k)^2}{h_j^k} a^{1/2}(\mu_i^2 I_i P_j^1 P_j v, v).
\]

Hence, combining (4.11), (4.17) and (4.18), we have

\[
(4.19) \quad a_j(P_j^k P_j \omega, P_j^k P_j \omega) \lesssim \left( \sum_{i=j+1}^{J} \sum_{m_i \in M_i, I_i, \phi_j^k \neq \emptyset \text{ on } E_j^k} \frac{h_i^k}{h_j^k} a^{1/2}(\mu_i^2 I_i P_j^1 P_j v, v) \right)^2
\]

\[
+ \left( \sum_{i=j+1}^{J} \sum_{m_i \in M_i, I_i, \phi_j^k \neq \emptyset \text{ on } E_j^k} \frac{(h_i^k)^2}{h_j^k} a^{1/2}(\mu_i^2 I_i P_j^1 P_j v, v) \right)^2
\]

\[
\lesssim \left( \sum_{i=j+1}^{J} \sum_{m_i \in M_i, I_i, \phi_j^k \neq \emptyset \text{ on } E_j^k} \frac{h_i^k}{h_j^k} a^{1/2}(\mu_i^2 I_i P_j^1 P_j v, v) \right)^2
\]

\[
+ \left( \sum_{i=j+1}^{J} \sum_{m_i \in M_i, I_i, \phi_j^k \neq \emptyset \text{ on } E_j^k} \frac{h_i^k}{h_j^k} a(\mu_i^2 I_i P_j^1 P_j v, v) \right)^2
\]

\[
\lesssim \left( \sum_{i=j+1}^{J} \sum_{m_i \in M_i, I_i, \phi_j^k \neq \emptyset \text{ on } \tilde{\Omega}_j^k} \frac{h_i^k}{h_j^k} a^{1/2}(\mu_i^2 I_i P_j^1 P_j v, v) \right)^2
\]
We set $\delta(m^1_i, m^k_j) = 1$, if $I_i \phi^j_k \neq 0$ on $\mathcal{E}_{j+1}^k$, and $\delta(m^1_i, m^k_j) = 0$, otherwise, $\tilde{\delta}(m^1_i, m^k_j) = 1$, if $I_i \phi^j_k \neq 0$ on $\tilde{\mathcal{E}}^k_j$, and $\tilde{\delta}(m^1_i, m^k_j) = 0$, otherwise. By (4.12) and (4.14), we obtain

\begin{equation}
\frac{J}{16} \leq \gamma \sum_{i=1}^{J} \sum_{j=1}^{m_i} a_{ij}(P_i^k P_j \omega, P_i^k P_j \omega) \leq \gamma \left( \sum_{i=1}^{J} \sum_{j=1}^{m_i} \sum_{i=j+1}^{J} \sum_{m_i, m_j} \left( \frac{h_i^k}{h_j^k} \right)^{1/2} a(\mu_i^2 I_i P_i^k P_j v, v) \right)
\end{equation}

A similar analysis can be used to derive

\begin{equation}
\frac{J}{16} \leq \gamma \sum_{i=1}^{J} \sum_{j=1}^{m_i} \left( \sum_{k=1}^{m_i} \left( \sum_{i=j+1}^{J} \sum_{m_i, m_j} \left( \frac{h_i^k}{h_j^k} \right)^{1/2} \tilde{\delta}(m_i^1, m_j^k) a(\mu_i^2 I_i P_i^k P_j v, v) \right) \right)
\end{equation}

Hence, (4.16) is verified. Combining (4.13-4.16), we obtain

\begin{equation}
\frac{J}{16} \leq \gamma \sum_{i=1}^{J} \sum_{j=1}^{m_i} \left( \sum_{k=1}^{m_i} \left( \sum_{i=j+1}^{J} \sum_{m_i, m_j} \left( \frac{h_i^k}{h_j^k} \right)^{1/2} \tilde{\delta}(m_i^1, m_j^k) a(\mu_i^2 I_i P_i^k P_j v, v) \right) \right) \leq \sum_{i=1}^{J} \sum_{j=1}^{m_i} a(T_i v, T_j u) \leq \left( \sum_{i=1}^{J} a(T_i v, v) \right)^{1/2} \left( \sum_{j=1}^{J} a(T_j u, u) \right)^{1/2}.
\end{equation}

