4.3 Hybrid iterative solver for the edge element discretized eddy currents equations

We are dealing with the edge element discretization of eddy currents equations in their weak formulation

\[ a(j, q) = \ell(q) \ , \quad q \in H_0(\text{curl}; \Omega) . \]  

(4.63)

Here, \( a(\cdot, \cdot) : H_0(\text{curl}; \Omega) \times H_0(\text{curl}; \Omega) \to \mathbb{R} \) is the bilinear form

\[ a(j, q) = \int_{\Omega} \left[ \alpha \text{ curl } j \cdot \text{ curl } q + \beta j \cdot q \right] \, dx \]  

(4.64)

with \( \alpha \) and \( \beta \) being symmetric, uniformly positive definite and positive semi-definite matrix-valued functions on \( \Omega \) with \( L^\infty \)-entries, respectively, whereas \( \ell(\cdot) : H_0(\text{curl}; \Omega) \to \mathbb{R} \) is the functional

\[ \ell(q) = \int_{\Omega} f \cdot q \, dx \]  

(4.65)

for some given \( f \in L^2(\Omega)^3 \) satisfying \( \text{div } f = 0 \).

Now, considering a simplicial triangulation \( T_h \) of \( \Omega \), we use the lowest order edge elements of Nédelec’s first family for the approximate solution of (4.63):

Find \( j_h \in Nd_{1,0}(\Omega, T_h) \) such that

\[ a(j_h, q_h) = \ell(q_h) \ , \quad q_h \in Nd_{1,0}(\Omega, T_h) . \]  

(4.66)

Assuming

\[ Nd_{1,0}(\Omega, T_h) = \text{span} \{ j_h^{(i)} \mid 1 \leq i \leq n_h \} \]

and setting

\[ j_h = \sum_{i=1}^{n_h} x_i j_h^{(i)} , \]

the finite dimensional variational equation (4.65) can be written as the linear algebraic system

\[ A \, x = b , \]  

(4.67)

where the stiffness matrix \( A = (a_{k\ell})_{k,\ell=1}^{n_h} \in \mathbb{R}^{n_h \times n_h} \) and the load vector \( b = (b_1, \ldots, b_{n_h})^T \in \mathbb{R}^{n_h} \) are given by

\[ a_{k\ell} := \int_{\Omega} \left[ \alpha \text{ curl } j_h^{(k)} \cdot \text{ curl } j_h^{(\ell)} + \beta j_h^{(k)} \cdot j_h^{(\ell)} \right] \, dx , \]  

(4.68)

\[ b_k := \int_{\Omega} f \cdot j_h^{(k)} \, dx . \]  

(4.69)
If $\beta = 0$, then
\[
\dim \ker A = n_h^0 = \dim Nd_{1,0}^0(\Omega, T_h),
\]
i.e., the kernel is given by the subspace of discrete irrotational vector fields. We remind that according to Lemma 4.6 we have
\[
Nd_{1,0}^0(\Omega, T_h) = \text{grad } S_{1,0}(\Omega, T_h).
\]
But even if $\beta \neq 0$, the convergence of a standard iterative solver such as SOR (Successive OverRelaxation) or SSOR (Symmetric Successive OverRelaxation) for (4.67) is affected by the kernel of the discrete curl-operator. As a remedy we suggest an additional iterative defect correction step on the subspace of discrete irrational vector fields. This leads to the following hybrid iterative solver:

**Algorithm:** Hybrid iterative solution of the edge element discretized eddy currents equations

**Step 1:** SOR iteration on the edge element discretized problem

Given $j_h^{(m)}$, $m \in \mathbb{N}_0$, compute $j_h^{(m+1/2)}$ by $\nu_1 \geq 1$ SOR iterations applied to the edge element discretized problem (4.66).

