Addendum 2  The construction of finite element spaces

The starting point for the construction of finite element spaces is a triangulation of the computational domain $\Omega$.

**Definition 2.1 Triangulation**

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with piecewise smooth boundary $\Gamma = \partial \Omega$. A triangulation $T_h$ is a partition of $\Omega$ in finitely many subdomains $K \subset \bar{\Omega}$, called elements, satisfying the following properties

- $(T_h)_1$ $\bar{\Omega} = \bigcup_{K \in T_h} K$,
- $(T_h)_2$ Each $K \in T_h$ is closed, i.e., $\bar{K} = K$,
- $(T_h)_3$ $K_1^{\text{int}} \cap K_2^{\text{int}} = \emptyset$ for all $K_1, K_2 \in T_h$, $K_1 \neq K_2$,
- $(T_h)_4$ For each $K \in T_h$ the boundary $\partial K$ is piecewise smooth.

The quantity $h := \max_{K \in T_h} \text{diam}(K)$ is called the granularity of the triangulation.

Examples for triangulations are triangulations by simplices and by rectangular elements.

**Definition 2.2 Simplices in $\mathbb{R}^d$**

A simplex $K \subset \mathbb{R}^d$ is the convex hull of $d + 1$ different points $x^{(i)} \in \mathbb{R}^d$, $1 \leq i \leq d + 1$:

$$K := \{ x \in \mathbb{R}^d \mid x = \sum_{i=1}^{d+1} \lambda_i x^{(i)}, \lambda_i \in [0, 1], 1 \leq i \leq d + 1, \sum_{i=1}^{d+1} \lambda_i = 1 \}.$$ 

A simplex is called non-degenerated, if the $d + 1$ points $x^{(i)}$, $1 \leq i \leq d + 1$, are not situated on a $d - 1$-dimensional manifold of $\mathbb{R}^d$. The points $x^{(i)}$, $1 \leq i \leq d + 1$, are called the vertices of $K$. The convex hull of any two vertices is a 1-dimensional simplex which is called an edge of $K$. The convex hull of any three vertices is called a face of $K$. In general, the convex hull of $2 \leq r < d$ vertices is an $r - 1$-dimensional simplex which is referred to as an $r - 1$-dimensional face of $K$.

A 2-dimensional simplex is a triangle, and a 3-dimensional simplex is a tetrahedron.

**Definition 2.3 Rectangular elements**

A rectangular element $K$ in $\mathbb{R}^d$, also called a $d$-rectangle, is the tensor product of $d$ intervals $[a_1^{(i)}, a_2^{(i)}]$, $1 \leq i \leq d$:

$$K := \prod_{i=1}^{d} [a_1^{(i)}, a_2^{(i)}].$$ 

The points $x^{(\ell)} = (x_1^{(\ell)}, ..., x_d^{(\ell)})^T$, $1 \leq \ell \leq 2^d$, where $x_j^{(\ell)} = a_1^{(i)}$ or $x_j^{(\ell)} = a_2^{(i)}$, $i \in \{1, ..., d\}$, are called the vertices of $K$. 

Let $1 \leq r < d$. Then, any set of the form
\[
\{(x_1, ..., x_d)^T \mid x_i \in [a_1^{(i)}, a_2^{(i)}], \ 1 \leq i \leq d \} ,
\]
\[
x_{ij} = a_1^{(ij)} \text{ or } x_{ij} = a_2^{(ij)}, \ i_j \in \{1, ..., d\}, \ 1 \leq j \leq r ,
\]
\[
x_{ik} \in [a_1^{(ik)}, a_2^{(ik)}], \ i_k \in \{1, ..., d\}, \ r+1 \leq k \leq d, \ i_k \neq i_j , \ 1 \leq j \leq r
\]
is called a $d-r$-dimensional face of $K$. A 1-dimensional face is referred to as an edge.

**Definition 2.4 Geometrically conforming triangulation**

A simplicial or rectangular triangulation $T_h$ of a bounded domain $\Omega \subset \mathbb{R}^d$ is said to be geometrically conforming, if for any $K_1, K_2 \in T_h, K_1 \neq K_2$, the intersection $K_1 \cap K_2$ is either empty, or a common vertex, or a common edge, or a common face of $K_1$ and $K_2$.