A similar analysis can be used to derive

\begin{equation}
\frac{J}{16} \leq \gamma \sum_{i=1}^{J} \sum_{j=1}^{m_i} \left( \sum_{k=1}^{m_i} \left( \sum_{i=j+1}^{J} \sum_{m_i, m_j} \left( \frac{h_i^k}{h_j^k} \right)^{1/2} \tilde{\delta}(m_i^1, m_j^k) a(\mu_i^2 I_i P_i^k P_j v, v) \right) \right) \leq \sum_{i=1}^{J} \sum_{j=1}^{m_i} a(T_i v, T_j u) \leq \left( \sum_{i=1}^{J} a(T_i v, v) \right)^{1/2} a(T_0 u, u)^{1/2},
\end{equation}

which, together with (4.21), completes the proof of the lemma. \hfill \Box

4.2. Local Gauss-Seidel smoother. In this subsection, we will verify assumptions A1-A3 for the multilevel methods with a local Gauss-Seidel smoother $R_i$ which is defined by

\begin{equation}
R_i := (I - E_i) A_i^{-1},
\end{equation}

where $E_i := (I - P_i^0) \cdots (I - P_i^1) = \prod_{k=i}^{n_i} (I - P_i^k)$. For brevity, we set $E_i := E_i^0$, since no confusion is possible. We have

\begin{equation}
T_0 = \mu_0 I_0 P_0 , \quad T_i = \mu_i I_i R_i A_i P_i = \mu_i I_i (I - E_i) P_i , \quad i = 1, \ldots, J.
\end{equation}

The decomposition of $v$ is the same as (4.2).
For $i = 1, \ldots, J$, let $E_i = I, E_{i-1}^k := (I - P_i^k) \cdots (I - P_i^1), k = 2, \ldots, \bar{n}_i$. It is easy to see that

\begin{equation}
I - E_i = \sum_{k=1}^{\bar{n}_i} P_i^k E_i^{k-1}.
\end{equation}

As in Lemma 4.5 in [33], there also holds

\begin{equation}
a_i(P_i v, P_i u) - a_i(E_i P_i v, E_i P_i u)
= \sum_{k=1}^{\bar{n}_i} a_i(P_i^k E_i^{k-1} P_i v, E_i^{k-1} P_i u), \quad v, u \in V_J.
\end{equation}

4.2.1. Verification of assumption A1. We consider the case $i \geq 1$, since for $T_0$ assumption A1 has been verified in Lemma 4.1.

Lemma 4.5. Let $T_i, i \geq 1$, be defined by (4.23). Then, $T_i$ is nonnegative in $V_J$ and there holds

$$a(T_i v, T_i v) \leq \omega_i a(T_i v, v), \quad v \in V_J, \quad \omega_i < 2.$$  

Proof. Due to (4.23) and (4.24) we have

$$a(T_i v, T_i v) = \mu_i^2 a(I_i (I - E_i) P_i v, I_i (I - E_i) P_i v)$$

$$= \mu_i^2 \sum_{k,m=1}^{\bar{n}_i} a(I_i (P_i^k E_i^{k-1} P_i v, I_i (P_i^m E_i^{m-1} P_i v)).$$

Using (4.25), the same techniques as in (4.5), and the stability of $I_i$, we obtain

\begin{equation}
a(T_i v, T_i v) \leq \mu_i^2 C_i \sum_{k=1}^{\bar{n}_i} a(I_i (P_i^k E_i^{k-1} P_i v, I_i (P_i^k E_i^{k-1} P_i v)
\leq \mu_i^2 C_0 C_i \sum_{k=1}^{\bar{n}_i} a_i(P_i^k E_i^{k-1} P_i v, P_i^k E_i^{k-1} P_i v)
= \mu_i^2 C_0 C_i ((a_i(P_i v, P_i v) - a_i(E_i P_i v, E_i P_i v))
= \mu_i^2 C_0 C_i (2a_i((I - (I - E_i)) P_i v, (I - (I - E_i)) P_i v))
= \mu_i^2 C_0 C_i (2a_i((I - (I - E_i)) P_i v, (I - (I - E_i)) P_i v))
\leq \mu_i^2 C_0 C_i (2a_i((I - E_i) P_i v, P_i v) - \frac{1}{C_0} a(I_i (I - E_i) P_i v, I_i (I - E_i) P_i v))
= 2\mu_i C_0 C_i a(T_i v, P_i v) - C_i a(T_i v, T_i v),$$

whence

$$a(T_i v, T_i v) \leq \frac{2\mu_i C_0 C_i}{1 + C_i} a(T_i v, v).$$

Obviously, the nonnegativeness of $T_i$ follows from the above inequality. Setting

$$\omega_i = \frac{2\mu_i C_0 C_i}{1 + C_i},$$

and choosing $0 < \mu_i < \frac{1 + C_i}{2C_0 C_i}$ such that $\omega_i < 2$, the lemma is proved.

We remark that we can choose $\mu_j = 1$, since $I_J$ is the identity. \hfill \square
4.2.2. Verification of assumption A2.

\textbf{Lemma 4.6.} Let \( \{T_i, i = 0, 1, ..., J\} \) be defined as in (4.23). There exists a constant \( K_0 \) such that
\[
a(v, v) \leq \frac{K_0}{\mu} a(Tv, v), \quad v \in V_J, \quad \mu = \min_{0 \leq i \leq J} \{\mu_i\}.
\]

\textbf{Proof.} In view of the decomposition of \( v \) in (4.2), we have \( a(v, v) = \sum_{i=0}^J a_i(v_i, P_i v) \).

