Exercise 5 (Block Gauss elimination)

Let $A \in \mathbb{R}^{N \times N}$, $N := \sum_{i=1}^{m} n_i$, $n_i \in \mathbb{N}$, $1 \leq i \leq m$ and $b \in \mathbb{R}^{N}$ be block structured according to

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ A_{21} & \cdots & A_{2m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{pmatrix}, \quad b = (b_1, \cdots, b_m)^T,$$

where $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $1 \leq i, j \leq m$, $b_i \in \mathbb{R}^{n_i}$, $1 \leq i \leq m$.

(i) Consider the solution of the linear algebraic system

$$Ax = b.$$

Use a corresponding structuring of the solution vector $x \in \mathbb{R}^{N}$ and give a block variant of Gauss elimination.

(ii) Give a block variant of the LR-decomposition for block tridiagonal matrices $A \in \mathbb{R}^{N \times N}$, i.e., $A_{ij} = 0$ for $|i-j| \geq 2$, $1 \leq i, j \leq m$.

Exercise 6 (Schur complement)

Let $A \in \mathbb{R}^{N \times N}$, $N := n_1 + n_2$, $n_i \in \mathbb{N}$, $1 \leq i \leq 2$ be a symmetric positive definite block matrix of the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Prove that the Schur complement $S = A_{22} - A_{21}^T A_{11}^{-1} A_{12}$ is also symmetric positive definite.

Exercise 7 (Positive definite symmetric Toeplitz matrices)

A matrix $A \in \mathbb{R}^{n \times n}$ is called a normalized symmetric Toeplitz matrix, if

$$a_{ij} = r_{|i-j|}, \quad 1 \leq i, j \leq n,$$
where \( r_0 = 1 \) and \( r = (r_1, \ldots, r_n)^T \in \mathbb{R}^n \) such that \( A \) is positive definite. The solution of the linear algebraic system

\[
Ax = -r
\]

is referred to as the Yule-Walker problem which plays a significant role in algorithms for the reconstruction of noisy signals.

Consider a partitioning of the matrix \( A \) according to

\[
A = \begin{pmatrix}
\tilde{A} & P_{n-1}\tilde{r} \\
\tilde{r}^T P_{n-1} & 1
\end{pmatrix},
\]

where \( \tilde{A} \in \mathbb{R}^{(n-1) \times (n-1)} \), \( \tilde{r} = (r_1, \ldots, r_{n-1})^T \) and

\[
P_{n-1} = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}.
\]

Use a corresponding partitioning of the right-hand side and of the solution vector. Develop a recursive algorithm for the Yule-Walker problem which computes the solution for the dimension \( n \), provided the solution for the dimension \( n - 1 \) is known.

**Exercise 8** (Cholesky decomposition of positive semidefinite matrices)

Assume that \( A \in \mathbb{R}^{n \times n} \) is positive semidefinite of rank \( r < n \). Prove the following two assertions:

(i) There exists an upper triangular matrix \( R \) with nonnegative diagonal elements such that

\[
A = R^T R.
\]

(ii) There exists a permutation matrix \( P \) such that \( P^T AP \) has a unique Cholesky decomposition of the form

\[
P^T AP = R^T R, \quad R = \begin{pmatrix}
R_{11} & R_{12} \\
0 & 0
\end{pmatrix},
\]

where \( R_{11} \) is an \( r \times r \) upper triangular matrix with positive diagonal elements.

**Delivery of the homework at latest on September 11, 2009.** The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.