Chapter 3  Continuous-Time Optimal Control

3.1 Example: Resource allocation as a bilinear control problem

We consider a producer who produces with production rate $y(t)$ at time $t \in [0, T], T > 0$. He allocates a certain fraction $0 \leq u(t) \leq 1$ of the production to reinvestment and the rest $1 - u(t)$ to the production of a storable good. The producer wants to choose $u = u(t)$ such that the total amount of the stored product is maximized.

$$\text{maximize } J(y, u) := \int_0^T (1 - u(t)) y(t) \, dt$$

We call $y = y(t)$ the state and $u = u(t)$ the control. According to our assumptions above, the state $y$ evolves in time according to the following initial-value problem for a first order ordinary differential equation

$$\dot{y}(t) = \gamma u(t)y(t), \quad t \in [0, T],$$
$$y(0) = y_0,$$

where $\dot{y} := dy/dt$ and $\gamma > 0$ and $y_0$ are given constants.
Continuous-Time Optimal Control

Problem: Given the data

\( h : \mathbb{R}^n \rightarrow \mathbb{R}, \ g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}, \ f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \ y_0 \in \mathbb{R}^n, \ U \subset L^\infty([0, T], \mathbb{R}^m), \)

find

\( (y, u) \in C^1([0, T]; \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m) \)

such that

\[
\text{(CTOC) \quad \text{minimize} \quad J(y, u) := h(y(T)) + \int_0^T g(y(t), u(t)) \, dt}
\]

subject to \( \dot{y}(t) = f(y(t), u(t)), \quad t \in [0, T], \)

\( y(0) = y_0, \)

\( u(t) \in U \quad \text{for almost all} \ t \in [0, T]. \)

Notations: \( y \) is called the state, \( u \) is called the control. The set \( U \) is called the control constraint set. If \( (y^*, u^*) \in C^1([0, T]; \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m) \) satisfies \( \text{(CTOC)} \), \( y^* \) is called the optimal state and \( u^* \) is called the optimal control.

The function \( J^*(t, y) := h(y(t)) + \int_0^t g(y(s), u^*(s)) \, ds \) is called the optimal value function.
Assumptions on the data:

- \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) and \( g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) are continuously differentiable in the first argument and continuous in the second argument.

- \( h : \mathbb{R}^n \to \mathbb{R} \) is continuously differentiable.

Example: Linear-Quadratic Optimal Control

Minimize

\[
J(y, u) := \langle y(T), Q_T y(T) \rangle + \int_0^T \left( \langle y(t), Qy(t) \rangle + \langle u(t), Ru(t) \rangle \right) dt,
\]

subject to

\[
\dot{y}(t) = Ay(t) + Bu(t), \quad t \in [0, T],
\]

\[
y(0) = y_0,
\]

\[
u(t) \in \mathcal{U} \quad \text{for almost all } t \in [0, T].
\]

Here, \( Q_T, Q \in \mathbb{R}^{n \times n} \) are symmetric, positive semidefinite, \( R \in \mathbb{R}^{m \times m} \) is symmetric, positive definite, and \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \).
Hamilton-Jacobi-Bellman (HJB) Equation

Theorem (HJB equation and optimal value function)
Let \( V(t,y), \ t \in [0,T], \ y \in \mathbb{R}^n, \) be a continuously differentiable solution of the HJB equation. Then it holds

\[
\begin{align*}
\text{(HJB)} \quad & \min_{u \in U} \ (g(y,u) + \nabla_t V(t,y) + \nabla_y V(t,y)^T f(y,u)) = 0, \\
& V(T,y) = h(y).
\end{align*}
\]

On the other hand, let \((\hat{y}(t),\hat{u}(t), \ t \in [0,T], \) be an admissible pair of states and controls for \((\text{CTOC})\) such that \(\hat{u}\) is piecewise continuous with \(\hat{u}(t) \in U\) for almost all \(t \in [0,T]\) and \(\hat{y}\) solves

\[
\hat{y}(t) = f(\hat{y}(t),\hat{u}(t)), \ t \in [0,T], \quad \hat{y}(0) = y_0.
\]

