Numerical Methods for Option Pricing
(Homework 4)

Exercise 10 (Continuous dividends)

In case of options on indices (e.g., on the Dow Jones etc.), dividends are paid throughout the entire year. Therefore, the assumption of continuous dividends represents a simple approach to consider dividends in the Black-Scholes model.

(i) Assume that the dividend is proportional to the value \( S \) of the asset and that after the time interval \( \Delta t \) the dividend

\[ D \Delta t \],

is paid, where \( D \) stands for the proportionality factor. In order to exclude arbitrage, the value of the asset must be reduced accordingly. Assuming that the value of the asset satisfies a geometric Brownian motion with \( \mu, \sigma \), determine the modified Itô process for the value of asset.

(ii) Consider a portfolio consisting of \( c_1 \) bonds \( B \) with risk-free interest rate \( r > 0 \), \( c_2 \) stocks \( S \), and a sold option \( V = V(S, t) \). Determine the associated Itô process under the assumption of a self-financing portfolio. Apply Itô’s formula to \( V \) and show that a risk-free portfolio satisfies the partial differential equation

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D) S \frac{\partial V}{\partial S} - r V = 0.
\]

What is the appropriate choice of \( c_2 \)?

(iii) In case of continuous dividends and European options, show that the put-call parity is of the form

\[
P_t - C_t = K \exp(-r(T - t)) - S \exp(-D(T - t)).
\]

What are the boundary conditions for the partial differential equations from (ii), if \( V = C \) is a European call?

6 Points

Exercise 11 (Greeks I)

(i) Using the Black-Scholes formula for the price \( V \) of a European call, compute the sensitivities

\[
\Delta = \frac{\partial V}{\partial S}, \quad \Gamma = \frac{\partial^2 V}{\partial S^2}.
\]
In order to do so, show first that the distribution function $\Phi$ of the standard normal distribution satisfies

$$S \Phi'(d_1) = K \exp(-r(T-t)) \Phi'(d_2).$$

(ii) Deduce from (i) that the price $V$ is a convex function. Use this property to show that the fraction of the bond

$$c_1(t) = \frac{1}{B(t)} (V(S(t),t) - S(t) \frac{\partial V}{\partial S})$$

is a strictly negative function.

[Hint: Use Taylor expansion of $V(0,t) = 0$ around $V(S,t).$]

(iii) Further, compute the remaining Greeks

$$\theta = \frac{\partial V}{\partial t}, \quad \rho = \frac{\partial V}{\partial r}, \quad \kappa = \frac{\partial V}{\partial \sigma}.$$

6 Points

Exercise 12 (Greeks II)

Write a MATLAB-program `greeks.m` which computes the greeks Delta, Gamma, Vega, Theta, and Rho of a European call and displays them in a graphics at $t = 0$ (dotted line), $t = 0.4$ (dashed line), and $t = 0.8$ (straight line) for

$$K = 100, \quad T = 1, \quad r = 0.1, \quad \sigma = 0.4.$$

6 Points

Exercises 10 and 11 are due on October 17, 2007. Exercise 12 is due on November 19, 2007. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class