**Definition 2.5 Barycentric coordinates and center of gravity**

Let $K$ be a non-degenerated simplex in $\mathbb{R}^d$ with vertices $x^{(i)} = (x_1^{(i)}, ..., x_d^{(i)})^T, 1 \leq i \leq d+1$. Then any point $x = (x_1, ..., x_d)^T \in \mathbb{R}^d$ can be written as a linear combination of the vertices $x^{(i)}$:
\[
x_j = \sum_{i=1}^{d+1} \lambda_i^{(i)} x_j^{(i)} , \ 1 \leq j \leq d , \quad \sum_{i=1}^{d+1} \lambda_i(x) = 1
\]

The $\lambda_i, 1 \leq i \leq d+1$, are called the barycentric coordinates. They are the unique solution of the linear system
\[
\begin{pmatrix}
x_1^{(1)} & \cdots & x_{d+1}^{(1)} \\
x_2^{(1)} & \cdots & x_{d+1}^{(2)} \\
\vdots & \ddots & \vdots \\
x_d^{(1)} & \cdots & x_{d+1}^{(d)} \\
1 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_{d+1}
\end{pmatrix}
= \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_d \\
1
\end{pmatrix}
\]

The point $x^{(c)} \in K$ for which
\[
x^{(c)}_i = \frac{1}{d+1} , \ 1 \leq i \leq d
\]
is called the center of gravity of $K$.

**Definition 2.6 Reference simplex**

The non-degenerated simplex $K_{ref}$ with vertices
\[
x^{(1)} = (0, 0, ..., 0)^T ,
\]
\[
x^{(i)} = (x_1^{(i)}, ..., x_d^{(i)})^T , \ x_j^{(i)} = \delta_{i-1,j} , \ 2 \leq i \leq d+1, \ 1 \leq j \leq d
\]
is called the reference simplex resp. the reference element in case of a simplicial triangulation $T_h$.

The barycentric coordinates are given by
\[
\lambda_i = x_i , \ 1 \leq i \leq d , \quad \lambda_{d+1} = 1 - \sum_{j=1}^{d} x_j
\]
**Definition 2.7 Bubble function**

The function \( B(x) := \prod_{i=1}^{d+1} \lambda_i(x) \) is a polynomial of degree \( d + 1 \). It has the property

\[
B(x) = 0, \quad x \in \partial K
\]

and is therefore referred to as \( d + 1 \)-dimensional bubble function.

For a triangulation \( T_h \) of \( \Omega \subset \mathbb{R}^d \) and \( D \subset \bar{\Omega} \), we refer to \( \mathcal{N}_h(D) \), \( \mathcal{E}_h(D) \), and \( \mathcal{F}_h(D) \) as the sets of vertices, edges, and faces of \( T_h \) in \( D \).

Further, for each \( K \in T_h \) we denote by

\[
P_K := \{ u : K \to \mathbb{R} \}
\]  

(2.1)

a linear space of functions on \( K \) with

\[
dim P_K = n_K < \infty. \tag{2.2}
\]

We define \( V_h \) as the linear space of functions on \( \bar{\Omega} \) whose restrictions to \( K \) are in \( P_K \):

\[
V_h := \{ u_h : \bar{\Omega} \to \mathbb{R} \mid u_h|_K \in P_K, \ K \in T_h \}. \tag{2.3}
\]

The following result gives sufficient conditions for \( V_h \) to be a subspace of \( H^1(\Omega) \).

**Lemma 2.8 Conformity in \( H^1(\Omega) \)**

Let \( T_h \) be a geometrically conforming triangulation of \( \Omega \) and assume that

\[
P_K \subset H^1(K), \quad K \in T_h, \tag{2.4}
\]

\[
V_h \subset C(\bar{\Omega}). \tag{2.5}
\]

Then

\[
V_h \subset H^1(\Omega). \tag{2.6}
\]

**Proof:** Given \( u_h \in V_h \), we have to show that \( u_h \) possesses weak partial derivatives \( D^\alpha u_h \in L^2(\Omega), |\alpha| = 1 \). In view of (2.4), \( D^\alpha u_h|_K \in L^2(K) \). Hence, we may apply Green’s theorem elementwise. Denoting by \( \alpha_i, i \in \{1, \ldots, d\} \) that component of \( \alpha \) with \( \alpha_i = 1 \), for \( \varphi \in C^\infty_0(\Omega) \) we obtain:

\[
\int_{\Omega} u_h D^\alpha \varphi \, dx = \sum_{K \in T_h} \int_{K} u_h|_K \, D^\alpha \varphi \, dx =
\]

\[
= - \sum_{K \in T_h} \int_{\partial K} D^\alpha u_h|_K \, \varphi \, ds + \sum_{K \in T_h} \int_{\partial K} n_{\alpha_i} u_h \varphi \, ds =
\]

\[
= - \sum_{K \in T_h} \int_{\partial K} D^\alpha u_h|_K \, \varphi \, ds + \sum_{F \in \mathcal{F}_h(\Omega)} [n_{F,i} u_h] \varphi \, d\sigma,
\]

where \( n_{\alpha_i} \) is the outward normal to \( \partial K \) at \( x \).
where 
$$[n_{F,i}u_h]_J := n_{F,i}u_h|_{K_1} - n_{F,i}u_h|_{K_2} \quad , \quad F = K_1 \cap K_2$$
with $n_{F,i}$ being the i-th component of the unit normal on $F$ directed towards $K_2$.