Moreover, assume that the minimum in \((\text{HJB})\) is attained for \(\hat{u}\). Then it holds

\[
V(t,y) = J^*(t,y), \quad \hat{u}(t) = u^*(t), \ t \in [0,T], \ y \in \mathbb{R}^n.
\]
**Example:** Consider the following linear-quadratic problem

\[
\begin{align*}
\text{minimize} & \quad J(y, u) := \frac{1}{2} y(T)^2, \\
\text{subject to} & \quad \dot{y}(t) = u(t), \quad t \in [0, T], \\
& \quad y(0) = y_0, \\
& \quad -1 \leq u(t) \leq 1, \quad t \in [0, T].
\end{align*}
\]

**Natural control policy:** Move the state \( y \) as fast as possible to zero and then keep it at zero.

\[
\hat{u}(t) = -\text{sgn}(y) = \begin{cases} 
1, & \text{if } y < 0 \\
0, & \text{if } y = 0 \\
-1, & \text{if } y > 0
\end{cases}
\]

Case 1:

\[
\begin{align*}
y(t) = y = 0, \quad t \in [0, T), & \quad \implies \\
\dot{y}(\tau) = 0, \quad \tau \in [t, T], & \quad \implies \quad y(\tau) = 0, \quad \tau \in [t, T], \quad \implies \\
y(T) = 0.
\end{align*}
\]
Case 2:

\[ y(t) = y > 0, \quad t \in [0, T), \quad \implies \]
\[ \dot{y}(\tau) = -1, \quad \tau \in [t, T], \quad y(t) = y \quad \implies \quad y(\tau) = (t - \tau) + y, \quad \tau \in [t, T]. \]

When does \( y \) reach zero?

\[ y(\tau^*) = (t - \tau^*) + y = 0 \quad \iff \quad \tau^* = t + y \]

Therefore:

\[ \dot{y}(\tau) = 0, \quad \tau \geq t + y \quad \implies \quad y(\tau) = 0, \quad \tau \geq t + y \quad \implies \]
\[ y(\tau) = \begin{cases} 
(t - \tau) + y, & t \leq \tau \leq t + y \\
0, & \tau \geq t + y 
\end{cases} \]

It follows that

\[ y(\tau) = \max (0, y + t - \tau), \quad \tau \in [t, T], \]
\[ y(T) = \max (0, y - (T - t)). \]
Case 3:

\[ y(t) = y < 0, \quad t \in [0, T), \quad \Rightarrow \]
\[ \dot{y}(\tau) = 1, \quad \tau \in [t, T], \quad y(t) = y \quad \Rightarrow \quad y(\tau) = (\tau - t) + y, \quad \tau \in [t, T]. \]

When does \( y \) reach zero?

\[ y(\tau^*) = (\tau^* - t) + y = 0 \quad \iff \quad \tau^* = t - y \]

Therefore:

\[ \dot{y}(\tau) = 0, \quad \tau \geq t - y \quad \Rightarrow \quad y(\tau) = 0, \quad \tau \geq t - y \quad \Rightarrow \]

\[ y(\tau) = \begin{cases} (\tau - t) + y, & t \leq \tau \leq t - y \\ 0, & \tau \geq t - y \end{cases} \]

It follows that

\[ y(\tau) = \min \{0, y + \tau - t\} = \max \{0, -y + t - \tau\}, \quad \tau \in [t, T], \]
\[ y(T) = \max \{0, -y - (T - t)\}. \]
Altogether:

\[ y(\tau) = \max (0, |y| + t - \tau) \]
\[ y(T) = \max (0, |y| - (T - t)) \]

The associated value function \( \hat{J}(t, y) = \frac{1}{2}y(T)^2 \) is given by

\[ \hat{J}(t, y) = \frac{1}{2} (\max (0, |y| - (T - t)))^2 \]

Compute the derivatives of \( \hat{J}(t, y) \):

\[ \nabla_t \hat{J}(t, y) = \max (0, |y| - (T - t)) \]
\[ \nabla_y \hat{J}(t, y) = \text{sgn}(y) \max (0, |y| - (T - t)) \]

Hence, the HJB equation reads as follows:

\[(\text{HJB}) \quad \min_{|u| \leq 1} (1 + \text{sgn}(y)) \max (0, |y| - (T - t))) = 0\]

Observations: The minimum is attained for \( \hat{u} \quad \Rightarrow \quad J^*(t, y) = \hat{J}(t, y), \quad u^* = \hat{u}. \)
HJB Equation for the Linear-Quadratic Optimal Control Problem

\[ \min_{u \in \mathbb{R}^m} \left( \langle y, Qy \rangle + \langle u, Ru \rangle + \nabla_t V(t, y) + \langle \nabla_y V(t, y), Ay + Bu \rangle \right) = 0 \]

\[ V(T, y) = \langle y, QTy \rangle \]

We try a solution of the form \( V(t, y) = y^T K(t)y \), \( K(t) \in \mathbb{R}^{n \times n} \) symmetric.