For \( i = 1, ..., J \), we also have (cf. (4.8))
\[
a_i(v_i, P_i v) \leq \left( \sum_{k=1}^{\tilde{n}_i} a_i(v_i(m_i^k)\phi_i^k, v_i(m_i^k)\phi_i^k) \right)^{1/2} \left( \sum_{k=1}^{\tilde{n}_i} a_i(P_i^k P_i v, P_i^k P_i v) \right)^{1/2}.
\]

Since \( I - E_i^{k-1} = \sum_{m=1}^{k-1} P_m E_i^{m-1} \), we deduce
\[
\sum_{k=1}^{\tilde{n}_i} a_i(P_i^k P_i v, P_i^k P_i v) = \sum_{k=1}^{\tilde{n}_i} a_i(P_i^k P_i v, P_i^k E_i^{k-1} P_i v) + \sum_{k=1}^{\tilde{n}_i} a_i(P_i^k P_i v, P_i^k P_m E_i^{m-1} P_i v) \leq \left( \sum_{k=1}^{\tilde{n}_i} a_i(P_i^k P_i v, P_i^k P_i v) \right)^{1/2} \left( \sum_{k=1}^{\tilde{n}_i} a_i(P_i^k E_i^{k-1} P_i v, E_i^{k-1} P_i v) \right)^{1/2} + \sum_{k,m=1}^{\tilde{n}_i} |a_i(P_i^k P_i v, P_m^m E_i^{m-1} P_i v)|.
\]

Furthermore, using the same technique as in (4.5), we have
\[
\sum_{k,m=1}^{\tilde{n}_i} |a_i(P_i^k P_i v, P_m^m E_i^{m-1} P_i v)| \leq \left( \sum_{k=1}^{\tilde{n}_i} a_i(P_i^k P_i v, P_i^k P_i v) \right)^{1/2} \left( \sum_{k=1}^{\tilde{n}_i} a_i(P_i^k E_i^{k-1} P_i v, E_i^{k-1} P_i v) \right)^{1/2},
\]

Then, it follows from (4.26) that
\[
\sum_{k=1}^{\tilde{n}_i} a_i(P_i^k P_i v, P_i^k P_i v) \leq \sum_{k=1}^{\tilde{n}_i} a_i(P_i^k E_i^{k-1} P_i v, E_i^{k-1} P_i v) \lesssim \frac{1}{\mu_i} a(T_i v, v).
\]

Hence,
\[
a_0(P_0 v, P_0 v) + \sum_{k=1}^{\tilde{n}_i} a_i(P_i^k P_i v, P_i^k P_i v) \lesssim \sum_{i=0}^{J} \frac{1}{\mu_i} a(T_i v, v).
\]

Finally, similar to the analysis of (4.9) and (4.10), we deduce that assumption A2 holds true.

\[\square\]

4.2.3. Verification of assumption A3.

\textbf{Lemma 4.7.} There exists a constant \( K_i \) independent of mesh sizes and mesh levels such that assumption A3 holds true for \( \{T_i, i = 0, 1, ..., J\} \) defined by (4.23).
Proof. We set $\xi_i = T_i v$. It follows from (4.23) that

$$
\sum_{i=1}^{J} \sum_{j=1}^{J-i-1} a(T_i v, T_j u) = \sum_{j=1}^{J} \sum_{i=j+1}^{J} a(\xi_i, \mu_j I_j (I - E_j) P_j u)
$$

$$
= \sum_{j=1}^{J} \sum_{i=j+1}^{J} \mu_j a_j(P_j \xi_i, (I - E_j) P_j u) = \sum_{j=1}^{J} \sum_{i=j+1}^{J} \mu_j \sum_{k=1}^{\tilde{n}_j} a_j(P_j \xi_i, P_j^k E_j^{k-1} P_j u)
$$

$$
= \sum_{j=1}^{J} \sum_{i=j+1}^{J} \mu_j a_j(P_j^k P_j \sum_{j=1}^{J} \sum_{i=j+1}^{J} \xi_i, P_j^k E_j^{k-1} P_j u).
$$

Further, Hölder’s inequality yields

(4.27) \[
\sum_{i=1}^{J} \sum_{j=1}^{J-i-1} a(T_i v, T_j u) \leq \left( \sum_{j=1}^{J} \sum_{k=1}^{\tilde{n}_j} a_j^2(P_j^k E_j^{k-1} P_j u, E_j^{k-1} P_j u) \right)^{1/2} \cdot \left( \sum_{j=1}^{J} \sum_{i=1}^{J} \sum_{k=1}^{\tilde{n}_j} a_j(P_j^k P_j \xi_i, P_j^k P_j \xi_i) \right)^{1/2}.
\]

In view of the estimate of (4.26) in Lemma 4.5 and $\mu_j < \frac{1 + C_0}{2 C_0 e^j} \lesssim 1$, for $j = 1, \ldots, J$, we find

