3.5 Panel-Clustering

The panel clustering provides an efficient algorithmic tool for the numerical approximation of matrix-vector products in boundary element methods for the weak formulation of boundary integral equations.

3.5.1 Cluster and Cluster-Trees

The fundamental concepts in panel clustering are based on the notions of clusters and cluster-trees.

**Definition 3.25 Cluster**

Let $T_h := \{\Delta_1, ..., \Delta_m\}$ be a triangulation of $\Gamma$. A cluster $\tau$ is a nonempty union of elements of $T_h$, i.e.,

$$\tau = \bigcup_{i \in I} \Delta_i , \quad \emptyset \neq I \subset \{1, ..., m\} . \quad (3.88)$$

For a cluster $\tau$ we denote by $B_{\rho_\tau}(z_\tau)$ the smallest ball containing $\tau$, i.e.,

$$\tau \subset B_{\rho_\tau}(z_\tau) , \quad \tau \subset B_{\tilde{\rho}}(\tilde{z}) \Rightarrow \tilde{\rho} = \rho_\tau , \quad \tilde{z} = z_\tau .$$

$z_\tau$ and $\rho_\tau$ are called the center and the radius of the cluster.

**Definition 3.26 Cluster-Tree**

A set

$$C := \{\tau_1, ..., \tau_m\}$$

of clusters is called a cluster-tree, if the following conditions are satisfied

$(C_1)$ $\Gamma \in C$ and $T_h \subset C$ ,

$(C_2)$ If $\tau_1, \tau_2 \in C$ are two different clusters in $C$, then either $\tau_1 \subset \tau_2$, or $\tau_2 \subset \tau_1$, or the intersection of their interiors is empty, $\tau_1^{int} \cap \tau_2^{int} = \emptyset$ ,

$(C_3)$ If $\sigma \in C$ and $\tau \in C \setminus T_h$, then $\sigma$ is called the son of $\tau$ ($\sigma = s(\tau)$), and $\tau$ is called the father of $\sigma$ ($\tau = f(\sigma)$), if

$$\sigma \subset \tau \quad \text{and} \quad \sigma \in \tau' \subset \tau \quad \Rightarrow \quad \sigma = \tau' ,$$

$(C_4)$ There exists a constant $C$ independent of $h$, such that for all $\tau \in C \setminus T_h$

$$1 < \text{card } s(\tau) \leq C . \quad (3.89)$$

The properties $(C_1) - (C_4)$ uniquely define a tree structure. The root of the tree is $\Gamma$ and the leaves are the elements of $T_h$. 
**Definition 3.27  Level of a cluster**

The level $\ell(\tau)$ of a cluster $\tau \in \mathcal{C}$ is the maximum distance to a leave in $\mathcal{C}$, i.e.,

$$
\ell(\tau) := \begin{cases} 
0, & \tau \in \mathcal{T}_h, \\
1 + \max_{\sigma = s(\tau)} \ell(\sigma), & \tau \in \mathcal{C} \setminus \mathcal{T}_h.
\end{cases}
$$

(3.90)

**Definition 3.28  Admissible clusters, near- and far-field**

For $\eta \in (0, 1)$, a cluster $\tau \in \mathcal{C}$ is called $\eta$-admissible with respect to $\Delta \in \mathcal{T}_h$, if

$$
\eta \, \text{dist}(\Delta, z_\tau) \geq \rho_\tau.
$$

(3.91)

The set of $\eta$-admissible clusters w.r.t. $\Delta \in \mathcal{T}_h$ will be denoted by $\mathcal{A}_\eta(\Delta)$.