**Step 2:** Defect correction on the subspace of irrotational vector fields

Evaluate the residuum on the subspace of irrotational vector fields
\[
r_h(v_h) := \ell(\text{grad } v_h) - a(j_h^{(m+1/2)}, \text{grad } v_h), \quad v_h \in S_{1,0}(\Omega, T_h) \tag{4.72}
\]
and compute $u_h^{(\nu_2)} \in S_{1,0}(\Omega, T_h)$ by $\nu_2 \geq 1$ SOR iterations applied to the defect correction equation
\[
\int_{\Omega} \beta \text{grad } u_h \cdot \text{grad } v_h \, dx = r_h(v_h), \quad v_h \in S_{1,0}(\Omega, T_h). \tag{4.73}
\]
The new iterate $j_h^{(m+1)}$ is then given by
\[
j_h^{(m+1)} = j_h^{(m+1/2)} + \text{grad } u_h^{(m+1)}. \tag{4.74}
\]

**Remark.** If $\beta = 0$, we are faced with an electrostatic or magnetostatic problem and solve the corresponding elliptic boundary problems (cf. Chapter 1.3 and Chapter 1.4) numerically by standard finite element methods.
4.4 Mesh refinement based on a posteriori error estimation

Finite element triangulations should reflect the behavior of the solution in so far as the finite element mesh should be fine in regions with pronounced changes in the solution, whereas a coarser mesh should be used where the solution does not change much.

Typical examples are problems where the solution has a singularity. Singularities may occur due to the geometry of the domain, as for instance, in case of edge or corner singularities (cf. Figures 4.1 and 4.2 below).

![Figure 4.1: Domain with an edge singularity](image1)

![Figure 4.2: Domain with a corner singularity](image2)
Singularities can further be caused by internal layers due to a strongly discontinuous behavior of the coefficients in the equation (e.g., jumps in the material parameters).

The aim of mesh adaptivity is to refine the mesh locally in order to keep the global discretization error equidistributed. Since the solution is in general not known, information about the global discretization error has to be provided by the results of actual computations. This is usually done by so-called a posteriori error estimators which should be cheaply computable and provide local information about the behavior of the discretization error.

In case of edge element approximations $\mathbf{j}_h \in Nd_{1,0}(\Omega, \mathcal{T}_h)$ of the eddy current problem (4.63) we are interested in a quantity $\eta$ such that

$$\gamma \eta \leq \|\mathbf{j}_h - \mathbf{j}\|_{\text{curl}, \Omega} \leq \Gamma \eta$$

with constants $0 < \gamma \leq \Gamma$ that do not depend on the granularity $h$ of the triangulation.

Moreover, we want that quantity $\eta$ to be given in terms of local, elementwise contributions $\eta_k$, $K \in \mathcal{T}_h$, according to

$$\eta = \left( \sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{1/2}.$$  

(4.76)

We use the following refinement strategy:

**Refinement strategy:** We compute the average of the element-oriented contributions

$$\eta_{av} := \frac{1}{n_h^K} \sum_{K \in \mathcal{T}_h} \eta_K,$$  

(4.77)

where $n_h^K$ is the number of elements of the triangulation.

An element $K \in \mathcal{T}_h$ is marked for refinement, if

$$\eta_K > \sigma \eta_{av}$$

with $0 < \sigma < 1$ being a user-specified safety factor.

![Figure 4.3: Refinement of a tetrahedron by bisection](image)
There are several ways to refine an element: One way is to subdivide a tetrahedron into two tetrahedra as shown in Figure 4.3. Usually, the longest edge of the element is chosen and the bisection is done by connecting the midpoint of that edge with the opposite vertices.

In order to maintain geometrical conformity of the refined triangulation, adjacent elements that share subdivided faces have to be refined too, even if they have not been marked for refinement.

Another way than refinement by bisection is the so-called "red-green" refinement (cf., e.g., J. Bey, Tetrahedral grid refinement. Computing 55:355–378, 1995).

Given a tolerance $tol$, the adaptive numerical solution process is stopped when

$$\eta < tol.$$  \hfill (4.79)

Error estimators $\eta$ that satisfy

$$\|j_h - j\|_{\text{curl}, \Omega} \leq \Gamma \eta$$

are called reliable, since the stopping criterion (4.79) guarantees that the global discretization error is of the same order of magnitude as the given tolerance $tol$.

However, the estimator $\eta$ can provide an overestimation of the error which results in unnecessary refinements and thus a waste of computational time. On the other hand, if the estimator $\eta$ satisfies

$$\gamma \eta \leq \|j_h - j\|_{\text{curl}, \Omega},$$

it is called efficient, because unnecessary refinements are avoided.

Obviously, estimators $\eta$ satisfying (4.75) are both efficient and reliable.

