3.2 Finite dimensional subspaces of $\mathbf{H}(\mathbf{div};\Omega)$

We consider conforming finite element approximations of the space $\mathbf{H}(\mathrm{div};\Omega)$ based on both simplicial and rectangular triangulations $\mathcal{T}_h$ of a simply-connected bounded domain $\Omega \subset \mathbb{R}^d$ in case $d = 2$ and $d = 3$. Throughout the following we denote by $\mathcal{N}_h, \mathcal{E}_h$, and $\mathcal{F}_h$ the sets of vertices, edges, and faces of the triangulation $\mathcal{T}_h$. In particular, for $D \subset \Omega$, we refer to $\mathcal{N}_h(D), \mathcal{E}_h(D)$, and $\mathcal{F}_h(D)$ as the sets of vertices, edges, and faces in D.
We further denote by $P_k(D)\ ,\ k \in \mathbb{N}_0$ and $\tilde{P}_k(D)\ ,\ k \in \mathbb{N}_0$ the set of polynomials of degree $\le k$ and the set of homogeneous polynomials of degree k on D. Moreover, $Q_{\ell,m,n}(D)\ ,\ \ell,m,n \in \mathbb{N}_0$ refers to the set of polynomials in $(x_1,x_2,x_3)^T \in D$ the maximum degree of which are ℓ in x_1, m in x_2, and n in x_3. If $\ell = m = n = k$, we simply write $Q_k(D)$ instead of $Q_{k,k,k}(D)$. Obvious modifications apply in case $D \subset \mathbb{R}^2$.

3.2.1 Conforming elements for $\mathbf{H}(\mathrm{div};\Omega)$

Let $\mathcal{T}_h$ be a triangulation of Ω and

$$P_K \ := \ \{\ \mathbf{q} = (q_1,...,q_d)^T \mid q_i : K \to \mathbb{R}\ ,\ 1 \le i \le d\ \}\ , \qquad (3.20)$$
$$\mathbf{V}_h(\Omega) \ := \ \{\ \mathbf{q}_h : \bar{\Omega} \to \mathbb{R} \mid \mathbf{q}_h|_K \in P_K\ ,\ K \in \mathcal{T}_h\ \}\ . \qquad (3.21)$$

The following result gives sufficient conditions for $\mathbf{V}_h(\Omega) \subset \mathbf{H}(\mathrm{div};\Omega)$.

Theorem 3.7 Sufficient conditions for conformity
Let $\mathcal{T}_h$ be a triangulation of Ω and let $P_K\ ,\ K \in \mathcal{T}_h$, and $\mathbf{V}_h(\Omega)$ be given by (3.20) and (3.21), respectively. Assume that

$$P_K \subset \mathbf{H}(\mathrm{div};K)\ ,\ K \in \mathcal{T}_h\ , \qquad (3.22)$$

$$[\mathbf{n}\cdot\mathbf{q}|_e] = 0 \quad \text{for all } e \ = \ K_i \cap K_j \in \mathcal{E}_h(\Omega)\ ,\ \mathbf{q} \in V_h(\Omega), d = 2\ , \qquad (3.23)$$
$$[\mathbf{n}\cdot\mathbf{q}|_f] = 0 \quad \text{for all } f \ = \ K_i \cap K_j \in \mathcal{F}_h(\Omega)\ ,\ \mathbf{q} \in V_h(\Omega), d = 3\ , \qquad (3.24)$$

where $[\mathbf{n}\cdot\mathbf{q}|_e]$ and $[\mathbf{n}\cdot\mathbf{q}|_f]$ denote the jump of $\mathbf{n}\cdot\mathbf{q}$ across e and f, i.e.,

$$[\mathbf{n}\cdot\mathbf{q}|_g] := \mathbf{n}\cdot\mathbf{q}|_{g\cap K_i} - \mathbf{n}\cdot\mathbf{q}|_{g\cap K_j}\ ,\ g := e\ (d=2) \text{ and } g := f\ (d=3)\ . \qquad (3.25)$$

Then $\mathbf{V}_h(\Omega) \subset \mathbf{H}(\mathrm{div};\Omega)$.