(4.28) \[
\sum_{j=1}^{J} \sum_{i=1}^{J-i-1} a_j^2(P_j^k E_j^{k-1} P_j u, E_j^{k-1} P_j u) \leq \frac{\mu_j a(T_j u, u)}{C_0} \quad \text{for } j = 1, \ldots, J
\]

$$
\leq 2 \sum_{j=1}^{J} \mu_j a(T_j u, u) \lesssim a(T_j u, u),
$$

whence

$$
\sum_{j=1}^{J} \sum_{k=1}^{\tilde{n}_j} a_j^2(P_j^k E_j^{k-1} P_j u, E_j^{k-1} P_j u) \lesssim \sum_{j=1}^{J} a(T_j u, u).
$$

Next, we show that

(4.29) \[
\sum_{j=1}^{J} \sum_{k=1}^{\tilde{n}_j} a_j(\sum_{i=j+1}^{J} P_j^k P_j \xi_i, \sum_{i=j+1}^{J} P_j^k P_j \xi_i) \lesssim \sum_{i=2}^{J} a(T_i v, v).
\]

We note that $P_j^k P_j \xi_i = \frac{a_j(P_j \xi_i, \phi_j^k)}{a_j(\phi_j^k, \phi_j^k)} \phi_j^k \approx a_j(P_j \xi_i, \phi_j^k) \phi_j^k$, and similarly $P_i^l E_i^{l-1} P_i v \approx a_v(E_i^{l-1} P_i v, \phi_i^l) \phi_i^l$. Then, there holds

$$
a_j(\sum_{i=j+1}^{J} P_j^k P_j \xi_i, \sum_{i=j+1}^{J} P_j^k P_j \xi_i) \lesssim (\sum_{i=j+1}^{J} a_j(P_j \xi_i, \phi_j^k))^2.
$$
Moreover,

\[ a_j(P_j \xi_i, \phi^k_j) = a_j(P_j \mu_i (I - E_i) P_i v, \phi^k_j) = \mu_i a((I - E_i) P_i v, I_j \phi^k_j) \]

\[ = \mu_i \sum_{l=1}^{\tilde{n}_i} a_l(P_l^i E^{l-1}_i P_i v, P_l I_j \phi^k_j) \approx \mu_i \sum_{l=1}^{\tilde{n}_i} a_l(\phi^k_i, P_l I_j \phi^k_j) a_l(E^{l-1}_i P_i v, \phi^k_l) \]

\[ = \mu_i \sum_{l=1}^{\tilde{n}_i} a(I_j \phi^k_i, I_j \phi^k_j) a_l(E^{l-1}_i P_i v, \phi^k_l). \]

Similar to the analysis of (4.20) in Lemma 4.4 in the Jacobi case, and due to Lemma 4.3, we have

\[ \sum_{J=1}^{J} \sum_{k=1}^{J} \sum_{i=j+1}^{J} a_j(\xi, \phi^k_j)^2 \]

\[ \leq \sum_{J=1}^{J} \sum_{k=1}^{J} \sum_{i=j+1}^{J} \sum_{m_j^l \in M_j, I_j \phi^k_l \neq 0 \text{ on } \hat{\Omega}_j^k} \left( \frac{h_j^k}{h_j^k} \right)^{1/2} \mu^2_i a_i(P_i^l E^{l-1}_i P_i v, E^{l-1}_i P_i v) \]

\[ + \sum_{J=1}^{J} \sum_{k=1}^{J} \sum_{i=j+1}^{J} \sum_{m_j^l \in M_j, I_j \phi^k_l \neq 0 \text{ on } \hat{\Omega}_j^k} \frac{h_j^k}{h_j^k} \mu^2_i a_i(P_i^l E^{l-1}_i P_i v, E^{l-1}_i P_i v) \]

\[ \leq \sum_{i=2}^{J} \sum_{m_j^l \in M_j} \mu^2_i a_i(P_i^l E^{l-1}_i P_i v, E^{l-1}_i P_i v) \sum_{j=1}^{J} \sum_{m_j^l \in M_j} \left( \frac{h_j^k}{h_j^k} \right)^{1/2} \tilde{\theta}(m_j^l, m_j^k) \]

\[ + \sum_{i=2}^{J} \sum_{m_j^l \in M_j} \mu^2_i a_i(P_i^l E^{l-1}_i P_i v, E^{l-1}_i P_i v) \sum_{j=1}^{J} \sum_{m_j^l \in M_j} \frac{h_j^k}{h_j^k} \tilde{\theta}(m_j^l, m_j^k) \]

\[ \leq \sum_{i=2}^{J} \sum_{m_j^l \in M_j} \mu^2_i a_i(P_i^l E^{l-1}_i P_i v, E^{l-1}_i P_i v)(1 + \sqrt{h_j}) \leq \sum_{i=2}^{J} a(T_i v, v). \]

Hence, (4.29) is verified. In view of (4.26), (4.27) and (4.29), it follows that

\[ (4.30) \quad \sum_{i=1}^{J} a(T_i v, T_j u) \leq (\sum_{i=2}^{J} a(T_i v, v))^{1/2} (\sum_{j=1}^{J} a(T_j u, u))^{1/2}. \]

We further deduce

\[ \sum_{i=1}^{J} a(T_i v, T_0 u) \leq (\sum_{i=1}^{J} a(T_i v, v))^{1/2} a(T_0 u, u)^{1/2}, \]

which, together with (4.30), implies Lemma 4.7.