Computation of the derivatives:

\[ \nabla_t V(t, y) = y^T \dot{K}(t)y \]
\[ \nabla_y V(t, y) = 2K(t)y \]

Substitution into (HJB-LQ) yields

\[ 0 = \min_{u} \mathcal{L}(y, u) := y^T Qy + u^T Ru + y^T \dot{K}(t)y + 2y^T K(t)Ay + 2y^T K(t)Bu \]
Optimality condition:

\[ \mathcal{L}_u(y, u) = 2B^T K(t)y + 2Ru = 0 \quad \implies \]

\[ \star \quad u = -R^{-1}B^T K(t)y \]

Substituting (\star) into (HJB - LQ) yields

\[ 0 = y^T(\dot{K}(t) + K(t)A + A^T K(t) - K(t)BR^{-1}B^T K(t) + Q)y \quad \implies \]

The matrix \( K(t) \) must satisfy the continuous-time Riccati equation

\[ \dot{K}(t) = -K(t)A - A^T K(t) + K(t)BR^{-1}B^T K(t) - Q, \]

\[ K(T) = Q_T. \]
3.2 Pontrjagin’s Minimum Principle (Heuristic Approach)

We first provide a heuristic derivation of the minimum principle based on the HJB equation

$$(HJB) \quad 0 = \min_{u \in U} \left( g(y, u) + \nabla_t J^*(t, y) + \nabla_y J^*(t, y)^T f(y, u) \right), \quad t \in [0, T], \ y \in \mathbb{R}^n,$$

$$J^*(T, y) = h(y), \quad y \in \mathbb{R}^n.$$

**Lemma (Differentiation of min-functions).**

Assume that $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuously differentiable and $U \subset \mathbb{R}^m$ is a convex subset. Assume further that $\mu^* : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable such that

$$\mu^*(t, y) = \arg\min_{u \in U} F(t, y, u).$$

Then, it holds

$$\nabla_t \min_{u \in U} F(t, y, u) = \nabla_t F(t, y, \mu^*(t, y)), \quad t \in [0, T], \ y \in \mathbb{R}^n,$$

$$\nabla_y \min_{u \in U} F(t, y, u) = \nabla_y F(t, y, \mu^*(t, y)), \quad t \in [0, T], \ y \in \mathbb{R}^n.$$
Proof. We set $z := (t, y)$ and $F(z, u) := F(t, y, u)$ as well as $\mu^*(z) := \mu^*(t, y)$.

Since $\min_{u \in \mathcal{U}} F(z, u) = F(z, \mu^*(z))$, we have

$$\nabla \min_{u \in \mathcal{U}} F(z, u) = \nabla_z F(z, \mu^*(z)) + \nabla_{\mu^*(z)} \nabla_u F(z, \mu^*(z)).$$

We will show: $\nabla \mu^*(z) \nabla_u F(z, \mu^*(z)) = 0$.

Since $\mathcal{U} \subset \mathbb{R}^m$ is convex, the optimality condition gives rise to the variational inequality

$$(u - \mu^*(z))^T \nabla_u F(z, \mu^*(z)) \geq 0, \quad u \in \mathcal{U}.$$  

Choose $u = \mu^*(z + \Delta z) \Rightarrow \mu^*(z) + \nabla \mu^*(z)^T \Delta z + o(||\Delta z||) \implies$

$$(\nabla \mu^*(z)^T \Delta z + o(||\Delta z||))^T \nabla_u F(z, \mu^*(z)) \geq 0 \quad \text{for all } \Delta z \implies$$

$$\nabla \mu^*(z) \nabla_u F(z, \mu^*(z)) = 0.$$
Adjoint State and Adjoint State Equation

Consider the HJB equation and assume that the minimum is attained for $\mu^*(t,y)$, which is continuously differentiable, and that $\mathcal{U} \subset \mathbb{R}^m$ is a convex set.