Proof: We prove the result in case $d = 3$. The case $d = 2$ can be shown in literally the same way.
Given $\mathbf{q}_h \in \mathbf{V}_h(\Omega)$, we have to show that $\mathrm{div}\mathbf{q}_h$ is well defined and $\mathrm{div}\mathbf{q}_h \in L^2(\Omega)$. In other words, we have to find $z_h \in L^2(\Omega)$ such that

$$\int_\Omega \mathbf{q}_h \cdot \mathbf{grad}\ \varphi\ d\mathbf{x} \ = \ - \int_\Omega z_h\ \varphi\ d\mathbf{x} \quad , \quad \varphi \in \mathcal{D}(\Omega)\ .$$

In view of (3.22), Green's formula can be applied elementwise:

$$\begin{aligned} \int_\Omega \mathbf{q}_h \cdot \mathbf{grad}\, \varphi \, d\mathbf{x} \; &= \; \sum_{K \in \mathcal{T}_h} \int_K \mathbf{q}_h \cdot \mathbf{grad}\, \varphi \, d\mathbf{x} \; = \\ = \; - \sum_{K \in \mathcal{T}_h} \int_K \mathrm{div} \mathbf{q}_h \, \varphi \, d\mathbf{x} \; &+ \; \sum_{K \in \mathcal{T}_h} \int_{\partial K} \mathbf{n} \cdot \mathbf{q}_h|_{\partial K} \, \varphi \, d\sigma \; = \\ = \; - \sum_{K \in \mathcal{T}_h} \int_K \mathrm{div} \mathbf{q}_h \, \varphi \, d\mathbf{x} \; &+ \; \sum_{f \in \mathcal{F}_h(\Omega)} \int_f [\mathbf{n} \cdot \mathbf{q}_h|_f] \, \varphi \, d\sigma \; . \end{aligned}$$

Taking advantage of (3.24), the assertion follows for z_h with $z_h|_K := \mathrm{div}\mathbf{q}_h$, $K \in \mathcal{T}_h$.

3.2.2 Raviart-Thomas elements $\mathbf{RT}_k(K)$

Let us first consider the case of simplicial triangulations $\mathcal{T}_h$ of Ω. For $K \in \mathcal{T}_h$ and $k \in \mathbb{N}_0$ we set

$$Rk(\partial K) \; := \; \{ \; \varphi \in L^2(\partial K) \; | \; \begin{cases} \varphi|_e \in P_k(e) \; , \; e \in \mathcal{E}_h(K) & , \quad d = 2 \\ \varphi|_f \in P_k(f) \; , \; f \in \mathcal{F}_h(K) & , \quad d = 3 \end{cases} \} \; .$$

Definition 3.8 Raviart-Thomas elements $\mathbf{RT}_k(K)$
Let K be a d-simplex. The Raviart-Thomas element $\mathbf{RT}_k(K), k \in \mathbb{N}_0$, is defined by

$$\mathbf{RT}_k(K) \; = \; P_k(K)^d \; + \; \mathbf{x} \; \tilde{P}_k(K) \; . \tag{3.26}$$

For $\mathbf{q} \in \mathbf{RT}_k(K)$, the degrees of freedom Σ_K are given by

$$\int_{\partial K} \mathbf{q} \cdot \mathbf{n} \; p_k \; d\sigma \quad , \quad p_k \in R_k(\partial K) \; , \tag{3.27}$$

$$\int_K \mathbf{q} \cdot \mathbf{p}_{k-1} \; d\mathbf{x} \quad , \quad \mathbf{p}_{k-1} \in P_{k-1}(K)^d \; . \tag{3.28}$$

We have

$$\dim \mathbf{RT}_k(K) \; = \; \begin{cases} (k+1)(k+3) & , \quad d = 2 \; , \\ \frac{1}{2} \; (k+1)(k+2)(k+4) & , \quad d = 3 \end{cases} \; . \tag{3.29}$$

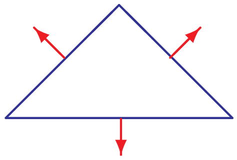

Fig. 3.1: $\mathbf{RT}_0(K)$ (d = 2)

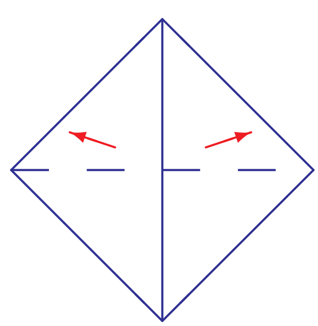

Fig. 3.2: $\mathbf{RT}_0(K)$ (d = 3)