**Numerical results**

In this section, for selected test examples we present numerical results that illustrate the optimality of algorithm 4.1 and algorithm 4.2. The implementation is based on the FFW toolbox [8]. The local error estimators and the strategy **MARK** for the selection of elements and edges for refinement have been realized.
as in the algorithm ANFEM II in [12]. In the following examples, both \textbf{LMPA} and \textbf{LMAA} are considered as preconditioners for the conjugate gradient method, i.e., a symmetric version of \textbf{LMPA (SLMPA)} has been used in the computations. Likewise, a symmetric version of \textbf{LMAA (SLMAA)} is employed when the smoother is nonsymmetric, otherwise, \textbf{LMAA} is directly applied. The algorithms \textbf{LMPA} and \textbf{LMAA} require \(O(N \log N)\) and \(O(N)\) operations respectively, where \(N\) is the number of degrees of freedom (DOFs) (cf. [26]).

The estimate (A.1) in the appendix indicates that the prolongation operator \(I_i\) from \(V_i\) to \(V_J\) would increase the energy by a constant \(C_0\) at worst, which is essential in the convergence analysis of the local multilevel methods. We can weaken the influence by a well chosen scaling number \(\mu_{J,i}\) in (3.2). As seen from Theorem 4.1 and Theorem 4.2, the uniform convergence rate of \textbf{LMPA} or the preconditioned conjugate gradient method by \textbf{LMAA} will deteriorate for decreasing scaling number \(\mu = \min_{0 \leq i < J} \{\mu_{J,i}\}\). This property will be observed in the following Example 6.1. We always choose \(\mu_{J,J} = 1\) in the computations.

For the preconditioned conjugate gradient method, the iteration stops when it satisfies
\[
\|r^0_i - A r^n_i\|_{0, \Omega} \leq \epsilon \|r^0_i\|_{0, \Omega}, \quad \epsilon = 10^{-6},
\]
where \(\{r^k_i : k = 1, 2, \ldots\}\) stands for the set of iterative solutions of the residual equation \(A_i x = r^0_i\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Grid_on_Level_13.png}
\caption{Locally refined mesh at the 13-th refinement level (Example 6.1).}
\end{figure}

At the \(i\)-th level, let \(u^0_i = u_{i-1}, r^n_i = f_i - A_i u^n_i\), and set
\[
\epsilon_0 = (r^0_i)^T B_i r^0_i, \quad \epsilon_n = (r^n_i)^T B_i r^n_i,
\]
where \(B_i\) is the local multilevel iteration. The number of iteration steps required to achieve the desired accuracy is denoted by \textbf{iter}. We further denote by \(\rho = (\sqrt{\epsilon_n/\epsilon_0})^{1/\text{iter}}\) the average reduction factor.
Example 4.1. On the L-shaped domain \( \Omega = [-1,1] \times [-1,1] \setminus (0,1] \times [-1,0) \), we consider the following elliptic boundary value problem
\[
-\Delta (0.5u) + u = f(x, y) \quad \text{in} \; \Omega,
\]
\[
u = g(x, y) \quad \text{on} \; \partial \Omega,
\]
where \( f \) and \( g \) are chosen such that \( u(r, \theta) = r^{3 \theta} \sin(\frac{\theta}{2}) \) is the exact solution (in polar coordinates).

Table 1. Number of iterations and average reduction factor \( \rho \) on each level for the respective algorithms with scaling number \( \mu_{J,0} = 0.8 \) and \( \mu_{J,J} = 1 \), \( 0 \leq i \leq J - 1 \), \( J \geq 1 \). For the conjugate gradient method without preconditioning, only the number of iterations is given (Example 6.1).

<table>
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<th>Level</th>
<th>DOFs</th>
<th>CG iter</th>
<th>SLMPA-GS iter</th>
<th>SLMPA-Jacobi iter</th>
<th>SLMAA-GS iter</th>
<th>LMAA-Jacobi iter</th>
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Table 2. Average reduction factors \( \rho \) (SLPMA-GS) for different scaling numbers (Example 6.1).

<table>
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<th>( \alpha = 1 )</th>
<th>( \alpha = 0.5 )</th>
<th>( \alpha = 0.2 )</th>
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<td>0.2310</td>
<td>0.2423</td>
<td>0.3949</td>
<td>0.5058</td>
</tr>
</tbody>
</table>