Applying the previous lemma gives

\begin{align*}
(*)_1 & \quad 0 = \nabla_t \min_{u \in \mathcal{U}} F(t,y,\mu^*(t,y)) = \nabla_{tt}^2 J^*(t,y) + \nabla_{yt}^2 J^*(t,y)^T f(y,\mu^*(t,y)), \\
(*)_2 & \quad 0 = \nabla_t \min_{u \in \mathcal{U}} F(t,y,\mu^*(t,y)) = \nabla_y g(y,\mu^*(t,y)) + \nabla_{yt}^2 J^*(t,y) + \\
& \quad \nabla_{yy}^2 J^*(t,y)^T f(y,\mu^*(t,y)) + \nabla_y f(y,\mu^*(t,y)) \nabla_y J^*(t,y),
\end{align*}

where $\nabla_y f$ is the Jacobian of $f$ with respect to $y$. 
Consider (*) for an optimal state $y^*$ and an optimal control $u^*$, i.e.

$$\dot{y}^*(t) = f(y^*(t), u^*(t)), \quad t \in [0, T].$$

Observing

$$\frac{\partial}{\partial t} (\nabla_y J^*(t, y^*(t))) = \nabla_{yt} J^*(t, y^*(t)) + \nabla_{yy} J^*(t, y^*(t))^{T} \frac{d}{dt} y^*(t) \quad ,$$

it follows that

(a) \quad \nabla_{yt}^2 J^*(t, y^*(t)) + \nabla_{yy}^2 J^*(t, y^*(t))^{T} f(y^*(t), u^*(t)) = \frac{d}{dt} \nabla_y J^*(t, y^*(t)) =: p(t).$$

On the other hand, observing

$$\frac{\partial}{\partial t} (\nabla_t J^*(t, y^*(t))) = \nabla_{tt} J^*(t, y^*(t)) + \nabla_{yt}^2 J^*(t, y^*(t))^{T} \frac{d}{dt} y^*(t) \quad ,$$

it follows that

(b) \quad \nabla_{tt}^2 J^*(t, y^*(t)) + \nabla_{yt}^2 J^*(t, y^*(t))^{T} f(y^*(t), u^*(t)) = \frac{d}{dt} \nabla_t J^*(t, y^*(t)) =: p_0(t).$$
Inserting \((a),(b)\) into \((*)_1, (*)_2\) gives
\[
\begin{align*}
(+)_1 \quad \dot{p}(t) &= -\nabla_y f(y^*(t), u^*(t)) p(t) - \nabla_y g(y^*(t), u^*(t)), \quad t \in [0, T], \\
p(T) &= \nabla_y J^*(T, y^*(T)) = \nabla h(y^*(T)).
\end{align*}
\]
The function \(p\) is called the \textit{adjoint state} and \((+)_1\) is called the \textit{adjoint state equation}.

\[
(+)_2 \quad \dot{p}_0(t) = 0 \quad \implies \quad p_0(t) = \text{const., } t \in [0, T].
\]
Moreover, we obtain
\[
\begin{align*}
u^*(t) &= \arg\min_{u \in U} \left( g(y^*(t), u(t)) + p(t)^T f(y^*(t), u(t)) \right), \quad t \in [0, T]. \\
&= \nabla_y J^*(t, y^*(t))
\end{align*}
\]
Hamiltonian

The Hamiltonian $H(y, u, p)$ associated with (CTOC) is given by

$$H(y, u, p) = g(y, u) + p^T f(y, u).$$

Theorem (Pontrjagin’s Minimum Principle)