For $\Delta \in \mathcal{T}_h$ the sets

$$
\mathcal{F}_\eta(\Delta) := \{ \tau \in \mathcal{C} \setminus \mathcal{T}_h \mid \tau \in \mathcal{A}_\eta(\Delta), \ f(\tau) \notin \mathcal{A}_\eta(\Delta) \},
$$

(3.92)

$$
\mathcal{N}_\eta(\Delta) := \{ \hat{\Delta} \in \mathcal{T}_h \mid f(\hat{\Delta}) \notin \mathcal{A}_\eta(\Delta) \}
$$

(3.93)

are called the far-field of $\Delta$ and the near-field of $\Delta$, respectively.

The far-field $\mathcal{F}_\eta(\Delta)$ admits a partition

$$
\mathcal{F}_\eta(\Delta) = \bigcup_{k=1}^L S^{(k)}_\eta(\Delta),
$$

(3.94)
where
\[ S^{(k)}_{\eta}(\Delta) := \{ \tau \in F_{\eta}(\Delta) \mid \ell(\tau) = k \} , \quad 1 \leq k \leq L . \] (3.95)

The union of the near- and the far-field
\[ N_{\eta}(\Delta) \cup F_{\eta}(\Delta) \] (3.96)
is called an \( \eta \)-admissible covering of \( \Gamma \).

The following result illustrates the dependence of the near- and the far-field on \( \eta \):

**Theorem 3.29** Assume that \( C \) is a cluster tree and \( \eta \in (0,1) \). Then, there exists a constant \( C_{\Gamma} \) independent of \( h \), such that
\[ |N_{\eta}(\Delta)| \leq C_{\Gamma} \left( \frac{1}{\eta} \right)^2 , \] (3.97)
\[ |F_{\eta}(\Delta)| \leq C_{\Gamma} \left( \frac{1}{\eta} \right)^2 \log m . \] (3.98)


---

### 3.5.2 Recursive computation of matrix-vector products

We now consider the efficient evaluation of the matrix-vector product \( Au \) where \( u = (u_1, \ldots, u_n)^T \in \mathbb{C}^n \) and \( A = (a_{ij})_{i,j=1}^n \in \mathbb{C}^{n \times n} \) with
\[ a_{ij} = \int \int k(x, y) \psi_i(x) \psi_j(y) \, d\sigma(y) \, d\sigma(x) . \]

Note that \( V_h := \text{span} \{ \psi_1, \ldots, \psi_N \} \).

We assume that we are given a hierarchical partition of \( \Gamma \) into clusters. Then, the evaluation of \( Au \) typically amounts to the computation of terms of the form
\[ (Au)_i = \sum_{j=1}^N \left[ \int \int_{\Delta \cap \tau} k(x, y) \psi_i(x) \psi_j(y) \, d\sigma(y) \, d\sigma(x) \right] u_j , \] (3.99)
where \( \Delta \subset \text{supp} (\psi_i) \) and \( \tau \subset \Gamma \).

We further assume that for \( \tau \in F_{\eta}(\Delta) \) we know an approximation of the kernel \( k(x, y) \) by means of an appropriate system of functions \( \{\omega_\nu\} \) and \( \{\sigma_\nu\} \) according to
\[ k(x, y) \approx \sum_{|\nu|=0}^{p-1} \sum_{\mu=0}^{\nu} \kappa_{\nu,\mu}(z_\Delta - z_\tau) \, \omega_\nu(x) \, \sigma_\mu(y) , \] (3.100)
where \( \nu, \mu \in \mathbb{N}_0^3 \) are multiindices and the second sum in (3.100) has to be understood componentwise.

Using (3.100) in (3.99), we obtain

\[
(Au)_i \approx \sum_{|\nu|=0}^{p-1} \sum_{\mu=0}^{\nu} \kappa_{\nu,\mu}(z_\Delta - z_\tau) \int_\Delta \omega_\nu(x) \psi_i(x) \, d\sigma(x) \cdot \kappa_{\nu,\mu}(\Delta, \tau, i) \\
+ \sum_{j=1}^N \left[ \int_\Gamma \sigma_\mu(y) \psi_j(y) \, d\sigma(y) \right] u_j .
\]

= J_\mu^\tau(u)

We note that the expansion coefficients \( \kappa_{\nu,\mu}(\Delta, \tau, i) \) do not depend on \( u \) and thus can be computed previous to the matrix-vector product.