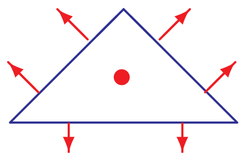

Fig. 3.3: $\mathbf{RT}_1(K)$ (d = 2)

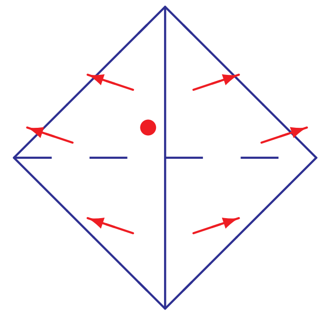

Fig. 3.4: $\mathbf{RT}_1(K)$ (d = 3)

Lemma 3.9 Range of the discrete divergence operator
For $\mathbf{q} \in \mathbf{RT}_k(K)$ we have

$$\operatorname{div} \mathbf{q} \in P_k(K) \ , \tag{3.30}$$
$$\mathbf{n} \cdot \mathbf{q}|_{\partial K} \in R_k(\partial K) \ . \tag{3.31}$$

Proof: According to (3.27), $\mathbf{q}$ can be written as

$$\mathbf{q} \ = \ \mathbf{p}_k \ + \ \mathbf{x} p_k \quad , \quad \mathbf{p}_k \in P_k(K)^d \ , \ p_k \in \tilde{P}_k(K) \ . \tag{3.32}$$

Then

$$\operatorname{div} \mathbf{q} \ = \ \operatorname{div} \mathbf{p}_k \ + \ \mathbf{x} \cdot \mathbf{grad}\ p_k \ + \ 3\ p_k \ = \ \operatorname{div} \mathbf{p}_k \ + \ (k+3)\ p_k \tag{3.33}$$

which gives (3.30).
Further, for $\mathbf{n} = (n_1, ..., n_d)^T$ we get

$$\mathbf{n} \cdot \mathbf{q} \ = \ \mathbf{n} \cdot \mathbf{p}_k \ + \ p_k \ \sum_{i=1}^d n_i \ x_i \ .$$

Since $\sum_{i=1}^d n_i \ x_i = const.$ on $e \subset \partial K$ resp. $f \subset \partial K$, this shows (3.31).

Lemma 3.10 Auxiliary result for unisolvence
Let $\mathbf{q} \in P_k(K)^d$ and assume

$$\int\limits_{\partial K} \mathbf{q} \cdot \mathbf{n}\ p_k \ d\sigma \ = \ 0 \quad , \quad p_k \in R_k(\partial K) \ , \tag{3.34}$$
$$\int\limits_{K} \mathbf{q} \cdot \mathbf{p}_{k-1} \ d\mathbf{x} \ = \ 0 \quad , \quad \mathbf{p}_{k-1} \in P_{k-1}(K)^d \tag{3.35}$$

Then

$$\operatorname{div} \mathbf{q} \ = \ 0 \ .$$

Proof: We apply Green's formula

$$\int\limits_K \operatorname{div} \mathbf{q}\, p_{k-1} \, d\mathbf{x} \ = \ - \underbrace{\int\limits_K \mathbf{q} \cdot \mathbf{grad}\ p_{k-1} \ d\mathbf{x}}_{=0} + \underbrace{\int\limits_{\partial K} \mathbf{n} \cdot \mathbf{q}\ p_{k-1} \ d\sigma}_{=0} \quad , \quad p_{k-1} \in P_{k-1}(K) \ .$$

Since div $\mathbf{q} \in P_{k-1}(K)$, we may choose $p_{k-1} = \text{div } \mathbf{q}$ which gives the assertion.

Theorem 3.11 Unisolvence of the $\mathbf{RT}_k(K)$ element
The element $(K, \mathbf{RT}_k(K), \Sigma_K)$ is unisolvent.

Proof: We have to show that the relations (3.34) and (3.35) imply $\mathbf{q} = 0$.
Due to (3.30) we have $\mathbf{n} \cdot \mathbf{q} \in R_k(\partial K)$ and hence, (3.34) implies $\mathbf{n} \cdot \mathbf{q} = 0$ on each edge resp. face.
In the same way as in Lemma 3.10 we deduce div $\mathbf{q} = 0$. Observing Lemma 3.10 and (3.33), we get $p_k = 0$.
Taking advantage of the affine equivalence, we consider the reference tetrahedron (cf. Figure 3.5).