Let $(y^*, u^*)$ be the optimal state and the optimal control for (CTOC) and assume that (A1) and (A2) are satisfied. Then, there exist an optimal adjoint state $p^*$ satisfying the following terminal value problem for the adjoint state equation

$$\dot{p}^*(t) = -\nabla_y H(y^*(t), u^*(t), p^*(t)), \quad t \in [0, T],$$

$$p^*(T) = \nabla h(y^*(T)).$$

Moreover, it holds

$$u^*(t) = \arg\min_{u \in U} H(y^*(t), u(t), p^*(t)), \quad t \in [0, T],$$

$$H(y^*(t), u^*(t), p^*(t)) = C = \text{const.}, \quad t \in [0, T].$$
Example: Linear-quadratic optimal control

\[
\text{minimize } J(y, u) = \frac{1}{2} y(T)^2 + \frac{1}{2} \int_0^T u(t)^2 \, dt,
\]
\[
\text{subject to } \dot{y}(t) = y(t) + u(t), \quad t \in [0, T],
\]
\[
y(0) = 1.
\]

We have

\[
f(y, u) = y + u, \quad g(y, u) = \frac{1}{2} u^2, \quad h(y(T)) = \frac{1}{2} y(T)^2.
\]

Hence, the Hamiltonian

\[
H(y, u, p) = g(y, u) + pf(y, u)
\]

reads as follows

\[
H(y, u, p) = \frac{1}{2} u^2 + p(y + u).
\]
For the adjoint equation

\[ \dot{p}(t) = -\nabla_y H(y(t), u(t), p(t)), \quad t \in [0, T] \]
\[ p(T) = \nabla h(y(T)) \]

we thus obtain

\[ \dot{p}(t) = -p(t), \quad t \in [0, T] \]
\[ p(T) = y(T) \]

Moreover, for the optimal control \( u^* \) it holds

\[ u^*(t) = \arg\min_u H(y^*(t), u(t), p^*(t)), \quad t \in [0, T] \]

and hence,

\[ u^*(t) = \arg\min_u \left( \frac{1}{2}u^2 + p^*(t)(y^*(t) + u(t)) \right) \]
The optimality condition gives
\[ u^*(t) + p^*(t) = 0, \]
whence
\[ (*) \quad u^*(t) = -p^*(t), \quad t \in [0, T]. \]

Elimination of the control from the state equation results in
\[ (+) \quad \dot{y}^*(t) = y^*(t) - p^*(t), \quad t \in [0, T], \]
\[ y^*(0) = 1. \]

On the other hand, an elementary integration of the adjoint equation yields
\[ (++) \quad p^*(t) = c \exp(-t), \quad t \in [0, T], \]
where
\[ p^*(T) = c \exp(-T) = y^*(T) \quad \implies \quad c = \exp(T) \cdot y^*(T). \]
Inserting \((++)\) into \((+))\) gives

\[
\dot{y}^*(t) = y^*(t) - \exp(T-t) \: y^*(T), \quad t \in [0, T],
\]

\[
y^*(0) = 1.
\]

Integration yields

\[
y^*(t) = \exp(t) - \int_0^t \exp(t-s) \: F(s) \: ds
\]

\[
= \exp(t) - \int_0^t \exp(t-s) \: \exp(T-s) \: y^*(T) \: ds
\]

\[
= \exp(t) - \exp(T) \: y^*(T) \: \int_0^t \exp(t-2s) \: ds
\]

\[
= \exp(t) + \frac{1}{2} \exp(T) \: y^*(T) \: \left[ \exp(t-2s) \right]_0^t
\]

\[
= \exp(t) + \frac{1}{2} \exp(T) \: y^*(T) \: (\exp(-t) - \exp(t)).
\]
For $y^*(T)$ we obtain

$$y^*(T) = \exp(T) + \frac{1}{2} \exp(T) y^*(T) (\exp(-T) - \exp(T))$$

$$= \exp(T) + \frac{1}{2} \exp(T) y^*(T) \exp(-T) (1 - \exp(2T)) \implies$$

$$(1 - \frac{1}{2}(1 - \exp(2T))) y^*(T) = \exp(T) \implies$$

$$y^*(T) = (1 + \frac{1}{2}(\exp(2T) - 1))^{-1} \exp(T).$$
Pontrjagin’s Minimum Principle (Variational Approach)

\[
\begin{align*}
\text{(CTOC)} \quad & \text{minimize} \quad J(y, u) := h(y(T)) + \int_0^T g(y(t), u(t)) \, dt \\
\text{subject to} \quad & \dot{y}(t) = f(y(t), u(t)), \quad t \in [0, T], \\
& y(0) = y_0, \\
& u(t) \in \mathcal{U} \quad \text{for almost all } t \in [0, T].
\end{align*}
\]