The far-field coefficients \( J_\mu^\tau(u) \) can be efficiently computed using the tree structure of the cluster:

For \( \Delta \in T_h \) we set

\[
J_\Delta^\mu(u) := \sum_{j=1}^N \left[ \int_\Delta \sigma_\mu(y) \psi_j(y) \, d\sigma(y) \right] u_j .
\]

Then, for all clusters \( \tau \in \mathcal{C} \) we obtain the recursive scheme

\[
J_\tau^\mu(u) = \sum_{\tau' \in s(\tau)} J_{\tau'}^\mu(u) .
\]

**Panel clustering with polynomial systems**

The expansion (3.101) shows that polynomial systems are not appropriate in the approximation of strongly oscillating kernels:

Let us assume that the cluster \( \tau \) consists of \( m_\tau \) elements of the triangulation \( T_h \). Then, the quasiuniformity of \( T_h \) implies

\[
m_\tau \approx |\tau| \approx \rho_\tau^2 \implies m_\tau \approx \left( \frac{\rho_\tau}{h} \right)^2 ,
\]

where \( \rho_\tau \) denotes the radius of the cluster.

For strongly oscillating kernels we have \( kh \approx 1 \). We already know that the degree \( p \) of the polynomials has to satisfy

\[
p \geq k \rho_\tau .
\]

Hence, for the approximation of the

\[
O(m_\tau) = O((\frac{\rho_\tau}{h})^2) = O((k\rho_\tau)^2)
\]

elements we need \( O((k\rho_\tau)^3) \) terms in the expansion.
Panel clustering with systems of spherical functions

The approximation of kernel functions by systems of spherical functions has the advantage that only two-dimensional expansions have to be considered which additionally can be performed by a local separation of the variables $x$ and $y$.

Given a cluster $\tau \in C$ with radius $\rho_\tau$ and center $z_\tau$, for $d \in \mathbb{N}$ and $\eta \in (0, 1)$ we define an index set

$$\Lambda_\tau := \{ (\nu_1, \nu_2) \in \mathbb{Z}^2 \mid 0 \leq \nu_1 \leq k\rho_\tau + d, |\nu_2| \leq \nu_1 \}.$$  \hspace{1cm} (3.103)

Then, for $x \in \Delta_x$ and $y \in \tau, \tau \in F_\eta(\Delta_x)$, the kernel function can be approximated as follows:

$$k(x, y) = k_d(x, y) + R_d = \sum_{\nu \in \Lambda_d} G_\nu(x - z_\nu) I_\nu(y - z_\tau, y) + R_d(x, y),$$  \hspace{1cm} (3.104)

where the remainder $R_d$ satisfies the estimate

$$|R_d(x, y)| \leq C \eta_d |k(x, z_\tau)|.$$  \hspace{1cm} (3.105)

According to Theorem 3.xx, the $|\Lambda_\tau|$-values of $G_\nu$ and $I_\nu$ can be computed by $O(|\Lambda_\tau|)$ operations.

However, a price has to be paid for the two-dimensional approximation of the kernel:

The integrals of $G_\nu$ w.r.t. $\Delta \in T_h$ have to be computed separately for each cluster.