For ease of notation, we refer to \( \text{SLMPA-GS}, \text{SLMAA-GS} \) and \( \text{SLMPA-Jacobi}, \text{LMAA-Jacobi} \) as the preconditioned conjugate gradient method by \( \text{SLMPA} \) and \( \text{SLMAA} \) with local Gauss-Seidel smoothing and local Jacobi smoothing, respectively. For the Jacobi iteration, the scaling factor is chosen according to \( \gamma = 0.8 \).
At first, we choose $\mu_{J,i} = 0.8$ ($0 \leq i < J$) to illustrate the optimality of our algorithms. Figure 2 displays the locally refined mesh at the 13-th refinement level. As seen from Table 1, the number of iterative steps of the conjugate gradient method without preconditioning (CG) increases quickly with the mesh levels. However, for the algorithms $\text{SLMPA-GS}$, $\text{SLMPA-Jacobi}$, $\text{SLMAA-GS}$ and $\text{LMAA-Jacobi}$ we observe that the number of iteration steps and the average reduction factors are all bounded independently of the mesh sizes and the mesh levels. These results and Figure 3, displaying the CPU times (in seconds) for the respective algorithms, demonstrate the optimality of the algorithms and thus confirm the theoretical analysis.

Next, we choose different scaling numbers to illustrate how they influence the convergence behavior of the local multilevel methods. We only list the results for $\text{SLMPA-GS}$. A similar behavior can be observed for the other algorithms. We choose $\mu_{J,0} = \cdots = \mu_{J,J-1} = \alpha$ and $\mu_{J,J} = 1$, and thus $\mu = \min\{\alpha, 1\}$. Table 2 shows that for a fixed $\alpha$, $\text{SLMPA-GS}$ converges almost uniformly. The last four numbers of each row in Table 2 show that for a fixed level the average reduction factor of $\text{SLMPA-GS}$ deteriorates for decreasing $\mu$. If $\alpha \geq 1$, then $\mu = \min\{\alpha, 1\} = 1$, and the convergence rate will also deteriorate as $\alpha$ increases. This is also observed for the first numbers of each row in Table 2. In particular, for $\mu = 1$ the convergence rate of $\text{SLMPA-GS}$ deteriorates only with respect to $\omega$ (the spectral bound of $T_i$), which increases linearly with $\mu_{J,i}$.

Figure 3. CPU times for $\text{SLMPA-GS}$, $\text{SLMPA-Jacobi}$, $\text{SLMAA-GS}$, and $\text{LMAA-Jacobi}$ (from left to right and top to bottom)
Example 4.2. We consider Poisson’s equation

\[-\Delta u = 1 \quad \text{in } \Omega,\]

with Dirichlet boundary conditions on a domain with a crack, namely \(\Omega = \{(x, y) : |x| + |y| \leq 1\} \setminus \{(x, y) : 0 \leq x \leq 1, y = 0\}\). The exact solution is \(r^{1/2} \sin(\theta/2) - \frac{1}{4}r^2\) (in polar coordinates).

In this example, we choose \(\mu_{J,J} = 1, \mu_{J,i} = 1\) and \(\mu_{J,i} = 0.8\) \((0 \leq i < J, J \geq 1)\), respectively, for the local multilevel methods with local Gauss-Seidel smoothing and local Jacobi smoothing.

![Figure 4. Locally refined mesh at the 24-th refinement level (Example 6.2).](image)

Table 3. Number of iterations and average reduction factors \(\rho\) on each level for the respective algorithms with scaling numbers \(\mu_{J,J} = 1, \mu_{J,i} = 1, \mu_{J,i} = 0.8, 0 \leq i < J - 1, J \geq 1\). For the conjugate gradient method without preconditioning, only the number of iterations is given (Example 6.2).

<table>
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<tr>
<th>Level</th>
<th>DOFs</th>
<th>CG iter</th>
<th>SLMPA-GS (\rho)</th>
<th>SLMAA-GS (\rho)</th>
<th>SLMPA-Jacobi (\rho)</th>
<th>LMAA-Jacobi (\rho)</th>
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<td>18206</td>
<td>11 0.2610</td>
<td>51 0.7720</td>
<td>14 0.3756</td>
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<td>50 0.7662</td>
<td>13 0.3507</td>
<td>65 0.8151</td>
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<td>55 0.7854</td>
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<td>54 0.7852</td>
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<td>70 0.8275</td>
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</tr>
</tbody>
</table>

Figure 4 displays the locally refined mesh at the 24-th refinement level. The numbers in Table 3 and the CPU times (in seconds) displayed in Figure 5 show a similar behavior as in the previous example and thus also support the theoretical findings.
Appendix

In this appendix, we analyze the stability of $I_i$ and provide the proof of Lemma 4.3.

Proof of the stability result of the prolongation operator $I_i$. Let $T_{i+1}$ be the refined triangulation obtained from $T_i$ by the algorithm stated in section 2 or by the newest vertex bisection. Then, there exist constants $C_0$ and $\tilde{C}_0$ such that

\[(I_i v, I_i v) \leq \tilde{C}_0 (v, v), \quad a(I_i v, I_i v) \leq C_0 a_i (v, v), \quad v \in V_i.\]  

(A.1)

Since the analysis for the first bisection algorithm has been done in [26], we only give the proof for the refinement by the newest vertex bisection.

The first inequality in (A.1) is trivial for $I_i v$ being defined by local averaging. It suffices to derive the second one.