Assumptions:

(A1) Convexity

For every state \( y \in \mathbb{R}^n \) the set

\[
D := \{ f(y, u) \mid u \in \mathcal{U} \}
\]

is convex.
(A2) Regularity

Let $u_1(t), u_2(t) \in \mathcal{U}$ be two admissible controls and let $y_1$ be the state associated with $u_1$. For $\varepsilon \in [0, 1]$ consider the system

$$\dot{y}_\varepsilon = (1 - \varepsilon)f(y_\varepsilon(t), u_1(t)) + \varepsilon f(y_\varepsilon(t), u_2(t)), \quad t \in [0, T]$$

$$y_\varepsilon(0) = y_1(0)$$

Motivation: Assume that $y_\varepsilon$ is of the form

$$y_\varepsilon(t) = y_1(t) + \varepsilon \xi(t) + o(\varepsilon).$$

What is the equation that $\xi(t)$ must satisfy?
By Taylor expansion we find:

\[
\dot{y}_\varepsilon(t) = \dot{y}_1(t) + \varepsilon \dot{\xi}(t) = (1 - \varepsilon)f(y_1(t) + \varepsilon \xi(t) + o(\varepsilon), u_1(t)) + \varepsilon f(y_1(t) + \varepsilon \xi(t) + o(\varepsilon), u_2(t))
\]

\[
\dot{y}_1(t) = f(y_1(t), u_1(t))
\]

\[
f(y_1(t) + \varepsilon \xi(t) + o(\varepsilon), u_i(t)) = f(y_1(t), u_i(t)) + \varepsilon \nabla_y f(y_1(t), u_i(t)) \xi(t) + o(\varepsilon), \quad 1 \leq i \leq 2
\]

It follows that

\[
\varepsilon \dot{\xi}(t) = (1 - \varepsilon)(f(y_1(t), u_1(t)) + \varepsilon \nabla_y f(y_1(t), u_1(t)) \xi(t) + o(\varepsilon)) + \varepsilon (\nabla_y f(y_1(t), u_2(t)) \xi(t) + o(\varepsilon)) - f(y_1(t), u_1(t))
\]

Collecting all terms of order \(\varepsilon\) yields

\[
(+): \quad \dot{\xi}(t) = \nabla_y f(y_1(t), u_1(t)) \xi(t) + f(y_1(t), u_2(t)) - f(y_1(t), u_1(t)), \quad t \in [0, T],
\]

\[
\xi(0) = 0
\]

\[
(A2) \quad \text{Any solution } y_\varepsilon \text{ of } (*) \text{ is of the form } (o), \text{ where } \xi \text{ satisfies } (+).
\]
Example: Linear Systems

\[ \dot{y}(t) = Ay(t) + Bu(t) \]
\[ y(0) = y_0 \]

Then, (\*) and (\+) take the form

\[
\begin{align*}
(\ast) & \quad \dot{y}_\varepsilon(t) = Ay_\varepsilon(t) + Bu_1(t) + \varepsilon B(u_2(t) - u_1(t)) \\
(+) & \quad \dot{\xi}(t) = A\xi(t) + B(u_2(t) - u_1(t))
\end{align*}
\]

It follows that

\[
\begin{align*}
\dot{y}_\varepsilon(t) - \varepsilon \dot{\xi}(t) = & \quad A(y_\varepsilon(t) - \varepsilon \xi(t)) + Bu_1(t) \\
\dot{y}_1(t) = & \quad Ay_1(t) + Bu_1(t) \\
y_\varepsilon(t) = & \quad y_1(t) + \varepsilon \xi(t)
\end{align*}
\]
Proof of Pontrjagin’s Minimum Principle

Let us first assume $J(y, u) = h(y(T))$ (terminal cost only). The convexity assumption (A1) implies:

For any admissible control $u(t) \in U$, $t \in [0, T]$, and any $\varepsilon \in [0, 1]$ there exists $\bar{u}(t)$, $t \in [0, T]$, such that for the state $y_\varepsilon$ satisfying

$$\dot{y}_\varepsilon(t) = (1 - \varepsilon)f(y_\varepsilon(t), u^*(t)) + \varepsilon f(y_\varepsilon(t), u(t)), \quad t \in [0, T]$$
$$y_\varepsilon(0) = y_0$$

there holds

$$f(y_\varepsilon(t), \bar{u}(t)) = (1 - \varepsilon)f(y_\varepsilon(t), u^*(t)) + \varepsilon f(y_\varepsilon(t), u(t)), \quad t \in [0, T]$$

i.e., the state $y_\varepsilon$ corresponds to the control $\bar{u}(t)$.