In particular, for the $i$-th component of the matrix-vector product $Au$ we obtain

$$(Au)_i = \sum_{\Delta_x \subseteq \text{supp } \psi_i} \int_{\Delta_x} \left[ \sum_{j=1}^{N} \int_{\text{supp } \psi_j} k(x, y) \psi_j(y) \, d\sigma(y) \right] u_j \, d\sigma(x) =$$

$$= \int_{\Delta_x \subseteq \text{supp } \psi_i} \left[ \sum_{\Delta_y} \sum_{j: \Delta_y \subseteq \text{supp } \psi_j} \int_{\Delta_y} k(x, y) \psi_j(y) \, d\sigma(y) \right] u_j \, d\sigma(x) =$$

$$= \int_{\Delta_x \subseteq \text{supp } \psi_i} \left[ \sum_{\Delta_y \in \mathcal{N}_\eta(\Delta_x)} \sum_{j: \Delta_y \subseteq \text{supp } \psi_j} \int_{\Delta_y} k(x, y) \psi_j(y) \psi_j(y) \, d\sigma(y) \right] u_j \, d\sigma(x) +$$

$$+ \sum_{\Delta_x \subseteq \text{supp } \psi_i} \left[ \sum_{\tau \in F_\eta(\Delta_x)} \sum_{j: \tau \subseteq \Delta_y \subseteq \text{supp } \psi_j} \int_{\Delta_y} k(x, y) \psi_j(y) \psi_j(y) \, d\sigma(y) \right] u_j \, d\sigma(x) =$$

$$(Nu)_i + (Fu)_i, \quad 1 \leq i \leq N,$$

where $(Nu)_i$ and $(Fu)_i$ represent the near-field and the far-field contributions, respectively.

Using the kernel approximation (3.104), for the far-field contribution $(Fu)_i$ we
get

\[(F \mathbf{u})_i = \sum_{\Delta_x \subset \text{supp} \psi} \int_{\Delta_x} \psi_i(x) \left[ \sum_{\tau \in \mathcal{F}_n(\Delta_x)} \sum_{\Delta_y \subset \text{supp} \psi_j} \left\{ \int_{\Delta_y} k(x, y) \psi_j(y) d\sigma(y) \right\} u_j \right] d\sigma(x) =
\]

\[= \sum_{\Delta_x \subset \text{supp} \psi} \int_{\Delta_x} \psi_i(x) \left[ \sum_{\tau \in \mathcal{F}_n(\Delta_x)} \sum_{\nu \in \Lambda_{\tau}} G_{\nu}(x - z_\tau) \right. \]

\[\left. \sum_{\Delta_y \subset \text{supp} \psi_j} \left\{ \int_{\Delta_y} I_{\nu}(y - z_\tau, y) \psi_j(y) d\sigma(y) \right\} u_j \right] d\sigma(x) + \bar{R}_i =
\]

\[= \sum_{\tau \in \mathcal{C}} \sum_{\nu \in \Lambda_{\tau}} c_{i, (\tau, \nu)} F_{(\tau, \nu)} + \bar{R}_i,
\]

where \(c_{i, (\tau, \nu)}, 1 \leq i \leq n, (\tau, \nu) \in \mathcal{C} \times \Lambda_{\tau},\) are the elements of the cluster matrix \(C:\)

\[c_{i, (\tau, \nu)} := \sum_{\Delta_x \subset \text{supp} \psi} \int_{\Delta_x} \psi_i(x) G_{\nu}(x - z_\tau) d\sigma(x) . \quad (3.107)\]

Note that the terms \(F_{(\tau, \nu)}, (\tau, \nu) \in \mathcal{C} \times \Lambda_{\tau},\) are referred to as the far-field coefficients which can be computed by means of the transformation matrix \(T = (t_{(\tau, \nu), j}), (\tau, \nu) \in \mathcal{C} \times \Lambda_{\tau}, 1 \leq j \leq n,\) given by

\[F_{(\tau, \nu)} = \sum_{j=1}^{n} t_{(\tau, \nu), j} u_j . \quad (3.108)\]

Consequently, the panel clustering can be interpreted as an approximate factorization of the stiffness matrix \(A\) according to

\[A \simeq N + C T , \quad (3.109)\]
3.5.3 Computational work and numerical results

We want to estimate the amount of computational work for the implementation of panel clustering using spherical functions in the approximation of the integral kernels.