In the sequel, we will be concerned with an efficient and reliable residual based a posteriori error estimator $\eta$ for the global discretization error $\|j_h - j\|_{\text{curl}, \Omega}$ that can be derived by an appropriate evaluation of the residual

$$r(q) := (f, q)_{0, \Omega} - (\alpha \text{curl} j_h, \text{curl} q)_{0, \Omega} - (\beta j_h, q)_{0, \Omega}, \quad q \in H_0(\text{curl}, \Omega).$$

with respect to the computed approximation. It is easily seen that the discretization error $e := j_h - j$ satisfies the error equation

$$a(e, q) = r(q), \quad q \in H_0(\text{curl}, \Omega).$$  \hfill (4.80)

The construction of the error estimator relies on the Helmholtz decomposition

$$H_0(\text{curl}, \Omega) = H^0_0(\text{curl}, \Omega) \oplus H^\perp_0(\text{curl}, \Omega)$$  \hfill (4.81)

into the subspace

$$H^0_0(\text{curl}, \Omega) = \{ q \in H_0(\text{curl}, \Omega) \mid \text{curl } q = 0 \}$$

of irrotational vector fields and its orthogonal complement with respect to the weighted $L^2$-inner product $(\beta \cdot, \cdot)_{0, \Omega}$:

$$H^\perp_0(\text{curl}, \Omega) = \{ q \in H_0(\text{curl}, \Omega) \mid (\beta q, q^0)_{0, \Omega} = 0, \quad q^0 \in H^0_0(\text{curl}, \Omega) \}.$$
Vector fields in $H_0^1(\text{curl}, \Omega)$ are called $\beta$-weakly solenoidal (in particular, weakly solenoidal, if $\beta = 1$), since they are orthogonal to the gradients of functions in $H_0^1(\Omega)$.

According to the Helmholtz decomposition (4.81), we split the error into an irrotational part $e^0$ and a $\beta$-weakly solenoidal part $e^\perp$:

$$e = e^0 + e^\perp, \quad e^0 \in H_0^0(\text{curl}, \Omega), \quad e^\perp \in H_0^1(\text{curl}, \Omega).$$

(4.82)

It follows readily from (4.80) that $e^0$ and $e^\perp$ are the unique solutions of the variational problems

$$\int_\Omega \beta e^0 \cdot q^0 \, dx = r(q^0), \quad q^0 \in H_0^0(\text{curl}, \Omega),$$

(4.83)

$$a(e^\perp, q^\perp) = r(q^\perp), \quad q^\perp \in H_0^1(\text{curl}, \Omega).$$

(4.84)

Remark. In view of

$$H_0^0(\text{curl}, \Omega) = \text{grad} \, H_1^0(\Omega),$$

and taking into account that $f$ is solenoidal, an application of Green’s formula reveals

$$r(\text{grad} \, v) = \sum_{K \in \mathcal{T}_h} \int_K (f - \beta j_h) \cdot \text{grad} \, v \, dx =$$

$$= \sum_{K \in \mathcal{T}_h} \int_{\partial K} \text{div}(\beta j_h) \, v \, ds - \sum_{F \in \mathcal{F}_h(\Omega)} \int_F [n \cdot (\beta j_h)]_J v \, d\sigma, \quad v \in H_1^0(\Omega),$$

where $[n \cdot (\beta j_h)]_J$ denotes the jump of $n \cdot (\beta j_h)$ across the face of adjacent elements.

On the other hand, the residual $r(q^\perp), \quad q^\perp \in H_0^1(\text{curl}, \Omega)$ can be written as

$$r(q^\perp) = \sum_{K \in \mathcal{T}_h} \int_K (f - \alpha \text{curl} \, j_h \cdot \text{curl} \, q^\perp - \beta j_h \cdot q^\perp) \, dx =$$

$$= \sum_{K \in \mathcal{T}_h} \int_K (f - \text{curl} \, \alpha \text{curl} \, j_h - \beta j_h) \cdot q^\perp \, dx +$$

$$+ \sum_{F \in \mathcal{F}_h(\Omega)} \int_F [\alpha \text{curl} \, j_h \wedge n]_J \cdot q^\perp \, d\sigma.$$

A proper evaluation of both residuals in (4.83),(4.84) reveals that the following quantities $\eta^{(0)}$, $\eta^{(\perp)}$ and $\eta^{ho}$ constitute the desired error estimator where $\eta^{(0)}$ is related to the irrotational part of the error, whereas $\eta^{(\perp)}$ is associated with the $\beta$-weakly solenoidal part and $\eta^{ho}$ refers to a higher order term.