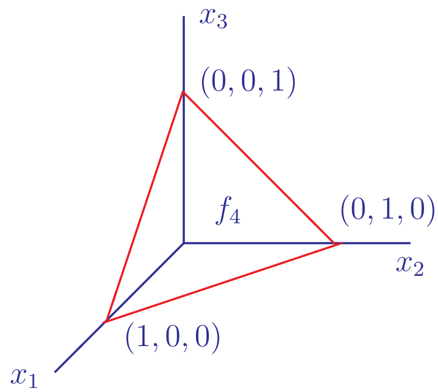

Fig. 3.5: The reference tetrahedron

In lights of the results obtained so far, we may assume

$$\hat{q}_i \ = \ \hat{x}_i \ \hat{\psi}_i \quad , \quad \hat{\psi}_i \in P_{k-1}(\hat{K}) \ , \ 1 \leq i \leq 3 \ .$$

If we choose $\hat{\mathbf{p}}_{k-1} := (\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3)^T$ in (3.33), we obtain

$$\sum_{i=1}^{3} \int_{\hat{K}} \hat{x}_i \ \hat{\psi}_i^2 \ d\mathbf{x} \ = \ 0$$

whence $\hat{\psi}_i = 0 \ , \ 1 \leq i \leq 3$, and thus $\hat{\mathbf{q}} = 0$.

Definition 3.12 The Raviart-Thomas finite element space $\mathbf{RT}_k(\Omega; \mathcal{T}_h)$
The Raviart-Thomas finite element space $\mathbf{RT}_k(\Omega; \mathcal{T}_h)$ is given by

$$\mathbf{RT}_k(\Omega; \mathcal{T}_h) \ := \ \{ \ \mathbf{q} \in L^2(\Omega)^d \mid \mathbf{q}|_K \in \mathbf{RT}_k(K) \ , \ K \in \mathcal{T}_h \ \} \ . \tag{3.36}$$

It is a finite dimensional subspace of $\mathbf{H}(\text{div}; \Omega)$.

3.2.3 Raviart-Thomas elements $\mathbf{RT}_{[k]}(K)$

Let $\mathcal{T}_h$ be a rectangular triangulation of Ω. In much the same way as before we define

$$Q_k(\partial K) \;:=\; \{\mathbf{q} \in L^2(K) \mid \left\{ \begin{array}{ll} \mathbf{q}|_e \in P_k(e)^2 \;,\; e \in \mathcal{E}_h(K) & ,\quad d=2 \\ \mathbf{q}|_f \in Q_k(f) \;,\; f \in \mathcal{F}_h(K) & ,\quad d=3 \end{array} \right\} \;. \tag{3.37}$$

Moreover, we set

$$\Psi_k(K) \;:=\; \left\{ \begin{array}{ll} Q_{k-1,k}(K) \times Q_{k,k-1}(K) & ,\quad d=2\;, \\ Q_{k-1,k,k}(K) \times Q_{k,k-1,k}(K) \times Q_{k,k,k-1}(K) & ,\quad d=3 \end{array} \right. \;. \tag{3.38}$$

Definition 3.13 Raviart-Thomas elements $\mathbf{RT}_{[k]}(K)$
Let K be a d-rectangle. The Raviart-Thomas element $\mathbf{RT}_{[k]}(K)\;,\; k \in \mathbb{N}_0$, is defined by

$$\mathbf{RT}_{[k]}(K) \;=\; Q_k(K)^d \;+\; \mathbf{x}\; Q_k(K)\;. \tag{3.39}$$

For $\mathbf{q} \in RT_{[k]}(K)$, the degrees of freedom Σ_K are given by

$$\int_{\partial K} \mathbf{q} \cdot \mathbf{n}\; d\sigma \quad,\quad \mathbf{p}_k \in Q_k(\partial K)\;, \tag{3.40}$$

$$\int_K \mathbf{q} \cdot \mathbf{p}_k\; d\mathbf{x} \quad,\quad \mathbf{p}_k \in \Psi_k(K)\;. \tag{3.41}$$

We have

$$\dim\, \mathbf{RT}_{[k]}(K) \;=\; \left\{ \begin{array}{ll} 2\;(k+1)(k+2) & ,\quad d=2\;, \\ 3\;(k+1)^2(k+2) & ,\quad d=3 \end{array} \right. \;. \tag{3.42}$$

Theorem 3.14 Unisolvence of the $\mathbf{RT}_{[k]}(K)$ element
The element $(K, \mathbf{RT}_{[k]}(K), \Sigma_K)$ is unisolvent.