The origin of vertices of $T \in T_{i+1}$ includes four cases depending whether the vertex of $T$ is the midpoint of an edge or a node in $T_i$. In particular, let $m, n$ denote the number of vertices of $T \in T_{i+1}$ representing midpoints or nodes in $T_i$, respectively. Setting $S = \{(m, n) : m + n = 3, m, n = 0, 1, 2, 3\}$, we have $\# S = 4$.

We only consider one of the possible cases: the vertices of $T \in T_{i+1}$ are all nodes in $T_i$, i.e., $T$ is not refined in the transition from $T_i$ to $T_{i+1}$, e.g., $T_2 \in T_{i+1}$ is also $K_2 \in T_i$ in Figure 6. A similar analysis can be carried out in all other cases.

Figure 5. CPU times for SLMPA-GS, SLMAA-GS, SLMPA-Jacobi, and LMAA-Jacobi (from left to right and top to bottom)
Figure 6. The left figure illustrates a local grid from $T_i$, the right one displays its refinement as part of $T_{i+1}$.

Note that $a(I_i v, I_i v)|_{T_2}$ can be bounded by
\begin{equation}
C((I_i v(x_1) - I_i v(x_2))^2 + (I_i v(x_1) - I_i v(x_3))^2)
\end{equation}
for some constant $C$. We recall that $I_i v(x_i)$ is the average of $v$ at $x_i$ over the triangles $K_l$, $l = 1, ..., M_{x_i}$, where $M_{x_i}$ is the number of triangles containing $x_i$. Hence, the first term of (A.2) can be written as
\begin{equation}
\frac{1}{M_{x_1}} \sum_{l=1}^{M_{x_1}} (v|_{K_l}(x_1) - v(m_1)) + \frac{1}{M_{x_2}} \sum_{s=1}^{M_{x_2}} (v(m_1) - v|_{K_s}(x_2)).
\end{equation}
A similar result can be obtained for the second term of (A.2). Since $v|_{K_l}(x_1) - v(m_1) = v|_{K_l}(x_1) - v(m_l) + \sum_{j=1}^{l-1} (v(m_{j+1}) - v(m_j))$, it suffices to find a constant $C$ such that the first term of (A.3) can be bounded by
\begin{equation}
\sum_{l=1}^{M_{x_1}} (v|_{K_l}(x_1) - v(m_1) \leq C a_i(v, v)|_{\tilde{T}}.
\end{equation}
where $\tilde{K} = \bigcup_{l=1}^{M_{x_1}} K_l$. The same analysis can be carried out for the second term of (A.3). Following (A.2-A.4), we get
\begin{equation}
a(I_i v, I_i v)|_{T_4} \leq C a_i(v, v)|_{\tilde{T}_4}
\end{equation}
with some constant $C$, where $\tilde{T}_4$ is a patch of triangles in $T_i$ also containing the vertices of $T_2$.

For $T \in T_{i+1}$, $\partial T \cap \partial \Omega \neq \emptyset$, let us assume $\partial T_4 \cap \partial \Omega \neq \emptyset$. Then, $a(I_i v, I_i v)|_{T_4}$ can be bounded by
\begin{equation}
C((I_i v(m_3) - I_i v(x_4))^2 + (I_i v(m_3) - I_i v(x_5))^2) = 2C(v(m_3) - v(m_7))^2.
\end{equation}
Combining (A.5) and (A.6) and summing up all $T \in T_{i+1}$ completes the proof. □