Since $u_h$ is continuous across $F$ due to (2.5), we have
$$[n_{F,i}u_h]_J = 0$$
and hence, $D^\alpha w u_h$ exists and is square integrable due to
$$(D^\alpha w u_h)|_K = D^\alpha w u_h|_K \in L^2(K) , \quad K \in T_h$$

A function $p \in P_K, K \in T_h$ will be specified by a set $\Sigma_K$ of degrees of freedoms.

**Definition 2.9 Finite elements and unisolvence**

The triple $(K, P_K, \Sigma_K)$ is called a finite element. It is said to be unisolvent, if any $p \in P_K$ is uniquely determined by the set $\Sigma_K$ of degrees of freedom.

In case of a simplicial triangulation $T_h$ and Lagrangian finite elements we choose $P_K$ as the linear space of polynomials of degree $k$ on $K$:

$$P_K := P_k(K) = \{ p(x) = \sum_{|\alpha| \leq k} a_\alpha x^\alpha \mid x \in K \} , \quad k \in \mathbb{N} \quad , (2.7)$$

where $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$.

We note that
$$n_k := \dim P_k(K) = \binom{k+d}{k} = \frac{(k+d)!}{k! \; d!} \quad . \quad (2.8)$$

The degrees of freedom are the function values (nodal values) at selected points (nodal points) $x_i \in K, 1 \leq i \leq n_k$:

$$\Sigma_K := \{ p(x_i) \mid p \in P_k(K) , x_i \in K , 1 \leq i \leq n_k \} \quad . \quad (2.9)$$

In case of the reference element $K_{ref}$ the set of nodal points is given by

$$L_k(K_{ref}) := \{ x = \left( \frac{i_1}{k} , \frac{i_2}{k} , \ldots , \frac{i_d}{k} \right)^T \mid i_j \in \{0,1,\ldots,k\}, \quad 1 \leq j \leq d , \quad \sum_{j=1}^d i_j \leq k \} \quad . \quad (2.10)$$

**Definition 2.10 Lagrangian finite element of type $k$**

The triple $(K, P_k(K), \Sigma_K)$ with

$$\Sigma_K := \{ p(x) \mid p \in P_k(K) , x \in L_k(K) \} \quad (2.11)$$

is called a Lagrangian finite element of type $k$. 
Examples of Lagrangian finite elements

(i) $k = 1$, $d = 2$

$$\dim P_1(K_{ref}) = 3 \quad , \quad L_1(K_{ref}) = \{(0, 0)^T, (1, 0)^T, (0, 1)^T\} .$$

(ii) $k = 2$, $d = 2$

$$\dim P_2(K_{ref}) = 6$$

$$L_2(K_{ref}) = \{(0, 0)^T, (1, 0)^T, (0, 1)^T, \left(\frac{1}{2}, 0\right)^T, (0, \frac{1}{2})^T, \left(\frac{1}{2}, \frac{1}{2}\right)^T\} .$$

(iii) $k = 3$, $d = 2$

$$\dim P_3(K_{ref}) = 10$$

$$L_3(K_{ref}) = \{(0, 0)^T, (1, 0)^T, (0, 1)^T, \left(\frac{1}{3}, 0\right)^T, (0, \frac{1}{3})^T, (0, \frac{2}{3})^T, \left(\frac{1}{3}, \frac{1}{3}\right)^T, \left(\frac{2}{3}, 0\right)^T, \left(\frac{1}{3}, \frac{1}{3}\right)^T, (0, \frac{2}{3})^T, \left(\frac{2}{3}, \frac{1}{3}\right)^T\} .$$

Lemma 2.11 Unisolvence of Lagrangian finite elements of type $k$ in case $d = 2$

Let $(K_{ref}, P_k(K_{ref}), \Sigma_{K_{ref}})$ be a Lagrangian finite element of type $k$. Then, any $p \in P_k(K_{ref})$ is uniquely determined by its values in $L_k(K_{ref})$.