Proof: The proof is left as an exercise.

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Definition 3.15 The Raviart-Thomas finite element space $\mathbf{RT}_{[k]}(\Omega;\mathcal{T}_h)$
The Raviart-Thomas finite element space $\mathbf{RT}_k(\Omega;\mathcal{T}_h)$ is given by

$$\mathbf{RT}_{[k]}(\Omega;\mathcal{T}_h) \;:=\; \{\; \mathbf{q} \in L^2(\Omega)^d \mid \mathbf{q}|_K \in \mathbf{RT}_{[k]}(K)\;,\; K \in \mathcal{T}_h\;\}\;. \tag{3.43}$$

It is a finite dimensional subspace of $\mathbf{H}(\mathrm{div};\Omega)$.

3.3 Boundary element approximation of the boundary integral formulation of Maxwell's exterior domain problem

We remind that the variational formulation of the boundary integral approach to the exterior domain problem for the time-harmonic Maxwell equations is as follows:
Find $\mathbf{j} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma)$ such that for all $\mathbf{q} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma)$:

$$\begin{aligned}
&\int_\Gamma [\mathbf{n}_\mathbf{x}(\mathbf{x}) \wedge \int_\Gamma G(\mathbf{x},\mathbf{y})\ \mathbf{j}(\mathbf{y})\ d\boldsymbol{\sigma}(\mathbf{y}) \wedge \mathbf{n}_\mathbf{x}(\mathbf{x})] \cdot \overline{\mathbf{q}}(\mathbf{x})\ d\boldsymbol{\sigma}(\mathbf{x}) \ - \\
&- \frac{1}{k^2} \int_\Gamma \int_\Gamma G(\mathbf{x},\mathbf{y})\ \mathrm{div}_\Gamma\ \mathbf{j}(\mathbf{y})\ \mathrm{div}_\Gamma \overline{\mathbf{q}}(\mathbf{x})\ d\boldsymbol{\sigma}(\mathbf{y})\ d\boldsymbol{\sigma}(\mathbf{x}) \ = \\
&= \frac{1}{i\omega\mu_0} \int_\Gamma \boldsymbol{\pi}_\mathbf{t}(\mathbf{E}^{inc})(\mathbf{x}) \cdot \overline{\mathbf{q}}(\mathbf{x})\ d\sigma(\mathbf{x}) \ .
\end{aligned} \tag{3.44}$$

We assume that $\mathcal{T}_h$ is a geometrically conforming simplicial triangulation of Γ and approximate $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma)$ by the Raviart-Thomas finite element space $RT_k(\Gamma,\mathcal{T}_h), k \in \mathbb{N}_0$. The Galerkin approximation of (3.44) then reads as follows:
Find $\mathbf{j}_h \in RT_k(\Gamma,\mathcal{T}_h)$ such that for all $\mathbf{q}_h \in RT_k(\Gamma,\mathcal{T}_h)$:

$$\begin{aligned}
&\int_\Gamma [\mathbf{n}_\mathbf{x}(\mathbf{x}) \wedge \int_\Gamma G(\mathbf{x},\mathbf{y})\ \mathbf{j}_h(\mathbf{y})\ d\boldsymbol{\sigma}(\mathbf{y}) \wedge \mathbf{n}_\mathbf{x}(\mathbf{x})] \cdot \overline{\mathbf{q}_h}(\mathbf{x})\ d\boldsymbol{\sigma}(\mathbf{x}) \ - \\
&- \frac{1}{k^2} \int_\Gamma \int_\Gamma G(\mathbf{x},\mathbf{y})\ \mathrm{div}_\Gamma\ \mathbf{j}_h(\mathbf{y})\ \mathrm{div}_\Gamma \overline{\mathbf{q}}_h(\mathbf{x})\ d\boldsymbol{\sigma}(\mathbf{y})\ d\boldsymbol{\sigma}(\mathbf{x}) \ = \\
&= \frac{1}{i\omega\mu_0} \int_\Gamma \boldsymbol{\pi}_\mathbf{t}(\mathbf{E}^{inc})(\mathbf{x}) \cdot \overline{\mathbf{q}}_h(\mathbf{x})\ d\sigma(\mathbf{x}) \ .
\end{aligned} \tag{3.45}$$