Proof of Lemma 4.3. The proof is similar to Lemma 3.2 in [29]. We only prove the first estimate in (4.11) and (4.12). The second estimate in (4.11) and (4.12) can be obtained similarly.
For the proof of the first estimate in (4.11), we set
\[ \rho(m_i^l) = \left\lfloor \frac{\ln(h_i^l/h_0)}{\ln(1/2)} \right\rfloor, \quad h_0 = \max_{E \in \mathcal{E}_0} h_E, \]
which characterizes the actual number of refinements of edges in \( \mathcal{E}_i \). It is obvious that
\[ \frac{1}{2} \rho(m_i^l) + 1 h_0 < \rho(m_i^l) \leq \frac{1}{2} \rho(m_i^l) h_0. \]
Denoting by \( d(\tilde{\Omega}_k^i) \) the diameter of \( \tilde{\Omega}_k^i \), there exists a constant \( \beta > 1 \) depending only on the minimum angle \( \theta \) in (1.6) such that \( d(\tilde{\Omega}_k^i) \leq h_k^i \), whence
\[ h_k^i \leq d(\tilde{\Omega}_k^i) \leq d(\Omega_k^i) \leq \beta h_k^i. \]
Due to the definition of (A.7), we have \( \rho(m_i^l) \geq \rho(m_i^l) - n_0 \), where \( n_0 = [\ln(\beta/\ln 2)] + 1 \). Thus, for the left term in (4.11) we get
\[ \sum_{i=j+1}^{j} \sum_{m_i^l \in \mathcal{M}_l, L_i \phi_i^l \neq 0 \text{ on } \mathcal{E}_{j+1}^k} (h_i^l)^{3/2} \leq (h_0)^{3/2} \sum_{i=j+1}^{j} \sum_{m_i^l \in \mathcal{M}_l, L_i \phi_i^l \neq 0 \text{ on } \mathcal{E}_{j+1}^k} \left( \frac{1}{\sqrt{2}} \right)^{3 \rho(m_i^l)} \]
\[ \leq (h_0)^{3/2} \sum_{m=\rho(m_i^l) - n_0}^{\infty} \left( \frac{1}{\sqrt{2}} \right)^{3 m}, \]
where \( \sigma_1(m, m_i^l) = \{ (i, l) \mid \rho(m_i^l) = m, j + 1 \leq i \leq J, L_i \phi_i^l \neq 0 \text{ on } \mathcal{E}_{j+1}^k \} \). We define a new set according to
\[ \sigma_0(m, y) = \{ i : y \in \mathcal{M}_i, \rho(y) = m, 0 \leq i \leq J \}. \]
In view of the minimum angle condition in the newest vertex bisection, we easily see that
\[ \# \sigma_0(m, y) \leq 1. \]
Moreover,
\[ \# \sigma_1(m, m_i^l) \leq \max_{(i, j) \in \sigma_1(m, m_i^l)} \# \sigma_0(m, m_i^l) \cdot \frac{h_j^k}{(\frac{1}{2})^{m+1} h_0} \leq \frac{h_j^k}{(\frac{1}{2})^{m+1} h_0}. \]
Therefore, (A.8) and (A.9) imply that
\[ \sum_{i=j+1}^{j} \sum_{m_i^l \in \mathcal{M}_l, L_i \phi_i^l \neq 0 \text{ on } \mathcal{E}_{j+1}^k} (h_i^l)^{3/2} \leq (h_0)^{3/2} \sum_{m=\rho(m_i^l) - n_0}^{\infty} \left( \frac{1}{\sqrt{2}} \right)^{3 m} \left( \frac{1}{(\frac{1}{2})^{m+1} h_0} \right)^{3/2} \]
This proves the first estimate in (4.11).
For the second estimate in (4.12), we need to show
\[ \# \sigma_2(m, m_i^l) \leq 1, \]
where \( \sigma_2(m, m_i^l) = \{ (k, j) \mid m_j^k \in \mathcal{M}_j, \rho(m_j^k) = m, L_i \phi_i^l \neq 0 \text{ on } \mathcal{E}_{j+1}^k, 0 \leq j \leq i - 1 \} \).
Let
\[ \mathcal{N}(m, m_i^l) = \{ y : y \in \mathcal{M}_j, \rho(y) = m, |y - m_i^l| \leq d(\bar{\Omega}_y), m_i^l \in \bar{\Omega}_y, L_i \phi_i^l \neq 0 \text{ on } \mathcal{E}_{j+1}(\bar{\Omega}_y), 0 \leq j \leq i - 1 \}, \]
where for \( y \in \mathcal{M}_j \), \( \bar{\Omega}_y \) is the patch of triangles in \( T_j \) containing the vertices of \( \bar{\Omega}_y \).
For each $y \in \mathcal{N}(m, m_l^i)$, there exists a constant $\tilde{\beta}$ depending only on the minimum angle in (1.6) such that
\[
|y - m_l^i| \leq d(\tilde{\Omega}_y) \leq \tilde{\beta}(\frac{1}{2})^m h_0.
\]
On the other hand, for any $y_1, y_2 \in \mathcal{N}(m, m_l^i)$, we have $|y_1 - y_2| \gtrsim (\frac{1}{2})^{m+1} h_0$ and
\[
\#\mathcal{N}(m, m_l^i) \lesssim 1.
\]
Hence,
\[
\#\sigma_2(m, m_l^i) \lesssim \#\mathcal{N}(m, m_l^i) \cdot \max_{y \in \mathcal{N}(m, m_l^i)} \#\sigma_0(m, y) \lesssim 1,
\]
which proves (A.10). Taking advantage of the preceding estimates, we conclude the proof as follows:
\[
\sum_{j=1}^{i-1} \sum_{m_j^i \in \mathcal{M}_j, I, \phi \neq 0 \text{ on } E_{j+1}^i} (h_j^i)^{-1/2} \lesssim h_0^{-1/2} \sum_{j=1}^{i-1} \sum_{m_j^i \in \mathcal{M}_j, I, \phi \neq 0 \text{ on } E_{j+1}^i} \left(\frac{1}{\sqrt{2}}\right)^{-\rho(m_j^i)} \rho(m_j^i) \lesssim \sum_{m=0}^{n_0} \sum_{(k,j) \in \sigma_2(m, m_l^i)} \left(\frac{1}{\sqrt{2}}\right)^{-m} \lesssim h_0^{-1/2} \left(\sqrt{2}\right)^{\rho(m_l^i) + n_0} \lesssim (h_l^i)^{-1/2}.
\]

References


LOCAL MULTILEVEL METHODS FOR ADAPTIVE NFEM

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