**Proof:** Assume

$$p(x) = 0 \quad , \quad x \in L_k(K_{ref}) .$$

For any $E \in \mathcal{E}_h(K_{ref})$ we have

$$p|_E \in P_k(E) .$$

But any $E \in \mathcal{E}_h(K_{ref})$ contains $k + 1$ nodal points from $L_k(K_{ref})$ whence

$$p|_E = 0 \quad , \quad E \in \mathcal{E}_h(K) .$$

Consequently, $p$ is of the form

$$p = \lambda_1 \lambda_2 \lambda_3 q \quad , \quad q \in P_{k-3}(K_{ref}) ,$$

where $\lambda_i, 1 \leq i \leq 3$, are the barycentric coordinates of $K_{ref}$. We have

$$q(x) = 0 \quad , \quad x \in L_k(K_{ref}) \cap K_{ref}^{int} .$$

Since

$$\text{card } L_k(K_{ref}) \cap K_{ref}^{int} = \dim P_{k-3}(K_{ref}) ,$$

it follows that

$$q = 0 \quad \Rightarrow \quad p = 0 .$$
If $K \in \mathcal{T}_h$ is an arbitrary element, we may transform $K_{ref}$ to $K$ by means of an affine mapping

$$F_K(\hat{x}) = B_K \hat{x} + b_K, \quad B_K \in \mathbb{R}^{d \times d}, b_K \in \mathbb{R}^d.$$  \hfill (2.12)

If we define

$$\Sigma_K := \{p(x) \mid p \in P_k(K), x \in L_k(K) := \{x = F_K(\hat{x}), \hat{x} \in L_k(K_{ref})\}\},$$  \hfill (2.13)

the result of the previous lemma applies to $(K, P_K, \Sigma_K)$, i.e., any $p \in P_k(K)$ is uniquely determined by the set $\Sigma_K$ of degrees of freedom.

**Lemma 2.12** Affine equivalence of Lagrangian finite elements

Two Lagrangian finite elements $(\hat{K}, P_{\hat{K}}, \Sigma_{\hat{K}})$ and $(K, P_K, \Sigma_K)$ are affine equivalent in the sense that there exists an affine mapping

$$F_K : \hat{K} \rightarrow K$$  \hfill (2.14)

such that

$$\hat{x} \mapsto F_K(\hat{x}) = B_K \hat{x} + b_K$$  \hfill (2.15)

such that

$$K = F_K(\hat{K}),$$  \hfill (2.16)

$$P_K = \{p = \hat{p} \circ F_K^{-1} \mid \hat{p} \in P_k(\hat{K})\},$$  \hfill (2.17)

$$\Sigma_K = \{p(F_K(\hat{x}) \mid p \in P_k(K) \quad \hat{x} \in L_k(\hat{K})\}. $$  \hfill (2.18)

**Definition 2.13** Lagrangian finite element spaces and nodal bases

The linear space $V_h$ that is composed by Lagrangian finite elements of type $k$ will be denoted by

$$S_k(\Omega; \mathcal{T}_h) := \{u_h : \Omega \rightarrow \mathbb{R} \mid u_h|_K \in P_k(K), K \in \mathcal{T}_h\}, \quad k \in \mathbb{N}.$$  \hfill (2.19)

A basis $\{\varphi_h^{(i)}\}_{i=1}^{n_h}$, $n_h := \text{card } N_h(\bar{\Omega})$ of $S_k(\Omega; \mathcal{T}_h)$ is given by

$$\varphi_h^{(i)}(x^{(j)}) = \delta_{ij}, \quad x^{(j)} \in N_h(\bar{\Omega}), \quad 1 \leq i, j \leq n_h.$$  \hfill (2.20)

This basis is referred to as the nodal basis.

**Lemma 2.14** Conformity of Lagrangian finite element spaces

If the simplicial triangulation $\mathcal{T}_h$ is geometrically conforming, the Lagrangian finite element spaces $S_k(\Omega; \mathcal{T}_h)$ are $H^1$-conforming, i.e., they are finite dimensional subspaces of $H^1(\Omega)$.

**Proof:** Let $u_h \in S_k(\Omega; \mathcal{T}_h)$ and $F := K_1 \cap K_2$ be the common face of two adjacent elements of $\mathcal{T}_h$. Then $u_h|_F \in P_k(F)$ with

$$\dim P_k(F) = \binom{k+d-1}{k}.$$  

Since

$$\text{card } L_k(K_1) \cap F = \text{card } L_k(K_2) \cap F = \binom{k+d-1}{k},$$
we have
\[ u_h|_{K_1 \cap F} = u_h|_{K_2 \cap F} \]
Hence, \( u_h \) is continuous across \( F \) which proves that \( u_h \in C(\bar{\Omega}) \).

