Let \( A \in \mathbb{R}^{m \times n} \), \( m > n \), \( \text{rank} A = n \), \( b \in \mathbb{R}^m \). The linear least squares problem

\[
\| A x - b \|_2 = \min
\]

can be formulated as the linear algebraic system

\[
\begin{pmatrix}
I_m & A \\
A^T & 0
\end{pmatrix}
\begin{pmatrix}
  r \\
x
\end{pmatrix}
= 
\begin{pmatrix}
b \\
0
\end{pmatrix},
\]

where \( I_m \) stands for the \( m \times m \) unit matrix and \( r := b - A x \in \mathbb{R}^m \) is the residual.

(i) Using the normal equations, show that the component \( x \) of the solution of

\( (*) \)

(\(*\)) solves the linear least squares problem (\(*\)).

(ii) Given a decomposition of \( A \) according to

\[
A = Q \begin{pmatrix} R \\ 0 \end{pmatrix}
\]

with an orthogonal matrix \( Q \in \mathbb{R}^{m \times m} \) and a regular upper triangular matrix \( R \in \mathbb{R}^{n \times n} \) show that by orthogonal row and column transformations the linear system

\[
\begin{pmatrix}
I_m & A \\
A^T & 0
\end{pmatrix}
\begin{pmatrix}
p \\
z
\end{pmatrix}
= 
\begin{pmatrix}
f \\
g
\end{pmatrix}
\]

can be transformed to the form

\[
\begin{pmatrix}
I_n & 0 & R \\
0 & I_{m-n} & 0 \\
R^T & 0 & 0
\end{pmatrix}
\begin{pmatrix}
h \\
d \\
z
\end{pmatrix}
= 
\begin{pmatrix}
f_1 \\
d \\
g
\end{pmatrix}.
\]

Specify the relation between the vectors \( h, d, f_1 \) and \( f, p \).
Exercise 18 (Least squares polynomial approximation)

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. We are looking for a polynomial

$$P^{(m)}(t) := \sum_{k=0}^{n-1} x_k^{(m)} t^k,$$

such that for $t_\ell := \frac{\ell}{m}$, $m > n$, there holds:

$$\frac{1}{m} \sum_{\ell=0}^{m} (f(t_\ell) - P^{(m)}(t_\ell))^2 = \min .$$

(i) Formulate the problem as a linear least squares problem

$$\| A^{(m)} x^{(m)} - b^{(m)} \|_2 = \min$$

with respect to the coefficients $x^{(m)} = (x_0^{(m)}, ..., x_{n-1}^{(m)})$. In particular, specify $A^{(m)}$ and $b^{(m)}$.

(ii) Determine a linear system such that its solution is the limit of the sequence $x^{(m)}$ für $m \to \infty$.

Exercise 19 (Updating least squares problems)

Let $A \in \mathbb{R}^{m \times n}$, $m > n$, rank $A = n$ and $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$. Consider

$$\hat{A} := A + uv^T$$

and show: If rank $\hat{A} = n$ and if

$$A = Q \begin{pmatrix} R \\ 0 \end{pmatrix}$$

is the QR-decomposition of $A$, then the QR-decomposition of $\hat{A}$ is given by

$$\hat{A} = \hat{Q} \begin{pmatrix} \hat{R} \\ 0 \end{pmatrix} ,$$

where

$$\hat{Q} = QU, \quad U = F_{m-1} ... F_1 \hat{F}_1 ... \hat{F}_n$$

with suitable Given rotations $F_k$, $1 \leq k \leq m - 1$, $\hat{F}_\ell$, $1 \leq \ell \leq n$.

[Hint: First, compute $F_k$, $1 \leq k \leq m - 1$ such that

$$F_1^T ... F_{m-1}^T Q^T u = \alpha e_1, \quad \alpha = \pm \|Q^T u\|_2 .$$

Show that

$$\hat{H} = \hat{F}_1^T ... \hat{F}_n^T \begin{pmatrix} R \\ 0 \end{pmatrix} + (Q^T u)v^T$$

is an upper Hessenberg matrix. Then, choose $\hat{F}_\ell$, $1 \leq \ell \leq n$, dentsprechend.]
Exercise 20 (Downdating least squares problems)

For linear least squares problems, 'downdating' refers to the effect of eliminating an observation:

Let $A \in \mathbb{R}^{m \times n}$, $m \geq n$, mit rank $A = n$ and let

$$A = Q \begin{pmatrix} R \\ 0 \end{pmatrix}$$

be the QR-decomposition of $A$. Partition $A$ according to

$$A = \begin{pmatrix} a^T_1 \\ \tilde{A} \end{pmatrix},$$

where $\tilde{A} \in \mathbb{R}^{(m-1) \times n}$ and $a^T_1$ is the first row of $A$ and show:

The QR-decomposition of $\tilde{A}$ is given by

$$\tilde{A} = \tilde{Q} \begin{pmatrix} \tilde{R} \\ 0 \end{pmatrix}, \quad \tilde{Q} = Q F_{m-1} \cdots F_1 = \begin{pmatrix} \alpha \\ 0 \\ 0 \tilde{Q} \end{pmatrix}, \quad |\alpha| = 1$$

with suitably chosen Givens rotations $F_k$, $1 \leq k \leq m - 1$.

[Hint: Either apply Exercise 19 or show that the 'downdating' problem is equivalent to the QR-decomposition of the extended matrix $(e_1, A) \in \mathbb{R}^{m \times (n+1)}$.]

Delivery of the homework at latest on October 13, 2005. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.