(i) Estimation of the irrotational part of the error

$$\eta^{(0)} := \left( \sum_{K \in \mathcal{T}_h} (\eta_K^{(0)})^2 \right)^{1/2} + \left( \sum_{F \in \mathcal{F}_h(\Omega)} (\eta_F^{(0)})^2 \right)^{1/2},$$

(4.85)
where \( \eta_{K}^{(0)} \) and \( \eta_{F}^{(0)} \) are given by
\[
\eta_{K}^{(0)} := h_{K} \| \text{div}(\sqrt{\beta} j_{h})\|_{0,K}, \quad K \in T_{h}, \tag{4.86}
\]
\[
\eta_{F}^{(0)} := h_{F}^{1/2} \| \frac{1}{\sqrt{\beta_{av}}} [n \cdot (\beta j_{h})]_{J}\|_{0,F}, \quad F \in \mathcal{F}_{h}(\Omega), \tag{4.87}
\]
where \( h_{K} := \text{diam}(K) \), \( h_{F} := \text{diam}(F) \) and \( \beta_{av} \) is average of \( \beta \) on \( F = K_{i} \cap K_{j} \) according to \( \beta_{av} := (\beta|_{K_{i}} + \beta|_{K_{j}})/2 \).

(ii) **Estimation of the \( \beta \)-weakly solenoidal part of the error**
\[
\eta^{(\perp)} := \left( \sum_{K \in T_{h}} (\eta_{K,1}^{(\perp)})^{2} \right)^{1/2} + \left( \sum_{F \in \mathcal{F}_{h}(\Omega)} (\eta_{F}^{(\perp)})^{2} \right)^{1/2}, \tag{4.88}
\]
\[
\eta^{ho} := \left( \sum_{K \in T_{h}} (\eta_{K,2}^{(\perp)})^{2} \right)^{1/2}. \tag{4.89}
\]
The local contributions \( \eta_{K,1}^{(\perp)} \), \( \eta_{K,2}^{(\perp)} \) and \( \eta_{F}^{(\perp)} \) are given by
\[
\eta_{K,1}^{(\perp)} := h_{K} \| \frac{1}{\sqrt{\alpha}} (\pi_{h} f - \text{curl} \alpha \text{curl} j_{h} - \beta j_{h})\|_{0,K}, \quad K \in T_{h}, \tag{4.90}
\]
\[
\eta_{K,2}^{(\perp)} := h_{K} \| \frac{1}{\sqrt{\alpha}} (f - \pi_{h} f)\|_{0,K}, \quad K \in T_{h}, \tag{4.91}
\]
\[
\eta_{F}^{(\perp)} := h_{F}^{1/2} \| \frac{1}{\sqrt{\alpha_{av}}} [\alpha \text{curl} j_{h} \wedge n]_{J}\|_{0,F}, \quad F \in \mathcal{F}_{h}(\Omega), \tag{4.92}
\]
where \( \pi_{h} \) is the \( L^{2} \)-projection onto \( P_{0}(K)^{3} \) and \( \alpha_{av} \) is the average of \( \alpha \) on \( F \) defined in the same way as \( \beta_{av} \).

**Theorem 4.7** **Residual type a posteriori error estimator**
Let \( \eta^{(0)} \), \( \eta^{(\perp)} \) and \( \eta^{ho} \) be given by (4.85), (4.88) and (4.89), respectively, and define
\[
\eta := \eta^{(0)} + \eta^{(\perp)}. \tag{4.93}
\]
Then, there exist positive constants \( \gamma_{\nu}, 1 \leq \nu \leq 2 \), and \( \Gamma \) depending only on \( \Omega, \alpha, \beta \), and on the local geometry of \( T_{h} \) such that
\[
\gamma_{1} \eta - \gamma_{2} \eta^{ho} \leq \| e \|_{\text{curl}, \Omega} \leq \Gamma (\eta + \eta^{ho}). \tag{4.94}
\]

**Remark.** If the right-hand side \( f \) is a smooth vector field, then the error term \( \eta^{ho} \) is of higher order than the others and can be neglected.