The regularity assumption (A2) implies:

$$y_\varepsilon(t) = y^*(t) + \varepsilon x_i(t) + o(\varepsilon),$$

$$\dot{\xi}(t) = \nabla_y f(y^*(t), u^*(t))\xi(t) + f(y^*(t), u(t)) - f(y^*(t), u^*(t)),$$

(LS) $$\xi(0) = 0.$$
Further, the optimality of $y^*$ implies:

$$h(y^*(T)) = h(y\varepsilon(T)) = h(y^*(T) + \varepsilon\xi(T) + o(\varepsilon)) \implies h(y^*(T)) + \varepsilon\nabla h(y^*(T))^T\xi(T) + o(\varepsilon) \implies$$

\[\nabla h(y^*(T))^T\xi(T) \geq 0\]

Now, consider the Wronski matrix $W(t, \tau)$ associated with (LS):

(WM) \[\frac{\partial W(t, \tau)}{\partial \tau} = -W(t, \tau)\nabla y f(y^*(\tau), u^*(\tau))^T, \quad \tau \in [0, t],\]

\[W(t, t) = I\]

The solution of (LS) can thus be written according to

\[\xi(t) = W(t, \tau)\xi(\tau) + \int_{\tau}^{t} W(t, s)(f(y^*(s), u(s)) - f(y^*(s), u^*(s))) \, ds \implies\]

\[\xi(T) = W(T, 0)\xi(0) + \int_{0}^{T} W(T, s)(f(y^*(s), u(s)) - f(y^*(s), u^*(s))) \, ds\]
We define:

$$\begin{align*}
(\text{AS}) & \quad \dot{p}^*(t) := W(T, t)^T p^*(T), \quad t \in [0, T] \\
p^*(T) := \nabla h(y^*(T))
\end{align*}$$

Differentiation with respect to $t$ yields

$$\dot{p}^*(t) = \frac{\partial W(T, t)^T}{\partial t} p^*(T) \overset{(\text{WM})}{=} -W(T, t)^T \nabla_y f(y^*(t), u^*(t))^T p^*(T)$$

$$= -\nabla_y f(y^*(t), u^*(t))^T W(T, t)^T p^*(T) \overset{(\text{AS})}{=} p^*(t) = -\nabla_y H(y^*(t), u^*(t), p^*(t))$$

which is the adjoint state equation in case $J(y, u) = h(y(T))$. 
(MP) remains to be shown: $(+)$, $(o)$, and $(AS)$ imply

\[
(\times) \quad 0 \leq (\nabla h(y^*(T)))^T \xi(T) \leq p^*(T)^T \xi(T)
\]

\[
\leq p^*(T)^T \int_0^T W(T, t)(f(y^*(t), u(t)) - f(y^*(t), \hat{u}(t))) \, dt
\]

We will show: For all $t \in [0, T]$, where $u^*$ is continuous, it holds:

\[
p^*(t)^T f(y^*(t), u^*(t)) \leq p^*(t)^T f(y^*(t), u(t)), \quad u \in \mathcal{U}
\]

**Proof by contradiction:** There exist $\hat{u} \in \mathcal{U}$ and $t_0 \in [0, T]$ such that

\[
p^*(t_0)^T f(y^*(t_0), u^*(t_0)) > p^*(t_0)^T f(y^*(t_0), \hat{u}(t_0))
\]

It follows that there exists an interval $I(t_0) \subset [0, T]$, such that

\[
p^*(t)^T f(y^*(t), u^*(t)) > p^*(t)^T f(y^*(t), \hat{u}(t)), \quad t \in I(t_0).
\]

We define

\[
u(t) := \begin{cases} \hat{u}(t), & t \in I(t_0) \\ u^*(t), & t \notin I(t_0) \end{cases}
\]