For this purpose, we make the following assumptions on the cluster:

- The diameter of $\Omega$ is 1,
- The cluster tree $C$ is a quadruple tree of height $L$ with $|T_h| = m = 4^L$,
- For a cluster $\tau$ of stage $\ell$ there holds $\rho_\tau \leq 2^\ell h$,
- For the index set $\Lambda_\tau := \{(\nu_1, \nu_2) \in \mathbb{Z}^2 \mid 0 \leq |\nu_2| \leq \nu_1 \leq k\rho_\tau + d\}$ there holds $|\Lambda_\tau| = (k\rho_\tau + d)^2$ and $\eta^d = (kh)^{\sigma}$
- For all $\Delta \in T_h$ and $1 \leq \ell \leq L$ there holds $|S_\ell(\Delta)| \leq C_\Gamma / \eta^2$

Denoting by $c_i$ the number of nonvanishing entries in the $i$-th row of the cluster matrix $C$, we obtain the estimate

\[
c_i \leq \sum_{\Delta_x \subset \text{supp } \Psi_i} \sum_{\tau \in \mathcal{F}_i(\Delta_x)} |\Lambda_\tau| = \sum_{\Delta_x \subset \text{supp } \Psi_i} \sum_{\ell=1}^{L-1} \sum_{\tau \in \mathcal{S}_i(\Delta_x)} |\Lambda_\tau| =
\]

\[
= \sum_{\Delta_x \subset \text{supp } \Psi_i} \sum_{\ell=1}^{L-1} \sum_{\tau \in \mathcal{S}_i(\Delta_x)} (k\rho_\tau + d)^2 \leq \frac{C_\Gamma}{\eta^2} \sum_{\ell=1}^{L-1} (2^\ell k h + d^2) =
\]

\[
= \frac{C_\Gamma}{\eta^2} \left( (L-1) d^2 + \frac{4}{3} k^2 h^2 (4^{L-1} - 1) + 4 k h d (2^{L-1} - 1) \right) \leq \frac{C_\Gamma}{\eta^2} \left( d^2 \log_4 m + \tilde{C}_\Gamma k^2 + 2 k d \sqrt{\tilde{C}_\Gamma} \right) \quad \text{where } \tilde{C}_\Gamma \geq m h^2.
\]

In order to determine the asymptotic behavior we assume that the wave number $k$ is fixed and study the case $h \to 0$. We have

\[
d = O(k \log h) \quad \Rightarrow \quad c_i = O\left(\frac{\log^3 h}{\eta^3 \log^2 \eta} \right)
\]

The optimal value of $\eta$ is $\eta^* = 1/e$ with

\[
c_i \approx |\log^3 h| = O(\log^3 N)
\]

Likewise, to estimate the computational work associated with the transformation matrix $T$ we suppose that for each $\Delta \in T_h$ there exists a sequence of clusters $\tau_\ell(\Delta)$, $1 \leq \ell \leq L - 1$, such that

\[
\Delta \subset \tau_1(\Delta) \subset \tau_2(\Delta) \subset \ldots \subset \tau_{L-1}(\Delta).
\]

Then, for the number $t_j$ of nonvanishing elements in the $j$-th column of $T$ we
obtain

\[ t_j \leq \sum_{\Delta \subset \text{supp } \Psi_j} \sum_{\tau \in C, \Delta \subset \tau} |\Lambda_{\tau}| = \sum_{\Delta \subset \text{supp } \Psi_j} \sum_{\ell=1}^{L-1} |\Lambda_{\tau}(\Delta)| = \]

\[ = \sum_{\Delta \subset \text{supp } \Psi_j} \sum_{\ell=1}^{L-1} (k \rho_{\tau}(\Delta) + d)^2 \leq \tilde{C} \sum_{\ell=1}^{L-1} (k 2^\ell h + d)^2 = \]

\[ \leq \tilde{C} \left( \frac{k^2}{3} + d^2 \log_4 m + 2 k d \sqrt{\tilde{C}} \right) \implies t_j = O(|\log^3 h|) . \]

The following figure shows the execution times of the panel cluster method as a function of the parameter \( \eta \) confirming the theoretical results obtained above:

![Figure 3.7: Execution times as a function of \( \eta \)](image)

We now give numerical results for the panel clustering applied to the acoustic scattering of a plane incident wave by an

- octahedron,
- ikosaeder.