**Remark 2.15** Due to the previous result, Lagrangian finite elements of type 1 are also called continuous, piecewise linear elements, those of type 2 are referred to as continuous, piecewise quadratic elements etc.

If \( T_h \) is a triangulation by rectangular elements, for \( K \in T_h \) we choose \( P_K = Q_k(K), k \in \mathbb{N} \), where
\[
Q_k(K) := \sum_{\alpha_i \leq k} a_{\alpha} x^\alpha
\]
(2.21)
and note that
\[
\dim Q_k(K) = (k + 1)^d
\]
(2.22)
For the reference element \( K_{ref} \), which is the unit cube
\[
K_{ref} := [0,1]^d
\]
(2.23)
we define the set \( L[k](K_{ref}) \) of nodal points according to
\[
L[k](K_{ref}) := \{ x = (\frac{i_1}{k},...,\frac{i_d}{k})^T | i_j \in \{1,...,d\}, 1 \leq j \leq d \}
\]
(2.24)
and set
\[
\Sigma_{K_{ref}} := \{ p(x) | p \in Q_k(K), x \in L[k](K_{ref}) \}
\]
(2.25)
An arbitrary rectangular element \( K \in T_h \) can be obtained from \( K_{ref} \) by an affine mapping
\[
F_K : K_{ref} \rightarrow K
\]
\[
\hat{x} \mapsto F_K(\hat{x}) = B_K \hat{x} + b_K
\]
where \( B_K \in \mathbb{R}^{d \times d} \) is a diagonal mapping and \( b_K \in \mathbb{R}^d \).

As in the case of simplicial triangulations we set
\[
P_K := \{ p = \hat{p} \circ F_K^{-1} | \hat{p} \in Q_k(K_{ref}) \}
\]
\[
\Sigma_K := \{ p(F_K(\hat{x}) | p \in Q_k(K), \hat{x} \in L[k](K_{ref}) \}
\]

**Definition 2.16** Rectangular finite elements
The element \( (K, Q_k(K), \Sigma_K), k \in \mathbb{N} \), is called a rectangular finite element in \( \mathbb{R}^d \).

**Lemma 2.17** Unisolvency of rectangular finite elements
The rectangular finite element \( (K, Q_k(K), \Sigma_K), k \in \mathbb{N} \), is unisolvent, i.e., any
\( p \in Q_k(K) \) is uniquely determined by the degrees of freedom as given by the set \( \Sigma_K \).

**Proof:** The proofs follows the same line of arguments as in the case of Lagrangian finite elements of type \( k \).

### Examples of rectangular finite elements in case \( d = 2 \)

(i) \( k = 1 \), \( d = 2 \)

\[
\dim Q_1(K_{ref}) = 4 , \\
L_{[1]}(K_{ref}) = \{(0,0)^T, (1,0)^T, (0,1)^T, (1,1)^T\} ,
\]

(ii) \( k = 2 \), \( d = 2 \)

\[
\dim Q_2(K_{ref}) = 9 , \\
L_{[1]}(K_{ref}) = \{(0,0)^T, \frac{1}{2}, 0)^T, (1,0)^T, (0, \frac{1}{2})^T, (\frac{1}{2}, 1)^T, (1, \frac{1}{2})^T, (0,1)^T, (\frac{1}{2}, 1)^T, (1,1)^T\} .
\]

### Definition 2.18 Finite element spaces for rectangular elements and nodal bases

Let \( T_h \) be a triangulation of the bounded domain \( \Omega \subset \mathbb{R}^d \) by rectangular elements. The finite element space composed of rectangular elements \((K, Q_k(K), \Sigma_K), k \in \mathbb{N}\), will be denoted by

\[
S_{[k]}(\Omega; T_h) := \{ u_h : \bar{\Omega} \to \mathbb{R} \mid u_h|_K \in Q_k(K) , K \in T_h \} . \tag{2.26}
\]

The nodal basis \((\varphi_h^{(i)})_{i=1}^{n_h}, n_h := \text{card} \, \mathcal{N}_h(\bar{\Omega})\), is given by

\[
\varphi_h^{(i)}(x^{(j)}) = \delta_{ij} , \quad x^{(j)} \in L_{[k]}(K) , \quad 1 \leq i, j \leq n_h . \tag{2.27}
\]

In much the same way as for Lagrangian finite element spaces we can show:

### Lemma 2.19 Conformity of finite element spaces for rectangular elements

The finite element spaces \( S_{[k]}(\Omega; T_h), k \in \mathbb{N}\), are \( H^1\)-conforming, i.e., they are finite dimensional subspaces of \( H^1(\Omega) \).