Observing $p^*(t) = W(T, t)^T p^*(T)$, this is a contradiction to $(\times)$. 
Let us now consider the general case:

\[ J(y, u) = h(y(T)) + \int_0^T g(y(t), u(t)). \]

It can be reformulated as the terminal cost problem:

\[ \bar{J}(y, u) = h(y(T)) + y(T) \]

by introducing the additional differential equation

\[ \dot{y}(t) = g(y(t), u(t)), \quad t \in [0, T]. \]

**Homework:** Use similar arguments as before to conclude!
3.3 Brachistochrone (Minimum Time Problem)

In 1696 the Swiss mathematician Jacob Bernoulli challenged the mathematical world with the following problem which is nowadays known as the brachistochrone. Correct solutions have been submitted by Jacob Bernoulli (his brother), l’Hopital, Leibniz, and Newton.

Given two points A and B, find a curve $C_{AB}$ from A to B such that a body with mass $m$ moves along $C_{AB}$ under the force of gravity in minimum time T (gravitational acceleration: g).
Parametrization of $C_{AB}: y = y(t), t \in [0, T] \implies$

Length of $C_{AB}$ from $y(t)$ to $y(t + dt)$: $\sqrt{1 + (\dot{y}(t))^2}$

Velocity of the body: $v(t)$

$$\begin{align*}
\text{Kinetic energy: } E_{\text{kin}}(t) &:= \frac{1}{2}m v(t)^2 \\
\text{Potential Energy: } E_{\text{pot}}(t) &:= mg y(t)
\end{align*}$$

$\implies v(t) = \sqrt{2gy(t)}$

Time to reach $B$:

$$\begin{align*}
\int_0^T \frac{\ell_{C_{AB}}}{v(t)} \, dt &= \int_0^T \frac{\sqrt{1 + (\dot{y}(t))^2}}{\sqrt{2gy(t)}} \, dt
\end{align*}$$

Setting $u(t) := \dot{y}(t)$, we end up with the Minimum Time Problem:

$$\begin{align*}
\text{minimize } & J(y, u) := \int_0^T \frac{\sqrt{1 + u(t)^2}}{\sqrt{2gy(t)}} \, dt \\
\text{subject to } & \dot{y}(t) = u(t), \quad t \in [0, T], \\
& y(0) = 0, \quad y(T) = x_B
\end{align*}$$
Hamiltonian:

\[ H(y,u,p) = g(y,u) + pu, \quad g(y,u) = \frac{\sqrt{1 + u(t)^2}}{\sqrt{2gy(t)}} \]

Optimality condition:

\[ H_u(y^*, u^*, p^*) = \nabla_u g(y^*(t), u^*(t)) + p^*(t) = 0 \quad \Rightarrow \]
\[ p^*(t) = -\nabla_u g(y^*(t), u^*(t)) = -\frac{u^*(t)}{\sqrt{1 + u^*(t)^2}} \frac{\sqrt{2gy^*(t)}}{2gy^*(t)} \]

Pontrjagin’s minimum principle: Hamiltonian constant along optimal trajectory

\[ g(y^*(t), u^*(t)) + p^*(t)u^*(t) = \text{const.}, \quad t \in [0, T] \quad \Rightarrow \]
\[ \frac{\sqrt{1 + u^*(t)^2}}{\sqrt{2gy^*(t)}} - \frac{u^*(t)}{\sqrt{1 + u^*(t)^2} \sqrt{2gy^*(t)}} = \frac{1}{\sqrt{1 + u^*(t)^2} \sqrt{2gy^*(t)}} = \text{const.} \]
It follows that
\[ \frac{1}{\sqrt{2g y^*(t)}} = C_1 \sqrt{1 + u^*(t)^2}. \]

Taking \( u^*(t) = \dot{y}^*(t) \) into account, it follows that
\[ (1 + \dot{y}^*(t)^2)y^*(t) = \frac{1}{2g C_1^2} =: C \quad \Rightarrow \quad \dot{y}^*(t) = \sqrt{\frac{C - y^*(t)}{y^*(t)}}, \quad t \in [0, T] \]

The solution of the differential equation is a cycloid.

The parameters \( C, T \) follow from the boundary conditions
\[ y^*(0) = 0, \quad y^*(T) = x_B. \]