Both geometrical objects are displayed in Figure 3.8.
In the following tables, \( n \) stands for the number of elements of the finest triangulation \( T_h \), \( t_{\text{near}} \) stands for the computational time to evaluate the near-field matrix \( N \), \( t_{\text{far}} \) denotes the computational time for the evaluation of the far-field matrix \( F \) in the naive way using quadrature formulas for computing the integrals, and \( t_{\text{panel}} \) represents the computational time for the evaluation of the far-field matrix \( F \) using panel clustering as described in 3.5.2. Finally, 

Error refers to the relative error

\[
\frac{\| (F - CT)x \|_2}{\| Fx \|_2},
\]

where \( \| \cdot \|_2 \) is the Euclidean norm.

Table 3.1 contains the results in case where elementwise constant functions have been used in the approximation of the solution and the wave number \( k \) was fixed (\( k = 1 \)):

<table>
<thead>
<tr>
<th>Geometry</th>
<th>( n )</th>
<th>( kh )</th>
<th>( t_{\text{near}} )</th>
<th>( t_{\text{far}} )</th>
<th>( t_{\text{panel}} )</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iko</td>
<td>320</td>
<td>3.24E-01</td>
<td>11.0</td>
<td>9.9</td>
<td>0.6</td>
<td>9.0E-02</td>
</tr>
<tr>
<td>Iko</td>
<td>1280</td>
<td>1.64E-01</td>
<td>46.0</td>
<td>19.7</td>
<td>5.0</td>
<td>4.7E-02</td>
</tr>
<tr>
<td>Iko</td>
<td>5120</td>
<td>8.26E-02</td>
<td>592.0</td>
<td>355.2</td>
<td>37.0</td>
<td>2.5E-02</td>
</tr>
<tr>
<td>Octa</td>
<td>128</td>
<td>5.77E-01</td>
<td>1.0</td>
<td>0.05</td>
<td>0.04</td>
<td>5.8E-01</td>
</tr>
<tr>
<td>Octa</td>
<td>512</td>
<td>3.01E-01</td>
<td>17.9</td>
<td>2.45</td>
<td>0.90</td>
<td>1.3E-01</td>
</tr>
<tr>
<td>Octa</td>
<td>2048</td>
<td>1.52E-02</td>
<td>221.3</td>
<td>55.5</td>
<td>10.89</td>
<td>4.8E-02</td>
</tr>
<tr>
<td>Octa</td>
<td>8192</td>
<td>7.64E-02</td>
<td>981.1</td>
<td>910.2</td>
<td>79.79</td>
<td>3.5E-02</td>
</tr>
<tr>
<td>Octa</td>
<td>32768</td>
<td>3.84E-02</td>
<td>9315.2</td>
<td>15799.6</td>
<td>627.36</td>
<td>1.2E-02</td>
</tr>
</tbody>
</table>

Table 3.1: Execution times and relative error (\( k = 1 \) and \( \eta = 1/e \))

(elementwise constant functions)
Table 3.2 contains the corresponding results for variable wave number $k$, but constant value of $kh$:

<table>
<thead>
<tr>
<th>Geom.</th>
<th>n</th>
<th>k</th>
<th>kh</th>
<th>$t_{near}$</th>
<th>$t_{far}$</th>
<th>$t_{panel}$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iko</td>
<td>320</td>
<td>$\pi/2$</td>
<td>0.51</td>
<td>2.41</td>
<td>1.25</td>
<td>0.66</td>
<td>9.2E-02</td>
</tr>
<tr>
<td>Iko</td>
<td>1280</td>
<td>$\pi$</td>
<td>0.51</td>
<td>44.07</td>
<td>23.6</td>
<td>5.4</td>
<td>6.1E-02</td>
</tr>
<tr>
<td>Iko</td>
<td>5120</td>
<td>$2\pi$</td>
<td>0.51</td>
<td>171.22</td>
<td>393.8</td>
<td>39.91</td>
<td>9.2E-02</td>
</tr>
<tr>
<td>Iko</td>
<td>20480</td>
<td>$4\pi$</td>
<td>0.52</td>
<td>684.96</td>
<td>6162.6</td>
<td>300.2</td>
<td>5.2E-02</td>
</tr>
<tr>
<td>Octa</td>
<td>128</td>
<td>$\pi/4$</td>
<td>0.57</td>
<td>1.06</td>
<td>0.04</td>
<td>0.05</td>
<td>5.6E-01</td>
</tr>
<tr>
<td>Octa</td>
<td>512</td>
<td>$\pi/2$</td>
<td>0.47</td>
<td>4.2</td>
<td>3.18</td>
<td>1.07</td>
<td>1.2E-02</td>
</tr>
<tr>
<td>Octa</td>
<td>2048</td>
<td>$\pi$</td>
<td>0.48</td>
<td>72.0</td>
<td>60.76</td>
<td>8.65</td>
<td>6.2E-02</td>
</tr>
<tr>
<td>Octa</td>
<td>8192</td>
<td>$2\pi$</td>
<td>0.48</td>
<td>279.1</td>
<td>1007.11</td>
<td>62.35</td>
<td>8.3E-02</td>
</tr>
</tbody>
</table>

Table 3.2: Execution times and relative error ($kh \approx 0.5$ and $\eta = 1/e$) (elementwise constant functions)

Finally, Table 3.3 documents the results both for fixed $k$ and variable $k$ (but constant $kh$) in case of piecewise linear approximations:

<table>
<thead>
<tr>
<th>Geom.</th>
<th>n</th>
<th>k</th>
<th>kh</th>
<th>$t_{near}$</th>
<th>$t_{far}$</th>
<th>$t_{panel}$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iko</td>
<td>162</td>
<td>1</td>
<td>3.24E-01</td>
<td>35.56</td>
<td>11.6</td>
<td>2.5</td>
<td>5.7E-03</td>
</tr>
<tr>
<td>Iko</td>
<td>642</td>
<td>1</td>
<td>1.64E-01</td>
<td>173.19</td>
<td>244.63</td>
<td>24.46</td>
<td>2.9E-04</td>
</tr>
<tr>
<td>Iko</td>
<td>2562</td>
<td>1</td>
<td>8.26E-02</td>
<td>1158.89</td>
<td>4227.93</td>
<td>191.53</td>
<td>2.5E-04</td>
</tr>
<tr>
<td>Iko</td>
<td>162</td>
<td>$\pi/2$</td>
<td>0.51</td>
<td>28.57</td>
<td>11.6</td>
<td>1.6</td>
<td>1.9E-02</td>
</tr>
<tr>
<td>Iko</td>
<td>642</td>
<td>$\pi$</td>
<td>0.51</td>
<td>143.64</td>
<td>267.6</td>
<td>28.93</td>
<td>5.1E-02</td>
</tr>
<tr>
<td>Iko</td>
<td>2562</td>
<td>$2\pi$</td>
<td>0.51</td>
<td>317.25</td>
<td>4632.9</td>
<td>327.98</td>
<td>2.3E-02</td>
</tr>
</tbody>
</table>

Table 3.3: Execution times and relative error ($\eta = 1/e$) (piecewise linear functions)